Lesson - 1

FIRST ORDER FIRST DEGREE DIFFERENTIAL EQUATIONS : HOMOGENEOUS AND EXACT

1.1 OBJECTIVE OF THE LESSON

After studying this lesson, the student will be in a position to know about Homogeneous, Non-Homogeneous, Exact and Non-exact differential equations and how to solve them.

1.2 STRUCTURE OF THE LESSON

This lesson has the following components.

1.3 Definitions and Examples
1.4 Variables separable
1.5 Homogeneous and Non homogeneous differential equation.
1.6 Exact differential equations and differential equations reducible to exact form
1.7 Answers to Self-Assessment Questions
1.8 Summary
1.9 Technical Terms
1.10 Exercises
1.11 Answers to Exercises
1.12 Model Examination Questions
1.13 Reference Books

1.3.1 Definitions and Examples : Ordinary differential equations find a wide range of application in biological, physical, social and engineering systems which are dynamic in character. They can be used to affectively analyze the evolutionary trend of such systems, they also aid in the formulation of these systems and the qualitative examination of this stability under and adaptability to external stimuli.

1.3.2 Differential Equation : An equation involving dependent and independent variables and the differential coefficients (derivatives) of dependant variable with respect to one independent variables is called a Differential equation.

Eg : 1. \( \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + y = 0 \) 2. \( \frac{dy}{dx} + 4y = 0 \)
1.3.3 **Ordinary Differential Equation**: A differential equation which contains only one independent variable and the derivatives are with respect to this independent variable only is called an ordinary Differential Equations.

**Eg**: 1. \( \frac{d^2 y}{dx^2} + \frac{dy}{dx} + 5 = 0 \)  
2. \( \frac{dy}{dx} - 3x + 4y = 0 \)

1.3.4 **Order of a Differential Equation**:

**Def**: If \( n^{th} \) derivative is the highest derivative in a differential equation then the order of the differential equation, is said to be \( n \).

**Eg.**: For the equation \( \frac{d^2 y}{dx^2} + \frac{dy}{dx} + 7y = 0 \), the order is 2.

**Def 1.3.5**: Let \( F(x, y, y', y'', \ldots, y^m) \) be a differential equation of order \( m \). If the differential equation can be expressed as a rational integral algebraic equation in \( y^m \) by using algebraic operations then the integral power of the highest derivative is called degree of the Differential Equation. Here \( y', y'', \ldots, y^m \) denote the respectively 1st derivative, 2nd derivative,\ldots, \( m^{th} \) derivative with respect to \( x \).

**Eg.** \[ \frac{d^2 y}{dx^2} = \sqrt{1 + \frac{dy}{dx}} \]

\[ \Rightarrow \left( \frac{d^2 y}{dx^2} \right)^2 = 1 + \frac{dy}{dx} \]

:\ For this equation the order = 2 and the degree = 2.

**Note**: The successive differential co-efficients of \( y \) with respect to \( x \) may also be denoted as \( y_1, y_2, \ldots, y_n \).

**Solution 1.3.6**: A relation between the variables without derivatives of a differential equation is said to be a solution or integral of the differential equation if the derivatives obtained there from, the equation is satisfied.

1.3.7 **General solution of a differential equation**: Let \( \phi(x, y, y_1, y_2, \ldots, y_n) = 0 \) be a differential equation of \( n^{th} \) order. If \( F(x, y, c_1, c_2, \ldots, c_n) = 0 \). Where \( c_1, c_2, \ldots, c_n \) are \( n \) independent
arbitrary constants, is a solution of the given differential equation, then \( F(x, y, c_1, c_2, \ldots, c_n) = 0 \) is called general solution of the differential equation.

Particular and singular solution of a differential equation.

### 1.3.8 Particular solution:

**Def:** If \( \phi(x, y, c_1, c_2, \ldots, c_n) = 0 \) is the general solution of a differential equation \( F(x, y, y_1, y_2, \ldots, y_n) = 0 \) then \( \phi(x, y, K_1, K_2, \ldots, K_n) = 0 \), where \( K_1, K_2, \ldots, K_n \) are fixed constants, is called a particular solution of the differential equation \( F(x, y, y_1, y_2, \ldots, y_n) = 0 \).

### 1.3.9 Singular solution:

**Def:** A solution \( F(x, y) = 0 \) of the differential equation \( \phi(x, y, y_1, y_2, \ldots, y_n) \) is called a singular solution if \( F(x, y) \) does not contain arbitrary constants and \( F(x, y) \) is not obtained by giving particular values to arbitrary constants.

### 1.3.10 First Order First Degree Differential Equation:

\[
\frac{dy}{dx} = F(x, y)
\]

is called First order first degree differential equation.

First order First degree differential equation can be solved by using the following four methods.

1. Variable separable method
2. Homogeneous equations and equations reducible to homogeneous form
3. Exact equations and equations which can be made exact with the help of Integrating factors.
4. Linear equations and Bernoulli's form.

### 1.4.1 Variable's separable method:

If the differential equation \( \frac{dy}{dx} = F(x, y) \) can be expressed in the form \( \frac{dy}{dx} = \frac{F(x)}{g(y)} \) or \( \frac{dy}{dx} = f(x) \cdot g(y) \), where \( f \) and \( g \) are continuous functions of a single variable, then the differential equation \( \frac{dy}{dx} = f(x, y) \) is said to be of the form variables separable.

Some differential equations can be brought to variables separable form by some substitution.

**Eg - 1.4.2:** Solve \( \frac{dy}{dx} = e^{x-y} + x^2 e^{-y} \)
Sol. Given equation is \( \frac{dy}{dx} = e^{x-y} + x^2e^{-y} = \frac{e^x + x^2}{e^y} \)

using variables separable method we get

\[
\int e^y \, dy = \int \left( e^x + x^2 \right) \, dx
\]

\[
\Rightarrow e^y = e^x + \frac{x^3}{3} + c
\]

Eg - 1.4.3 : Solve \( (1 + x^3) \cdot xy \cdot \frac{dy}{dx} = (1 + y^2) \cdot (1 + x + x^2) \)

Sol : Given equation in \( (1 + x^2) \cdot xy \cdot \frac{dy}{dx} = (1 + y^2) \cdot (1 + x + x^2) \)

using variable separable method, we get

\[
\int \frac{y}{1 + y^2} \, dy = \int \frac{1 + x + x^2}{(1 + x^2) \cdot x} \, dx = \int \frac{1 + x^2 + x}{(1 + x^2) \cdot x} \, dx
\]

\[
\Rightarrow \frac{1}{2} \int \frac{2y}{1 + y^2} \, dy = \int \frac{1}{x} + \frac{1}{1 + x^2} \, dx
\]

General solution is \( \frac{1}{2} \log |1 + y^2| = \log |x| + \tan^{-1} x + c \)

Eg - 1.4.4 : Solve \( \frac{dy}{dx} = (4x + y + 1)^2 \)

Sol : Given equation is \( \frac{dy}{dx} = (4x + y + 1)^2 \) \( \cdots \cdots \cdots \cdots \cdots \) (1)

Let \( 4x + y + 1 = t \) \( \cdots \cdots \cdots \cdots \cdots \) (2)

Differentiating with respect to \( x \)

\[
4 + \frac{dy}{dx} = \frac{dt}{dx}
\]
\[ \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 4 \]  

(3)

Substituting (2) and (3) in (1) we get

\[ \frac{dt}{dx} - 4 = t^2 \]

\[ \Rightarrow \frac{dt}{dx} = t^2 + 4 \]

using variables separable method we get

\[ \int \frac{dt}{t^2 + 4} = \int dx \]

\[ \Rightarrow \frac{1}{2} \tan^{-1} \left( \frac{t}{2} \right) = x + c \]

general solution is

\[ \frac{1}{2} \tan^{-1} \left( \frac{4x + y + 1}{2} \right) = x + c \]

1.5 Homogeneous Differential Equations

1.5.1 Homogeneous Function: A function \( F(x, y) \) is called a homogeneous function of degree \( 'n' \) if \( F(Kx, Ky) = K^n F(x, y) \) for all values of \( K \).

Eg.: \( F(x, y) = \frac{x^2 + y^2}{x + y} \)

Now \( F(Kx, Ky) = \frac{K^2 x^2 + K^2 y^2}{Kx + Ky} \)

\[ = \frac{K^2 (x^2 + y^2)}{K(x + y)} \]

\[ = \frac{K(x^2 + y^2)}{(x + y)} = K \cdot F(x, y) \]
1.6

\[ F(x, y) \] is a homogeneous function of degree '1'.

\[ F(x, y) = \sin x + \cos y \]

Now \( F(Kx, Ky) = \sin Kx + \cos Ky \neq K \sin x + K \cos y \)

\[ \therefore F(x, y) \] is not homogeneous function.

### 1.5.2 Homogeneous differential equation

\( \frac{dy}{dx} = F(x, y) \) is called a Homogeneous differential equation if \( F(x, y) \) is a Homogeneous function of degree zero. Or \( F(Kx, Ky) = Kx + Ky \)

\[ = F(x, y), \forall K \in \mathbb{R} \setminus \{0\} . \]

Eg. (1) \( \frac{dy}{dx} = \frac{x + y}{x - y} \)

\[ F(x, y) = \frac{x + y}{x - y} \]

Now \( F(Kx, Ky) = \frac{Kx + Ky}{Kx - Ky} = \frac{x + y}{x - y} = K^0 F(x, y) \)

\[ \therefore F(x, y) \] is a Homogeneous function of degree zero.

\[ \therefore \frac{dy}{dx} = \frac{x + y}{x - y} \] is a Homogeneous differential equation.

**Working Rule for Homogeneous differential equation**:

1. Let \( \frac{dy}{dx} = F(x, y) \) be a homogeneous differential equation.

\[ \frac{dy}{dx} = F(x, y) \] \[ \text{--------- (1)} \]

\[ = F(1, y/x) \]

Put \( y = vx \) \[ \text{--------- (2)} \]

Differentiating with respect to 'x' \[ \therefore F(Kx, Ky) = F(x, y) \]

\[ \text{Put } K = 1/x \text{ then } \]

\[ F(x, y) = F(1, y/x) \]
\[
\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \text{--------- (3)}
\]

Substituting (2) and (3) in (1) we get

\[
v + x \frac{dv}{dx} = F(1, v)
\]

\[
\therefore x \frac{dv}{dx} = F(1, v) - v
\]

Separating variables and integrating. Now, we get \( F(x, v, c) = 0 \) \text{--------- (4)}

Put \( v = \frac{y}{x} \) in (4)

General solution is \( F\left(x, \frac{y}{x}, c\right) = 0 \)

2. Let \( \frac{dx}{dy} = F(x, y) \) be a homogeneous differential equation.

\[
\frac{dx}{dy} = F(x, y) = F\left(\frac{x}{y}, 1\right) \quad \text{--------- (1)}
\]

Put \( x = vy \)

i.e. \( v = \frac{x}{y} \) \text{--------- (2)}

Differentiating with respect to \( y \)'

\[
\frac{dx}{dy} = v + y \frac{dv}{dy} \quad \text{--------- (3)}
\]

Substituting (2) and (3) in (1) we get

\[
v + y \frac{dv}{dy} = F(v, 1)
\]

\[
= y \frac{dv}{dy} = f(v, 1) - v
\]
Separating variables and integrating

\[ \frac{d}{d(v,1)} F(v) = \int \frac{dy}{y} \]

Now, we get \( F(v, y, c) = 0 \) \( \quad \text{(4)} \)

Put \( v = \frac{x}{y} \) in \( (4) \)

Now General solution is \( F\left(\frac{x}{y}, y, c\right) = 0 \)

1.5.3 : Solve \( \frac{dy}{dx} = \frac{y + \tan \frac{y}{x}}{x} \)

**Sol** : Given equation is \( \frac{dy}{dx} = \frac{y + \tan \frac{y}{x}}{x} \) \( \quad \text{(1)} \)

Here \( F(x, y) = \frac{y}{x} + \tan \frac{y}{x} \)

Now, \( F(Kx, Ky) = \frac{Ky}{Kx} + \tan \left( \frac{Ky}{Kx} \right) = \frac{y}{x} + \tan \frac{y}{x} = F(x, y) \)

\( \therefore (1) \) is a Homogeneous differential equation.

Put \( y = vx \) \( \quad \text{(2)} \)

Differentiating with respect to \('x'\).

\[ \frac{dy}{dx} = v + \frac{v}{x} \frac{dx}{dx} \] \( \text{(3)} \)

Substituting (2) and (3) in (1), we get

\[ v + x \frac{v}{x} = \frac{v}{x} + \tan \frac{v}{x} = v + \tan v \]

\( \Rightarrow x \frac{v}{dx} = \tan v \)

Separating variables integrating
\[ \int \frac{dv}{\tan v} = \int \frac{dx}{x} \]

\[ \Rightarrow \int \cot v \, dv = \int \frac{dx}{x} \]

\[ \Rightarrow \log |\sin v| = \log |x| + \log c \]

\[ \Rightarrow \log |\sin v| = \log |cx| \]

\[ \Rightarrow \sin v = cx \]

\[ \therefore \text{ General solution is } \sin \left( \frac{y}{x} \right) = cx \]

2) 1.5.4 : Solve \( xdy = \left( y + x \cos^2 \frac{y}{x} \right) \, dx \)

**Sol** : Given equation is \( xdy = \left( y + x \cos^2 \frac{y}{x} \right) \, dx \)

\[ \Rightarrow \frac{dy}{dx} = \frac{y + x \cos^2 \frac{y}{x}}{x} \]

\[ = \frac{y}{x} + \cos^2 \left( \frac{y}{x} \right) \quad \text{(1)} \]

Now (1) is a Homogeneous differential equation.

Put \( y = vx \) \quad \text{(2)}

Differentiating with respect to ‘\( x \)’

\[ \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \text{(3)} \]

Substituting (2) and (3) in (1), we get

\[ v + x \frac{dv}{dx} = v + \cos^2 v \]

\[ \Rightarrow x \frac{dv}{dx} = \cos^2 v \]
separating variables and integrating

\[\int \frac{d\nu}{\cos^2 \nu} = \int \frac{dx}{x}\]

\[\Rightarrow \int \sec^2 \nu \, d\nu = \int \frac{dx}{x}\]

\[\Rightarrow \tan \nu = \log|x| + c\]

\[\therefore \text{ General solution is } \tan \frac{y}{x} = \log|x| + c\]

3) 1.5.5 : Solve \(\frac{dy}{dx} = \frac{x-y}{x+y}\)

\textbf{Sol.} : Given equation is

\[\frac{dy}{dx} = \frac{x-y}{x+y} \quad \text{(1)}\]

\[F(x,y) = \frac{x-y}{x+y}\]

\[F(Kx, Ky) = \frac{Kx-Ky}{Kx+Ky}\]

\[= \frac{x-y}{x+y} = F(x, y)\]

(1) is a Homogeneous differential equation.

Put \(y = \nu x \quad \text{(2)}\)

Differentiate with respect to \(x\)

\[\frac{dy}{dx} = \nu + x \frac{d\nu}{dx} \quad \text{(3)}\]

Substituting (2) and (3) in (1), we get

\[\nu + x \frac{d\nu}{dx} = \frac{x-\nu x}{x+\nu x} = \frac{1-\nu}{1+\nu}\]
\[ \Rightarrow x \frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-v-v^2}{1+v} \]

\[ \Rightarrow x \frac{dv}{dx} = -\left(\frac{v^2 + 2v - 1}{v+1}\right) \]

Separating variables and integrating

\[ \int \frac{v+1}{v^2 + 2v - 1} dv = -\int \frac{dx}{x} \]

\[ \Rightarrow \frac{1}{2} \int \frac{2v+2}{v^2 + 2v - 1} dv = -\int \frac{dx}{x} \]

\[ \Rightarrow \frac{1}{2} \log\left|v^2 + 2v - 1\right| = -\log|x| + \log c \]

\[ \Rightarrow \log\left(\frac{v^2 + 2v - 1}{x}\right) = \log c \]

\[ \Rightarrow \left(\frac{v^2 + 2v - 1}{x}\right)^\frac{1}{2} = \frac{c}{x} \]

Squaring on both sides, we get

\[ \left(\frac{v^2 + 2v - 1}{x}\right)^2 = \frac{c^2}{x^2} \]

Put \( v = \frac{y}{x} \) is ------ (4)

\[ \Rightarrow \frac{y^2}{x^2} + \frac{2y}{x} - 1 = \frac{c^2}{x^2} \]

General solution is \( y^2 + 2xy - x^2 = K \), where \( K = c^2 \)

4) 1.5.6 : Solve \( x(x-y)dy = y(x+y)dx \)

Solution : Given equation is \( x(x-y)dy = y(x+y)dx \)
\[ \Rightarrow \frac{dy}{dx} = \frac{y(x + y)}{x(x - y)} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{xy + y^2}{x^2 - xy} \quad \text{-------- (1)} \]

(1) is a Homogeneous differential equation.

Put \[ y = vx \quad \text{-------- (2)} \]

Differentiate with respect to 'x'

\[ \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \text{-------- (3)} \]

Substituting (2) and (3) in (1), we get

\[ v + x \frac{dv}{dx} = \frac{x \cdot vx + v^2 x^2}{x^2 - x^2} = \frac{v + v^2}{1 - v} \]

\[ \Rightarrow x \frac{dv}{dx} = \frac{v + v^2}{1 - v} - v = \frac{y + v^2 - y + v^2}{1 - v} = \frac{2v^2}{1 - v} \]

Separating variables and integrating

\[ \int \frac{1 - v}{2v^2} dv = \int \frac{dx}{x} \]

\[ \Rightarrow \frac{1}{2} \int \frac{1}{v^2} dv - \frac{1}{2} \int \frac{1}{v} dv = \int \frac{dx}{x} \]

\[ \Rightarrow \frac{1}{2} \left( -\frac{1}{v} \right) - \frac{1}{2} \log |v| = \log |x| + c \]

\[ = - \frac{1}{2v} - \frac{1}{2} \log |v| = \log |x| + c \]

\[ \Rightarrow - \frac{1}{2v} - \frac{1}{2} \log |v| = \log |x| + c \quad \text{-------- (4)} \]
Put $v = \frac{y}{x}$ in (4)

General solution is

$$-x - \frac{1}{2} \log \frac{y}{x} = \log |x| + c.$$  

5) 1.5.7 : Solve $\frac{dy}{dx} = \frac{-2x}{x + ye^y}$

**Sol :** Given equation is $\frac{dy}{dx} = \frac{y}{x + ye^y}$

$$\Rightarrow \frac{dx}{dy} = \frac{x + y}{y} \quad \text{-------- (1)}$$

(1) is a Homogeneous differential equation.

Put $x = vy$ \quad \text{-------- (2)}

Differentiate with respect to 'y'

$$\Rightarrow \frac{dx}{dy} = v + y \frac{dv}{dy} \quad \text{-------- (3)}$$

Substituting (2) and (3) in (1)

$$v + y \frac{dv}{dy} = v + e^{-2v}$$

$$\Rightarrow y \frac{dv}{dy} = e^{-2v}$$

Separating variables and integrating

$$\int \frac{dv}{e^{-2v}} = \int \frac{dy}{y}$$

$$\Rightarrow \int e^{2v} dv = \int \frac{dy}{y}$$
\[
\Rightarrow e^{2v} = \log|y| + c \quad \text{-------- (4)}
\]

Put \( v = \frac{x}{y} \) in (4)

General solution is

\[
\frac{2x}{y} = \frac{e^y}{2} = \log|y| + c
\]

6) 1.5.8 : Solve \( y \, dx = \left( x + \sqrt{y^2 - x^2} \right) \, dy \)

**Sol. :** Given equation is \( y \, dx = \left( x + \sqrt{y^2 - x^2} \right) \, dy \)

\[
\Rightarrow \frac{dx}{dy} = \frac{x + \sqrt{y^2 - x^2}}{y}
\]

\[
= \frac{x}{y} + \frac{\sqrt{y^2 - x^2}}{y} \quad \text{-------- (1)}
\]

(1) is a Homogeneous differential equation.

put \( x = v \, y \quad \text{-------- (2)} \)

Differentiate with respect to \( 'y' \)

\[
\frac{dx}{dy} = v + y \frac{dv}{dy} \quad \text{-------- (3)}
\]

Substituting (2) and (3) in (1), we get

\[
v + y \frac{dv}{dy} = v + \frac{\sqrt{y^2 - v^2} \, y^2}{y} = v + \sqrt{1 - v^2}
\]

\[
\Rightarrow y \frac{dv}{dy} = \sqrt{1 - v^2}
\]

Separating variables and integrating
First Order First Degree Differential Equations...

\[ \int \frac{d v}{\sqrt{1-v^2}} = \int \frac{dy}{y} \]

\[ \Rightarrow \sin^{-1} v = \log|y| + c \quad \text{------ (4)} \]

Put \( v = \frac{x}{y} \) in (4)

General solution is \( \sin^{-1} \left( \frac{x}{y} \right) = \log|y| + c \)

1.2 Non-Homogeneous Differential Equation (or) Equations Reducible to Homogeneous Form

1.5.9 : \[ \frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \] where \( c_1 \neq 0 \) or \( c_2 \neq 0 \) is called non homogeneous differential equation of first order in \( x \) and \( y \).

1.5.10 : General solution of non-homogeneous differential equation \[ \frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \]

Given equation is \[ \frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \quad \text{------ (1)} \]

(1) can be reduced to homogeneous form (or) variables separable form by change of variables.

Case 1 : The two equations \( a_1 x + b_1 y + c_1 = 0 \) and \( a_2 x + b_2 y + c_2 = 0 \) are such that \( a_1 b_2 - a_2 b_1 \neq 0 \)

\[ \therefore \exists \; \text{a unique solution for the given system of equations. Let} \; (h, K) \; \text{be that solution} \]

\[ \therefore a_1 h + b_1 K + c_1 = 0 \; \text{and} \; a_2 h + b_2 K + c_2 = 0 \]

Solving \[ h = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \quad K = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1} \quad \text{------ (2)} \]

Now, write \( x = X + h, \quad y = Y + K \quad \text{------ (3)} \)

where \( x \) and \( y \) are new variables.
From (3) \( \frac{dx}{dX} = 1, \frac{dy}{dY} = 1 \), so that \( \frac{dy}{dx} = \frac{dY}{dX} \) \( \text{-------- (4)} \)

Substituting (3) and (4) in (1)

\[
\frac{dY}{dX} = \frac{a_1(X + h) + b_1(Y + K) + c_1}{a_2(X + h) + b_2(Y + K + c_2)}
\]

\[
= \frac{a_1X + b_1Y + a_1h + b_1K + c_1}{a_2X + b_2Y + a_2h + b_2K + c_2}
\]

\[
= \frac{a_1X + b_1Y}{a_2X + b_2Y} \text{ \( \text{-------- (5)} \) (\( : a_1h + b_1K + c_1 = 0 \) and \( 0 = a_2h + b_2K + c_2 \))}
\]

Now, (5) is a Homogeneous differential equation and it can be solved by the method discussed.

\( \therefore \) The general solution of (5) is of the form

\[
\phi(X, Y, c) = 0
\]

\( \therefore \) General solution of given equation is of the form

\[
\phi(x - h, y - k, c) = 0
\]

**Case 2:** Suppose \( a_1x + b_1y + c_1 = 0 \) and \( a_2x + b_2y + c_2 = 0 \) are such that \( a_1b_2 - a_2b_1 = 0 \)

\[
\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = u
\]

\[
\Rightarrow a_1 = ua_2, \quad b_1 = ub_2 \text{ \( \text{-------- (6)} \)}
\]

substituting (6) in (1) of case 1

\[
\frac{dy}{dx} = \frac{ua_2x + ub_2y + c_1}{a_2x + b_2y + c_2} \text{ \( \text{-------- (7)} \)}
\]

If \( c_1 = uc_2 \). Then \( \frac{dy}{dx} = u \)

\( \therefore \) General solution of (1) is \( y = ux + K \)

If \( c_1 = 4c_2, \ c_1 \neq uc_2 \) then put \( a_2x + b_2y = t \) \( \text{-------- (8)} \)
Differentiating with respect to \( x \)

\[
a_2 + b_2 \frac{dy}{dx} = \frac{dt}{dx}
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{\frac{dt}{dx} - a_2}{b_2} \tag{9}
\]

substituting (8) and (9) in (7)

\[
\frac{1}{b_2} \left( \frac{dt}{dx} - a_2 \right) = \frac{ut + c_1}{t + c_2}
\]

\[
\Rightarrow \frac{dt}{dx} = \frac{b_2 (ut + c_1)}{t + c_2} + a_2 = \frac{b_2 (ut + c_1) + a_2 (t + c_2)}{t + c_2} \tag{10}
\]

Now (10) is in variables separable form and it can be solved.

General solution of (10) is of the form \( f(t, x, c) = 0 \).

\[\therefore \text{General solution of (1) is of the form} \]

\[f(a_2 x + b_2 y; x, c) = 0\]

**Eg. 1 - 1.5.11 :** Solve \((x + y - 1) \frac{dy}{dx} = x - y + 2\)

**Sol :** Given equation is \( \frac{dy}{dx} = \frac{x - y + 2}{x + y - 1} \tag{1} \)

Here \( a_1 = 1, b_1 = -1, a_2 = 1, b_2 = 1 \)

Now \( a_1 b_2 - a_2 b_1 = 1 - (-1) = 2 \neq 0 \)

Let \( (h, K) \) be the solution of \( x - y + 2 = 0 \) and \( x + y - 1 = 0 \)

\[\therefore h - K + 2 = 0 \tag{2} \quad h + K - 1 = 0 \tag{3}\]

Solving (2) and (3), \( h = -\frac{1}{2}, K = \frac{3}{2} \)
Now, put \( x = X - \frac{1}{2}, \ y = Y + \frac{3}{2} \) \( \ldots (4) \)

\[ \therefore \frac{dy}{dx} = \frac{dY}{dX} \quad \ldots (5) \]

Substituting (4) and (5) in (1)

\[ \frac{dY}{dX} = \frac{\left( X - \frac{1}{2} \right) - \left( Y + \frac{3}{2} \right) + 2}{\left( X - \frac{1}{2} \right) + \left( Y + \frac{3}{2} \right) - 1} = \frac{X - Y}{X + Y} \quad \ldots (6) \]

Here \( F(KX, KY) = F(X, Y) \)

\[ \therefore \frac{dY}{dX} = \frac{X - Y}{X + Y} \]

is a homogeneous differential equation

Put \( Y = VX \) \( \ldots (7) \)

Differentiate with respect to 'X'

\[ \frac{dY}{dX} = VX + \frac{dV}{dX} \quad \ldots (8) \]

Substituting (7) and (8) in (6)

\[ VX + \frac{dV}{dX} = \frac{X - VX}{X + VX} = \frac{1 - V}{1 + V} \]

\[ \Rightarrow X \frac{dV}{dX} = \frac{1 - V}{1 + V} - V = \frac{1 - V - V^2}{1 + V} = -\frac{(V^2 + 2V - 1)}{1 + V} \]

Using variables separable method, we get

\[ \int \frac{V + 1}{V^2 + 2V - 1} dV = - \int \frac{dX}{X} \]

\[ \Rightarrow \frac{1}{2} \int \frac{2V + 2}{V^2 + 2V - 1} dV = - \int \frac{dX}{X} \]
\[
\frac{1}{2} \log |V^2 + 2V - 1| = -\log |X| + \log C
\]
\[
\log |V^2 + 2V - 1|^\frac{1}{2} = \log \left| \frac{C}{X} \right|
\]
\[
V^2 + 2V - 1 = \frac{C^2}{X^2}
\]
\[
\frac{Y^2}{X^2} + \frac{2Y}{X} - 1 = \frac{C^2}{X^2}
\]

:. General solution of (6) is \( Y^2 + 2XY - X^2 = C^2 \)

:. General solution of (1) is

\[
\left( y - \frac{3}{2} \right)^2 + 2 \left( x + \frac{1}{2} \right) \left( y - \frac{3}{2} \right) - \left( x + \frac{1}{2} \right)^2 = c^2
\]

i.e. \( y^2 - x^2 + 2xy - 2y - 4x + \frac{1}{2} = c^2 \)

**Eg. 2 - 1.5.12**: Solve \( \frac{dy}{dx} = \frac{x - 2y + 3}{2x - y + 5} \)

**Solution**: Given equation is \( \frac{dy}{dx} = \frac{x - 2y + 3}{2x - y + 5} \) \( \cdots \cdots \) (1)

Here \( a_1 = 1, \ b_1 = -2, \ a_2 = 2, \ b_2 = -1 \)

Now, \( a_1b_2 - a_2b_1 = -1 + 4 = 3 \neq 0 \)

Let \( (h, K) \) be the solution of \( x - 2y + 3 = 0 \) and \( 2x - y + 5 = 0 \)

Now \( h - 2K + 3 = 0 \) \( \cdots \cdots \) (2)

\[ 2h - K + 5 = 0 \] \( \cdots \cdots \) (3)

Solving (2) and (3), \( h = -\frac{7}{3}, \ K = \frac{1}{3} \)
Now, \( x = X + h = X - \frac{7}{3}, \) \( y = Y + K = Y + \frac{1}{3} \) ------- (4)

\[ \therefore \frac{dY}{dX} = \frac{dy}{dx} \]  
----------- (5)

Substituting (4) and (5) in (1)

\[ \frac{dY}{dX} = \frac{X - \frac{7}{3} - 2\left(Y + \frac{1}{3}\right) + 3}{2\left(X - \frac{7}{3}\right) - \left(Y + \frac{1}{3}\right) + 5} = \frac{X - 2Y}{2X - Y} \]
----------- (6)

Now (6) is a Homogeneous differential equation

Put \( Y = VX \)  
----------- (7)

Differentiate with respect to 'X'

\[ \frac{dY}{dX} = V + X \frac{dV}{dX} \]
----------- (8)

Substituting (7) and (8) in (6)

\[ V + X \frac{dV}{dX} = \frac{X - 2VX}{2X - VX} = \frac{1 - 2V}{2 - V} \]

\[ \Rightarrow X \frac{dV}{dX} = \frac{1 - 2V}{2 - V} - V = \frac{1 - 2V - 2V + V^2}{2 - V} = \frac{V^2 - 4V + 1}{2 - V} \]

Using variables separable method, we get

\[ \int \frac{V - 2}{V^2 - 4V + 1} dV = - \int \frac{dX}{X} \]

\[ \Rightarrow \frac{1}{2} \int \frac{2V - 4}{V^2 - 4V + 1} dV = - \int \frac{dX}{X} \]

\[ \Rightarrow \frac{1}{2} \log \left(V^2 - 4V + 1\right) = - \log |X| + \log C \]

\[ \Rightarrow \log \left(V^2 - 4V + 1\right)^{\frac{1}{2}} = \log \left|\frac{C}{X}\right| \]
\[ V^2 - 4V + 1 = \frac{C^2}{X^2} \]

\[ \therefore \text{General solution of (6) is } Y^2 - 4XY + X^2 = C^2 \]

\[ \therefore \text{General solution of (1) is } \]

\[ \left( y - \frac{1}{3} \right)^2 - 4\left( x + \frac{7}{3} \right)\left( y - \frac{1}{3} \right) + \left( x + \frac{7}{3} \right)^2 = C^2 \]

i.e. \[ y^2 + x^2 - 4xy + 6x - 10y + \frac{26}{3} = C^2 \]

**Eg. 3 - 1.5.13**: Solve \( \left( 2x + 2y + 2 \right) \frac{dy}{dx} = x + y + 1 \)

**Sol**: Given equation is \[ \frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3} \quad \text{-------- (1)} \]

Here \( a_1 = 1, b_1 = 1, a_2 = 2, b_2 = 2 \)

Now \( a_1b_2 - a_2b_1 = 2 - 2 = 0 \)

Hence, we take \( (x + y) = u \quad \text{-------- (2)} \)

Differentiate with respect to \'x' \n
\[ 1 + \frac{dy}{dx} = \frac{du}{dx} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1 \quad \text{-------- (3)} \]

Substituting (2) and (3) in (1) \n
\[ \frac{du}{dx} - 1 = \frac{u + 1}{2u + 3} \]

\[ \Rightarrow \frac{du}{dx} = \frac{u + 1}{2u + 3} + 1 = \frac{u + 1 + 2u + 3}{2u + 3} = \frac{3u + 4}{2u + 3} \quad \text{-------- (4)} \]

Using variables separable method, we get
\[ \int \frac{2u + 3}{3u + 4} \, du = \int dx \]

\[ \Rightarrow \int \frac{2}{3}(3u + 4) + \frac{1}{3} \, du = \int dx \]

\[ \Rightarrow \int du + \frac{1}{3} \int \frac{1}{3u + 4} \, du = \int dx \]

\[ \Rightarrow \frac{2}{3} u + \frac{1}{3} \cdot \frac{1}{3} \log |3u + 4| = x + C \]

General solution of (4) is \[ 6u + \log |3u + 4| = 9x + K \], where \( K = 9C \)

\[ \therefore \text{General solution of (1) is } 6(x + y) + \log |3x + 3y + 4| = 9x + K \]

**Eg. 1.5.14 :** Solve \((2x + y + 1) \, dx + (4x + 2y - 1) \, dy = 0\)

**Solution :** Given equation is \((2x + y + 1) \, dx + (4x + 2y - 1) \, dy = 0\)

\[ \Rightarrow \frac{dy}{dx} = -\frac{(2x + y + 1)}{4x + 2y - 1} \quad \text{(1)} \]

Here \( a_1 = -2, \ b_1 = -1, \ a_2 = 4, \ b_2 = 2 \)

\[ a_1b_2 - a_2b_1 = -4 - (-4) = 0 \]

Hence we take \(2x + y = u\) \quad \text{(2)}

Differentiate with respect to 'x'

\[ 2 + \frac{dy}{dx} = \frac{du}{dx} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2 \quad \text{(3)} \]

Substituting (2) and (3) in (1)

\[ \frac{du}{dx} - 2 = -\frac{(u + 1)}{2u - 1} \]

\[ \Rightarrow \frac{du}{dx} = \frac{-u - 1 + 4u - 2}{2u - 1} = \frac{3u - 3}{2u - 1} \quad \text{(4)} \]
Using variables separable method, we get
\[
\int \frac{2u-1}{3u-3} \, du = \int dx
\]
\[
\Rightarrow \frac{2}{3}(3u-3) + 1
\]
\[
\Rightarrow \frac{2}{3} \, du + \frac{1}{3u-3} \, du = \int dx
\]
\[
\Rightarrow \frac{2}{3} \, du + \frac{1}{3u-3} \, du = \int dx
\]
\[
\Rightarrow \frac{2}{3} u + \frac{1}{3} \log |3u-3| = x + c
\]

General solution of (4) is \( 2u + \log |3u-3| = 3x + 3c \)

\[\therefore\] General solution of (1) is \( 2(2x + y) + \log |6x + 3y - 3| = 3x + K \)

where \( K = 3c \)

1.6.1 Exact Differential Equation: A differential equation \( Mdx + Ndy = 0 \). Where M and N are functions of \( x, y \) is called Exact differential equation if there exists a function \( u \) of \( x, y \) having continuous first partial derivatives such that \( \frac{\partial u}{\partial x} = M, \frac{\partial u}{\partial y} = N \) for such a function \( u \), we write

\[ Mdx + Ndy = du, \text{ where } du \text{ stands for } \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy. \]

1.6.2: The necessary and sufficient condition for the differential equation \( Mdx + Ndy = 0 \) to be exact is suppose \( Mdx + Ndy = 0 \) is exact \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \).

Proof: \( \exists f(x, y) \) having continuous first partial derivations such that

\[ M = \frac{\partial F}{\partial x} \text{ and } N = \frac{\partial F}{\partial y} \]

Differentiate Partially with respect to 'y'  
\[
\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \cdot \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \cdot \partial y}
\]
For functions having continuous first partial derivating it is true that

\[
\frac{\partial^2 F}{\partial y \cdot \partial x} = \frac{\partial^2 F}{\partial x \cdot \partial y}
\]

\[
\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
\]

Now suppose \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \)

To prove \( M \, dx + N \, dy = 0 \) is an exact differential equation. Let \( V(x, y) = \int_{x} M \, dx \)

(where \( \int_{x} M \, dx \) means while integrating keep 'y' constant)

Now, \( \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial^2 V}{\partial y \cdot \partial x} = \frac{\partial^2 V}{\partial x \cdot \partial y} \)

\[
\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial^2 V}{\partial x \cdot \partial y} = 0
\]

\[
\Rightarrow \frac{\partial}{\partial x} \left( N \frac{\partial V}{\partial y} \right) = 0
\]

\[
\Rightarrow N \frac{\partial V}{\partial y} \text{ does not contain } x \text{ terms, i.e., it is a function of } y \text{ only.}
\]

\[
\Rightarrow N \frac{\partial V}{\partial y} = \phi(y)
\]

\[
\Rightarrow N = \frac{\partial V}{\partial y} + \phi(y)
\]

Now \( M \, dx + N \, dy = \frac{\partial V}{\partial x} \, dx + \left( \frac{\partial V}{\partial y} + \phi(y) \right) \, dy \)

\[
= \frac{\partial V}{\partial x} \, dx + \frac{\partial V}{\partial y} \, dy + \phi(y) \, dy
\]

\[
= d\, V + \phi(y) \, dy
\]

\[
= d\left( V + \psi(y) \right) \text{ where } d(\psi(y)) = \phi(y) \, dy
\]
\[ M \, dx + N \, dy = 0 \] is exact differential equation.

If \( M \, dx + N \, dy = 0 \) is exact differential equation then the general solution is

\[ V + \int \phi(y) \, dy = c \quad \text{where} \quad V = \int M \, dx \]

keeping \( y \) constant.

\[ \phi(y) = N - \frac{\partial V}{\partial y}, \quad \text{where } \phi(y) \text{ is free from } x \text{ or independent of } x. \]

\textbf{Eg. 1.6.3 :} Solve \((hx + by + f) \, dy + (ax + hy + g) \, dx = 0\)

\textbf{Sol :} Given equation is \((ax + hy + g) \, dx + (hx + by + f) \, dy = 0\)

comparing with \( M \, dx + N \, dy = 0 \)

Here \( M = ax + hy + g \quad N = hx + by + f \)

\[ \frac{\partial M}{\partial y} = h, \quad \frac{\partial N}{\partial x} = h \]

\[ \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

Hence given equation is exact.

\[ V = \int M \, dx = \int (ax + hy + g) \, dx = \frac{ax^2}{2} + (hy + g) \, x \]

Keeping \( y \) constant

\[ \text{Keeping } y \text{ constant} \]

\[ N - \frac{\partial V}{\partial y} = (hx + by + f) - hx = by + f \]

Now, \( \int \left( N - \frac{\partial V}{\partial y} \right) \, dy = \int (by + f) \, dy = \frac{by^2}{2} + fy \)

\[ \therefore \text{ General solution is } V + \int \left( N - \frac{\partial V}{\partial y} \right) \, dy = C \]

\[ i.e. \quad \frac{ax^2}{2} + (hy + g) \, x + \frac{by^2}{2} + fy = C \]
Eg. 1.6.4 : Solve \((4x + 3y + 1)dx + (3x + 2y + 1)dy = 0\)

**Sol:** Given equation is \((4x + 3y + 1)dx + (3x + 2y + 1)dy = 0\)

Here \(M = 4x + 3y + 1\) \quad \(N = 3x + 2y + 1\)

\[
\frac{\partial M}{\partial y} = 3, \quad \frac{\partial N}{\partial x} = 3
\]

\[
\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
\]

Given equation is exact.

\[
V = \int M \, dx \quad \text{y constant} = \int (4x + 3y + 1) \, dx = 2x^2(3y + 1)x
\]

\[
N - \frac{\partial V}{\partial y} = (3x + 2y + 1) - 3x = 2y + 1
\]

Now, \[
\int \left( N - \frac{\partial V}{\partial y} \right) dy = \int (2y + 1) dy = y^2 + y
\]

\[
\therefore \text{General solution is } 2x^2 + (3y + 1)x + y^2 + y = C
\]

Eg. 1.6.5 : Solve \((x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0\)

**Sol:** Given equation is \((x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0\)

Here \(M = x^2 - 4xy - 2y^2\) \quad \(N = y^2 - 4xy - 2x^2\)

\[
\frac{\partial M}{\partial y} = -4x - 4y, \quad \frac{\partial N}{\partial x} = -4y - 4x
\]

\[
\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
\]

Given equation is exact.

\[
V = \int M \, dx \quad \text{y constant} = \int x^2 - 4xy - 2y^2 \, dx = \frac{x^3}{3} - 2x^2y - 2y^2x
\]
Now, \( N \frac{\partial V}{\partial y} = (y^2 - 4xy - 2x^2) - (-2x^2 - 4yx) \)

\[
\int N \frac{\partial V}{\partial y} \, dy = \int y^2 \, dy = \frac{y^3}{3}
\]

General solution is \( \frac{x^3}{3} - 2x^2y - 2y^2x + \frac{y^3}{3} = C \)

**INTEGRATING FACTORS**

**Definition**: If \( M(x, y) \, dx + N(x, y) \, dy = 0 \) is not an exact differential equation if \( M \, dx + N \, dy = 0 \) can be made exact by multiplying with suitable term \( f(x, y) \neq 0 \) then \( f(x, y) \) is called an integrating factor of the differential equation \( M(x, y) \, dx + N(x, y) \, dy = 0 \).

1.6.7 **Type I**: If \( M(x, y) \, dx + N(x, y) \, dy = 0 \) is a homogeneous differential equation and \( M \, dx + N \, dy = 0 \) is not exact and \( M \, x + N \, y \neq 0 \). Then \( \frac{1}{M \, x + N \, y} \) is an integrating factor of \( M \, dx + N \, dy = 0 \).

**Solved Problems**:

**Eg. 1.6.8**: Solve \( x + y \frac{dy}{dx} = y - x \frac{dy}{dx} \)

**Sol**: Given equation is \( x + y \frac{dy}{dx} = y - x \frac{dy}{dx} \)  

\[
\Rightarrow x \frac{dy}{dx} + y \frac{dy}{dx} = y - x \\
\Rightarrow (x + y) \frac{dy}{dx} = y - x \\
\Rightarrow (x + y) \, dy = (y - x) \, dx \\
\Rightarrow (x - y) \, dx + (x + y) \, dy = 0
\]

Here \( M = x - y \)  \( N = x + y \)

\( \frac{\partial M}{\partial y} = -1 \) and \( \frac{\partial N}{\partial x} = 1 \)
Given equation is not exact.

But (2) is a homogeneous differential equation

\[ Mx + Ny = x^2 - xy + xy + y^2 = x^2 + y^2 \neq 0 \]

\[ \therefore \text{Integrating factor} = \frac{1}{Mx + Ny} = \frac{1}{x^2 + y^2} \]

Multiplying (2) by \( \frac{1}{x^2 + y^2} \) we get

\[ \frac{x-y}{x^2 + y^2} \ dx + \frac{x+y}{x^2 + y^2} \ dy = 0 \quad \text{----------- (3)} \]

Here \( M_1 = \frac{x-y}{x^2 + y^2} \quad \text{and} \quad N_1 = \frac{x+y}{x^2 + y^2} \)

\[ \frac{\partial M_1}{\partial y} = \frac{(x^2 + y^2)(-1) - (x-y)(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2} \]

\[ \frac{\partial N_1}{\partial x} = \frac{(x^2 + y^2)(1) - (x+y)(\partial x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2} \]

\[ \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \]

\[ \therefore (3) \text{ is a exact differential equation.} \]

Now \( V = \int M_1 \ dx \quad = \quad \int \frac{x-y}{x^2 + y^2} \ dx \)

\( y \text{ constant} \quad y \text{ constant} \)

\[ = \frac{1}{2} \int \frac{2x}{x^2 + y^2} \ dx \quad - \quad y \int \frac{1}{x^2 + y^2} \ dx \]

\( y \text{ constant} \quad y \text{ constant} \)
= \frac{1}{2} \log \left| x^2 + y^2 \right| - y \cdot \frac{1}{y} \tan^{-1} \left( \frac{x}{y} \right)

\frac{\partial V}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2y - \frac{1}{1 + \left( \frac{x}{y} \right)^2} \cdot \left( -\frac{x}{y^2} \right)

= \frac{y}{x^2 + y^2} + \frac{x}{x^2 + y^2} = \frac{x+y}{x^2 + y^2}

N_1 - \frac{\partial V}{\partial y} = -\frac{x+y}{x^2 + y^2} - \frac{x-y}{x^2 + y^2} = 0

\int \left( N_1 - \frac{\partial V}{\partial y} \right) dy = \int 0 \ dy = C_1

General solution is \frac{1}{2} \log \left| x^2 + y^2 \right| - \tan^{-1} \left( \frac{x}{y} \right) = C

Eg. 1.6.9 : Solve \( xy \, dx - \left( x^2 + 2y^2 \right) \, dy = 0 \)

Given equation is \( xy \, dx - \left( x^2 + 2y^2 \right) \, dy = 0 \) \( (1) \)

\( M = xy \quad \text{N} = -x^2 - 2y^2 \)

\( \frac{\partial M}{\partial y} = x, \quad \frac{\partial N}{\partial x} = -2x \)

\( \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \)

Given equation is not exact.

But (1) is a homogeneous differential equation.

\( M \times y + N \times y = x^2 y - x^2 y - 2y^3 = -2y^3 \neq 0 \)

Integrating factor = \( \frac{1}{M \times y + N \times y} = -\frac{1}{2y^3} \)

Multiplying (1) with \( -\frac{1}{2y^3} \)
\[-\frac{xy}{2y^3} \, dx + \frac{x^2 + 2y^2}{2y^3} \, dy = 0\]

\[\Rightarrow \frac{-x}{2y^2} \, dx + \left( \frac{x^2}{2y^3} + \frac{1}{y} \right) \, dy = 0 \quad (2)\]

\[M_1 = \frac{-x}{2y^2}, \quad N_1 = \frac{x^2}{2y^3} + \frac{1}{y}\]

\[\frac{\partial M_1}{\partial y} = \frac{-x}{2} \times -\frac{2}{y^3} = \frac{x}{y^3}, \quad \frac{\partial N_1}{\partial x} = \frac{2x}{2y^3} = \frac{x}{y^3}\]

\[\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}\]

\[\therefore (2) \text{ is an exact differential equation.}\]

\[V = \int M_1 \, dx = \int \frac{-x}{2y^2} \, dx = \frac{-x^2}{4y^2}\]

\[\text{y constant } \quad y \text{ constant}\]

\[\frac{\partial V}{\partial y} = \frac{-x^2}{4} \left( \frac{-2}{y^3} \right) = \frac{x^2}{2y^3}\]

\[N_1 - \frac{\partial V}{\partial y} = \frac{x^2}{2y^3} + \frac{1}{y} - \frac{x^2}{2y^3} = \frac{1}{y}\]

\[\int \left( N_1 - \frac{\partial V}{\partial y} \right) \, dy = \int \frac{1}{y} \, dy = \log |y|\]

\[\therefore \text{ General solution is } \frac{-x^2}{4y^2} + \log |y| = C.\]

**Eg. 1.6.10 Method II:** If the differential equation \(M \, dx + N \, dy = 0\) is of the form \(yf(x, y) \, dx + xg(x, y) \, dy = 0\) and \(Mx - Ny \neq 0\). Then \(\frac{1}{Mx - Ny}\) is an integrating factor.
Eg. 1.6.11: (1) Solve \((2xy + 1)y\,dx + \left(1 + 2xy - x^3y^3\right)x\,dy = 0\)

Sol: Given equation is \(y(2xy + 1)dx + x\left(1 + 2xy - x^3y^3\right)dy = 0\) -------- (1)

\[
M = 2xy^2 + y \quad \quad \quad N = x + 2x^2y - x^4y^3
\]

\[
\frac{\partial M}{\partial y} = 4xy + 1 \quad \quad \quad \frac{\partial N}{\partial x} = 1 + 4xy - 4x^3y^3
\]

\[
\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{(1) is not exact.}
\]

\[
M\,x - N\,y = 2x^2y^2 + xy - xy - 2x^2y^2 + x^4y^4
\]

\[
= x^4y^4 \neq 0
\]

Integrating factor = \(\frac{1}{x^4y^4}\)

Multiplying (1) with \(\frac{1}{x^4y^4}\) we get

\[
\frac{2xy^2 + y}{x^4y^4}\,dx + \frac{x + 2x^2y - x^4y^3}{x^4y^4}\,dy = 0
\]

\[
\Rightarrow \left(\frac{2}{x^3y^2} + \frac{1}{x^4y^3}\right)dx + \left(\frac{1}{x^3y^4} + \frac{2}{x^2y^3} - \frac{1}{y}\right)dy = 0 \quad \text{--------- (2)}
\]

\[
M_1 = \frac{2}{x^3y^2} + \frac{1}{x^4y^3} \quad , \quad N_1 = \frac{1}{x^3y^4} + \frac{2}{x^2y^3} - \frac{1}{y}
\]

\[
\frac{\partial M_1}{\partial y} = \frac{-4}{x^3y^3} - \frac{3}{x^4y^4} \quad \quad \frac{\partial N_1}{\partial x} = \frac{-3}{x^4y^4} - \frac{4}{x^3y^3}
\]

\[
\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}
\]

(2) is a exact differential equation.
\[ V = \int M_1 \, dx = \int \frac{2}{x^4 y^2} + \frac{1}{x^4 y^3} \, dx = \frac{-1}{x^2 y^2} - \frac{1}{3x^3 y^3} \]

\[ \frac{\partial V}{\partial y} = \frac{2}{x^2 y^3} + \frac{1}{x^3 y^4} \]

\[ N_1 - \frac{\partial V}{\partial y} = \left( \frac{1}{x^3 y^4} + \frac{2}{x^2 y^3} - \frac{1}{y} \right) - \left( \frac{2}{x^2 y^3} + \frac{1}{x^3 y^4} \right) \]

\[ = \frac{-1}{y} \]

\[ \int \left( N_1 - \frac{\partial V}{\partial y} \right) \, dy = \int \frac{-1}{y} \, dy = -\log |y| \]

General solution is \[ \frac{-1}{x^2 y^2} - \frac{1}{3x^3 y^3} - \log |y| = C \]

i.e., \[ \frac{1}{x^2 y^2} + \frac{1}{3x^2 y^3} + \log |y| = K \]

where \( K = -c \)

**Eg. 1.6.12**: Solve \((1 + xy) \, x \, dy + (1 - xy) \, y \, dx = 0\)

**Solution**: Given equation is \[ y(1 - xy) \, dx + x(1 + xy) \, dy = 0 \] \[ (1) \]

\[ M = y - xy^2 \quad N = x + x^2 y \]

\[ \frac{\partial M}{\partial y} = 1 - 2xy \quad \frac{\partial N}{\partial x} = 1 + 2xy \]

\[ \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, (1) \text{ is not an exact equation.} \]

\[ M \, x - N \, y = \left( xy - x^2 y^2 \right) - \left( xy + x^2 y^2 \right) \]

\[ = -2x^2 y^2 \neq 0 \]
Integrating factor \( = \frac{1}{M x - N y} = \frac{-1}{2x^2 y^2} \)

Multiplying (1) with \( \frac{1}{x^2 y^2} \) (Neglecting constant and -Ve sign)

We get

\[
\left( \frac{y - xy^2}{x^2 y^2} \right) \, dx + \left( \frac{x + x^2 y}{x^2 y^2} \right) \, dy = 0 \quad \text{(2)}
\]

\[
M_1 = -\frac{1}{x^2 y^2} - \frac{1}{x} \quad N_1 = \frac{1}{xy^2} + \frac{1}{y}
\]

\[
\frac{\partial M_1}{\partial y} = -\frac{1}{x^2 y^2} \quad \frac{\partial N_1}{\partial x} = -\frac{1}{x^2 y^2}
\]

\[\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}\]

(2) is an exact differential equation.

\[
V = \int M_1 \, dx = \int \left( \frac{1}{x^2 y} - \frac{1}{x} \right) \, dx
\]

\[
= -\frac{1}{xy} - \log |x|
\]

\[
\frac{\partial V}{\partial y} = \frac{1}{xy^2}
\]

\[
N_1 - \frac{\partial V}{\partial y} = \left( \frac{1}{xy^2} + \frac{1}{y} \right) - \frac{1}{xy^2} = \frac{1}{y}
\]

\[\therefore \int \left( N_1 - \frac{\partial V}{\partial y} \right) \, dy = \int \frac{1}{y} \, dy = \log y\]

General solution is \( -\frac{1}{xy} - \log |x| + \log |y| = C \)
1.6.13 Method III: If the differential equation \( M \, dx + N \, dy = 0 \) is not exact and there exists a continuous single variable function \( f(x) \) such that \( f(x) = \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \) then \( e^{\int f(x) \, dx} \) is an integrating factor of \( M \, dx + N \, dy = 0 \).

Eg. 1.6.14: (1) Solve \( \left( y + \frac{y^3}{3} + \frac{x^2}{2} \right) \, dx + \frac{1}{4} \left( x + xy^2 \right) \, dy = 0 \)

Solution: Given equation is \( \left( y + \frac{y^3}{3} + \frac{x^2}{2} \right) \, dx + \frac{1}{4} \left( x + xy^2 \right) \, dy = 0 \) ----------- (1)

\[
M = y + \frac{y^3}{3} + \frac{x^2}{2} \\
N = \frac{1}{4} \left( x + xy^2 \right)
\]

\[
\frac{\partial M}{\partial y} = 1 + y^2 \\
\frac{\partial N}{\partial x} = \frac{1}{4} \left( 1 + y^2 \right)
\]

\[
\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}
\]

(1) is not an exact differential equation.

Now \[
\frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{4}{x + xy^2} \left( \left( 1 + y^2 \right) - \frac{1}{4} \left( 1 + y^2 \right) \right)
\]

\[
= \frac{4}{x \left( 1 + y^2 \right)} \times \frac{3}{4} \left( 1 + y^2 \right)
\]

\[
= \frac{3}{x} = f(x)
\]

Integrating factor \( = e^{\int f(x) \, dx} = e^{\int \frac{3}{x} \, dx} = e^{3 \log x} = e^{3 \log x} = x^3 \) \( e^{\log f(x)} = f(x) \)

Multiplying (1) with \( x^3 \) we get

\[
\left( x^3 y + \frac{x^3 y^3}{3} + \frac{x^5}{2} \right) \, dx + \frac{1}{4} \left( x^4 + x^4 y^2 \right) \, dy = 0 \) ----------- (2)
1.35 First Order First Degree Differential Equations

\[ M_1 = x^3 y + \frac{x^3 y^3}{3} + \frac{x^5}{2} \]
\[ N_1 = \frac{1}{4} \left( x^4 + x^4 y^2 \right) \]

\[ \Rightarrow \frac{\partial M_1}{\partial y} = x^3 + x^3 y^2 \]
\[ \frac{\partial N_1}{\partial x} = x^3 + x^3 y^2 \]

\[ \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \]

(2) is a exact differential equation.

\[ V = \int M_1 \, dx = \int x^3 y + \frac{x^3 y^3}{3} + \frac{x^5}{5} \, dx \]
\[ y \text{ constant} \quad y \text{ constant} \]

\[ = \frac{x^4 y}{4} + \frac{x^4 y^3}{12} + \frac{x^6}{12} \]

\[ \frac{\partial V}{\partial y} = \frac{x^4}{4} + \frac{x^4 y^2}{4} \]

\[ N_1 - \frac{\partial V}{\partial y} = \left( \frac{x^4}{4} + \frac{x^4 y^2}{4} \right) - \left( \frac{x^4}{4} + \frac{x^4 y^2}{4} \right) = 0 \]

\[ \int \left( N_1 - \frac{\partial V}{\partial y} \right) \, dy = \int 0 \, dy = 0 \]

General solution is \[ \frac{x^4 y}{4} + \frac{x^4 y^3}{12} + \frac{x^6}{12} = C \]

1.6.15 Method IV: If the differential equation \( M \, dx + N \, dy = 0 \) is not exact and there exists a continuous single variable function \( g(y) \) such that \( g(y) = \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \) then \( e^{\int g(y) \, dy} \) is an integrating factor of \( M \, dx + N \, dy = 0 \).

\( \text{Eg. 1.6.16:} \) Solve \( (3x^2 y^4 + 2xy) \, dx + (2x^3 y^3 - x^2) \, dy = 0 \)

\( \text{Solution:} \) Given equation is \( (3x^2 y^4 + 2xy) \, dx + (2x^3 y^3 - x^2) \, dy = 0 \) \[ \text{--------- (1)} \]
\[ M = (3x^2y^4 + 2xy) \quad N = 2x^3y^3 - x^2 \]
\[
\frac{\partial M}{\partial y} = 12x^2y^3 + 2x \quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x
\]
\[ : \: \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \text{ (1) is not exact.} \]
\[
\frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{3x^2y^4 + 2xy} \left( (6x^2y^3 - 2x) - (12x^2y^3 + 2x) \right)
\]
\[ = -\frac{6x^2y^3 - 4x}{3x^2y^4 + 2xy} \]
\[ = -\frac{2(3x^2y^3 + 2x)}{y(3x^2y^3 + 2x)} \]
\[ = -\frac{2}{y} = g(y) \]

Integrating factor \[ = e^{\int g(y)dy} = \int \frac{2}{y} dy \]
\[ = e^{-2\log y} \]
\[ = e^{-2} \cdot \frac{1}{y^2} = \frac{1}{y^2} \]

Multiplying (1) with \( \frac{1}{y^2} \) we get
\[
\left( \frac{3x^2y^4 + 2xy}{y^2} \right) dx + \left( \frac{2x^3y^3 - x^2}{y^2} \right) dy = 0 \quad (2)
\]
\[ M_1 = 3x^2y^2 + \frac{2x}{y} \quad N_1 = 2x^3y - \frac{x^2}{y^2} \]
\[
\frac{\partial M_1}{\partial y} = 6x^2y - \frac{2x}{y^2} \quad \frac{\partial N_1}{\partial x} = 6x^2y - \frac{2x}{y^2}
\]

\[
\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}, \text{ (2) is exact.}
\]

\[
V = \int M_1 \, dx = \int 3x^2y^2 + \frac{2x}{y} \, dx = x^3y^2 + \frac{x^2}{y}
\]

\[
\text{y constant} \quad \text{y constant}
\]

\[
\frac{\partial V}{\partial y} = 2x^3y - \frac{x^2}{y^2}
\]

\[
N_1 - \frac{\partial V}{\partial y} = \left(2x^3y - \frac{x^2}{y^2}\right) - \left(2x^3y - \frac{x^2}{y^2}\right) = 0
\]

\[
\int \left(N_1 - \frac{\partial V}{\partial y}\right) \, dy = \int 0 \, dy = 0
\]

\[
\therefore \text{General solution is } x^3y^2 + \frac{x^2}{y} = C
\]

1.6.17: Solve \((x + y + 1) \, y \, dx + (x + 3y + 2) \, x \, dy = 0\)

**Solution:** Given equation is \((x + y + 1) \, y \, dx + (x + 3y + 2) \, x \, dy = 0 \quad \text{--------- (1)}

Here \(M = (x + y + 1) \, y \quad N = (x + 3y + 2) \, x \)

\[
= xy + y^2 + y \quad = x^2 + 3xy + 2x
\]

\[
\frac{\partial M}{\partial y} = x + 2y + 1 \quad \frac{\partial N}{\partial x} = 2x + 3y + 2
\]

\[
\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \text{ (1) is not exact.}
\]

\[
\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy + y^2 + y} \left((2x + 3y + 2) - (x + 2y + 1)\right)
\]
\[
\frac{x+y+1}{y(x+y+1)} = \frac{1}{y} = g(y)
\]

Integrating factor \( = e^{\int g(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y\)

Multiplying (1) with 'y', we get

\[
(y^2x + y^3 + y^2)dx + (x^2y + 3xy^2 + 2xy)dy = 0 \quad \text{(2)}
\]

\[
M_1 = y^2x + y^3 + y^2 \\
N_1 = x^2y + 3xy^2 + 2xy
\]

\[
\frac{\partial M_1}{\partial y} = 2xy + 3y^2 + 2y, \quad \frac{\partial N_1}{\partial x} = 2xy + 3y^2 + 2y
\]

\[
\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}
\]

(2) is a exact differential equation.

\[
V = \int M_1 \, dx = \int (y^2x + y^3 + y^2) \, dx = \frac{x^2y^2}{2} + y^3x + y^2x
\]

\[
\text{y constant}
\]

\[
\frac{\partial V}{\partial y} = x^2y + 3xy^2 + 2xy
\]

\[
\left( N_1 - \frac{\partial V}{\partial y} \right) = (2xy + x^2y + 3xy^2) - (x^2y + 3xy^2 + 2xy) = 0
\]

\[
\therefore \int \left( N_1 - \frac{\partial V}{\partial y} \right) \, dy = \int 0 \, dy = 0
\]

General solution is \( V + \int \left( N_1 - \frac{\partial V}{\partial y} \right) \, dy = C \)

\[
i.e. \quad \frac{x^2y^2}{2} + y^3x + y^2x = C
\]

### 1.7 SAQ (HOMOGENEOUS)

Solve \( xy' = x + y \)
Solve \( xy' - 2y = 3x \)

Solve \( x^2 y' - 2y = 3x \)

Solve \( \frac{dy}{dx} = \frac{x^2 + y^2}{xy} \)

Solve \( x \frac{dy}{dx} = y + xe^{-x} \)

**ANSWERS TO SELF ASSESSMENT QUESTIONS**

1.7.1 : \( y' = \frac{x+y}{x} = \frac{1+y}{x} \) \( \quad \) \( \text{--------- (1)} \)

Put \( y = Vx \) \( \quad \) \( \text{--------- (2)} \)

Differentiate with respect to ' \( x \)'

\( y' = V + x \frac{dV}{dx} \) \( \quad \) \( \text{--------- (3)} \)

substituting (2) and (3) in (1)

\[ V + x \frac{dV}{dx} = 1 + V \]

\[ \Rightarrow x \frac{dV}{dx} = 1 \]

separating variables and integrating

\[ \int dV = \int \frac{dx}{x} \Rightarrow V = x + c \Rightarrow \frac{y}{x} = x + c \]

1.7.2 : \( \frac{dy}{dx} = y' = \frac{2y + 3x}{x} = \frac{2y}{x} + 3 \) \( \quad \) \( \text{--------- (1)} \)

Put \( y = Vx \) \( \quad \) \( \text{--------- (2)} \)

Differentiate with respect to ' \( x \)'

\[ \frac{dy}{dx} = V + x \frac{dV}{dx} \] \( \quad \) \( \text{--------- (3)} \)

substituting (2) and (3) in (1)

\[ V + x \frac{dV}{dx} = 2V + 3 \]
\[ x \frac{dV}{dx} = V + 3 \]

separating variables and integrating

\[ \int \frac{dV}{V+3} = \int \frac{dx}{x} \]

\[ \Rightarrow \log|V+3| = \log x + \log C = \log Cx \]

\[ \Rightarrow V + 3 = Cx \]

\[ \Rightarrow \frac{y}{x}+3 = Cx \Rightarrow y + 3x = Cx^2 \]

1.7.3 : \( y' = \frac{dy}{dx} = \frac{x^2 - xy + y^2}{x^2} = 1 - \frac{y}{x} + \frac{y^2}{x^2} \)

Put \( y = Vx \)

Differentiate with respect to \( x' \)

\[ \frac{dy}{dx} = V + x \frac{dV}{dx} \]

substituting (2) and (3) in (1)

\[ V + x \frac{dV}{dx} = 1 - V + V^2 \Rightarrow x \frac{dV}{dx} = V^2 - 2V + 1 = (V-1)^2 \]

Separating variables and Integrating

\[ \int \frac{dV}{(V-1)^2} = \int \frac{dx}{x} \]

\[ \Rightarrow -\frac{1}{V-1} = \log x + C \Rightarrow \frac{x}{x-y} = \log x + C \]

1.7.4 : \( x \frac{dy}{dx} = y + xe^x \)

\[ \Rightarrow \frac{dy}{dx} = \frac{y + xe^x}{x} = y + e^x \]

Put \( y = Vx \)

Differentiate with respect to \( x' \)
\[
\frac{dy}{dx} = V + x \frac{dV}{dx} \quad \text{(3)}
\]

Substituting (2) and (3) in (1)

\[
V + x \frac{dV}{dx} = V + e^V \Rightarrow x \frac{dV}{dx} = e^V
\]

Separating variables and integrating

\[
\int \frac{dV}{e^V} = \int \frac{dx}{x}
\]

\[
\Rightarrow -e^{-V} = \log x + C \Rightarrow -e^{-\frac{y}{x}} = \log x + C
\]

1.8 **SUMMARY**

In this lesson we discussed Homogeneous, Non-homogeneous, Exact, Non-Exact first order differential equations and the related problems.

1.9 **TECHNICAL TERMS**

Homogeneous Differential equation, Non-Homogeneous, Exact, Non-Exact, Integrating factor.

**Exercise - Ia**

1. Solve \( y - x \frac{dy}{dx} = x + y \frac{dy}{dx} \)

2. Solve \( \left(x^2 - y^2\right)dx + 2xy \ dy = 0 \)

3. Solve \( \frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \)

4. Solve \( \frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \)

5. Solve \( \left(x^3 + 3xy^2\right)dx + \left(y^3 + 3x^2y\right)dy = 0 \)

6. Solve \( x \sin \left(\frac{y}{x}\right) \frac{dy}{dx} = y \sin \left(\frac{y}{x}\right) - x \)

7. Solve \( \frac{dy}{dx} = \frac{(x + y)^2}{2x^2} \)
8. Solve \( y^2 \, dy = x(\sqrt{y} \, dx - \sqrt{x} \, dy) \)

9. Solve \( \left(1 + e^{\frac{x}{y}}\right) \, dx + e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) \, dy = 0 \)

10. Solve \( \left(x - y \tan^{-1}\left(\frac{y}{x}\right)\right) \, dx + \left(x \tan^{-1}\frac{y}{x}\right) \, dy = 0 \)

11. Solve \( \frac{dy}{dx} = y + x \, e^{\frac{y}{x}} \)

12. Solve \( (x - y \log y + y \log x) \, dx + x(\log y - \log x) \, dy = 0 \)

13. Solve \( \frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3} \)

14. Solve \( (3xy^2 - y^3) \, dx - (2x^2 y - xy^2) \, dy = 0 \)

Exercise IB

Solve the following differential equations.

1. \((12x + 5y - 9) \, dx + (5x + 2y - 4) \, dy = 0\)

2. \((x + 2y - 3) \frac{dy}{dx} = 2x - y + 1\)

3. \((2x + y + 6) \, dx = (y - x - 3)\)

4. \((6x + 7y - 4) \, dx + (7x - 4y + 3) \, dy = 0\)

5. \(\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}\)

6. \(\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}\)

7. \((2x + y + 1) \, dx + (4x + 2y - 1) \, dy = 0\)

8. \((4x + 6y + 5) \frac{dy}{dx} = 3y + 2x + 4\)

9. \((2x + 4y + 3) \, dy = (2y + x + 1) \, dx\)

10. \(\frac{dy}{dx} = \frac{6x - 4y + 3}{3x - 2y + 1}\)
Exercise IC

Solve the following differential equations.

1. \[ x\left(1 + y^2\right)dx + y\left(1 + x^2\right)dy = 0 \]
2. \[ (x^2 - ay)dx = \left(ax - y^2\right)dy \]
3. \[ y \sin 2x dx - \left(y^2 + \cos^2 x\right)dy = 0 \]
4. \[ \left(1 + e^{x/y}\right)dx + e^{x/y} \left(1 - \frac{x}{y}\right)dy = 0 \]
5. \[ \left(e^{-y} + 1\right)\cos x dx + e^{y} \sin x dy = 0 \]
6. \[ (2x^2 + 6xy - y^2)dx + \left(3x^2 - 2xy + y^2\right)dy = 0 \]
7. \[ \left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xy e^{xy^2} - 3y^2\right)dy = 0 \]
8. \[ (xe^{xy} + 2y)dx + ye^{xy} = 0 \]
9. \[ xdx + ydy = \frac{x dy - y dx}{x^2 + y^2} \]
10. \[ (2xy + y - \tan y)dx + \left(x^2 - x \tan^2 y + \sec^2 y\right)dy = 0 \]

Exercise ID

1. Solve \( \left(x^2 y - 2xy^2\right)dx - \left(x^3 - 3x^2 y\right)dy = 0 \)
2. Solve \( \left(3xy^2 - y^3\right)dx = \left(2x^2 y - xy^2\right)dy \)
3. Solve \( x^2 ydx = \left(x^3 + y^3\right)dy \)
4. Solve \( y^2 dx + \left(x^2 - xy - y^2\right)dy = 0 \)
Answers to IE

1. Solve \( y(1 + xy)dx + x(1 - xy)dy = 0 \)

2. Solve \( y(xy + 1)dx + x \left(1 + xy + x^2y^2\right)dy = 0 \)

3. Solve \( y(xy + 2x^2y^2)dx + x \left(xy - x^2y^2\right)dy = 0 \)

4. Solve \( y \left(x^2y^2 + xy + 1\right)dx + x \left(x^2y^2 - xy + 1\right)dy = 0 \)

Exercise IF

1. Solve \( (x^2 + y^2 + 2x)dx + 2y\ dy = 0 \)

2. Solve \( (2y^3 + 2)dx + 3xy^2\ dy = 0 \)

3. Solve \( (x^3 - 2y^2)dx + 2xy\ dy = 0 \)

4. Solve \( 2xy\ dy - \left(x^2 + y^2 + 1\right)dx = 0 \)

5. Solve \( (3xy - 2ay^2)dx + \left(x^2 - 2axy\right)dy = 0 \)

Exercise IG

1. Solve \( (y + y^2)dx + xy\ dy = 0 \)

2. Solve \( (x^3 + xy^4)dx + 2y^3\ dy = 0 \)

3. Solve \( (xy^3 + y)dx + 2 \left(x^2y^2 + x + y^4\right)dy = 0 \)

4. Solve \( (y^4 + 2y)dx + \left(xy^3 + 2y^4 - 4x\right)dx = 0 \)

Answers to IA :

1. \( \frac{1}{2} \log \left(\frac{x^2y^2}{x^2}\right) + \tan^{-1} \frac{y}{x} = -\log x + C \)

2. \( x^2 + y^2 = Cx \)

3. \( x^2 - y^2 = Cx \)

4. \( y = C e^{\frac{x^2}{2y^2}} \)
5. \( y^2 = C x \left( x^2 + y^2 \right) \)
6. \( e^{\cos \left( \sqrt{y/x} \right)} = C x \)
7. \( \tan^{-1} \left( \frac{y}{x} \right) = C x \)
8. \( e^{\frac{x}{y} \left( \frac{x}{y} - 1 \right)} \log y = C \)
9. \( ye^{\frac{x}{y}} + x = C \)
10. \( xe^{\frac{y}{x}} = C \)
11. \( e^{\frac{x}{y}} + \log |x| + C = 0 \)
12. \( y \log y + (x - y) \log x = y + C x \)
13. \( \log \frac{y}{x} - \left( \frac{x^3}{3y^3} \right) + \log x = C \)
14. \( y^2 = C x^3 e^{\frac{y}{x}} \)

Answers to IB:

1. \( 6x^2 + 5xy + y^2 - 9x - 4y = C \)
2. \( y^2 + xy - x^2 - x - 3y = C \)
3. \( (2x + y - 3) = C(x - y - 3)^4 \)
4. \( 3x^2 + 7xy - 2y^2 - 4x + 3y = C \)
5. \( \left( y - \frac{7}{5} \right)^2 + \left( x - \frac{1}{5} \right) \left( y - \frac{7}{5} \right) - \left( x - \frac{1}{5} \right)^2 = C \)
6. \( e^{y-x} = C(x + y) \)
7. \( x + 2y + \log |2x + y - 1| = C \)
8. \( 14(2x + 3y) - 9 \log |14x + 21y + 22| = 49x + C \)
9. \( 4x + 8y + 5 = C e^{4x - 8y} \)
10. \( C e^{2x-y} = 3x - 2y + C \)

Answers to IC:

1. \( (1 + x^2)(1 + y^2) = C \)
2. \( x^3 - 3axy + y^3 = C \)
3. \(3y \cos 2x + 2y^3 + 3y = C\)  
4. \(x + ye^y = C\)

5. \((e^y + 1) \sin x = C\)  
6. \(2x^3 - 9x^2y - 3xy^2 + y^3 = C\)

7. \(e^{xy^2} + x^4 - y^3 = C\)  
8. \(e^{\frac{x}{y}} + y^2 = C\)

9. \(x^2 + y^2 + 2 \tan^{-1}\left(\frac{x}{y}\right) = K\)  
10. \(x^2y + (y - \tan y)x + \tan y = C\)

**Answers to ID:**

1. \(\frac{x}{y} - 2 \log|x| + 3 \log|y| = C\)  
2. \(3 \log|x| - 2 \log|y| + \frac{y}{x} = C\)

3. \(y = Ce^{\frac{x^3}{3y^3}}\)  
4. \(y^2(x - y) = C^2(x + y)\)

**Answers to IE:**

1. \(\frac{x}{y} = Ce^{xy}\)  
2. \(\frac{1}{xy} + \frac{1}{2x^2y^2} = \log|Cy|\)

3. \(x^2 = Cy e^{\frac{1}{xy}}\)  
4. \(\left(xy - \frac{1}{xy}\right) + \log\frac{x}{y} = C\)

**Answers to IF:**

1. \(\left(x^2 + y^2\right)e^x = C\)  
2. \(x^2y^3 + x^2 = C\)

3. \(x^3 + y^2 = Cx^2\)  
4. \(x^2 - y^2 - 1 = Cx\)

5. \(x^3y - ax^2y^2 = C\)

**Answers to IG:**

1. \(x + xy = C\)  
2. \(\left(x^2 + y^4 - 1\right)e^{x^2} = C\)

3. \(\left(x^2 + y^4 - 1\right)e^{x^2} = C\)  
4. \(xy + \frac{2x}{y^2} + y^2 = C\)
1.12 MODEL QUESTIONS

1. Solve \( \left( 1 + e^{\frac{x}{y}} \right) dx + \left( 1 - \frac{x}{y} \right) dy = 0 \)

2. Solve \( \left( y^2 - 2xy \right) dx + \left( 2xy - x^2 \right) dy = 0 \)

3. Solve \( \frac{dy}{dx} = \frac{6x + 5y + 7}{2x + 18y - 14} \)

4. Solve \( x^2 y dx - \left( x^3 + y^3 \right) dy = 0 \)

5. Solve \( \left( xy^2 + 2xy^2 y^3 \right) dx + \left( x^2 y - x^3 y^2 \right) dy = 0 \)

1.13 REFERENCE BOOKS


Introduction to Ordinary Differential Equations by Earl.A. Coddington

Lesson Writer

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Lesson - 2

LINEAR AND BERNOULLI'S FORM

2.1 OBJECTIVE OF THE LESSON

After studying this lesson, the student will be in a position to know about, linear differential equation, Bernoulli's form and how to solve them.

2.2 STRUCTURE OF THE LESSON

This lesson has the following components.

2.3 Definitions and Theorems
2.4 Linear and Bernoulli's form
2.5 Answers to Self Assessment Questions
2.6 Summary
2.7 Technical Terms
2.8 Exercises
2.9 Answers to Exercises
2.10 Model Examination Questions
2.11 Reference Books

2.3.1 Definition : A differential equation of the form \( \frac{dy}{dx} + P(x)\cdot y = Q(x) \) where \( P(x) \) and \( Q(x) \) are functions of \( x \), is called a linear differential equation of first order in \( y \).

2.3.2 Theorem : The general solution of linear differential equation \( \frac{dy}{dx} + P(x)\cdot y = Q(x) \) is given by \( y \cdot e^{\int P \,dx} = \int Q \cdot e^{\int P \,dx} \,dx + C \).

Proof : Given equation is \( \frac{dy}{dx} + P\cdot y = Q \) ---------- (1)

\[ \Rightarrow dy + P \, y \, dx = Q \, dx \]

\[ \Rightarrow (P \, y - Q) \, dx + dy = 0 \] ---------- (2)

Here \( M = P \, y - Q \) \quad \( N = 1 \)
\[
\frac{\partial M}{\partial y} = P, \quad \frac{\partial N}{\partial x} = 0
\]

\[\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} (2) \text{ is not exact.}\]

Now \[\frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = P - 0 = P \quad \text{(function of } x \text{ only)}\]

\[\therefore \text{Integrating factor } = e^{\int P \, dx}\]

Multiplying (1) by \(e^{\int P \, dx}\)

\[e^{\int P \, dx} \left( \frac{dy}{dx} + P \cdot y \right) = e^{\int P \, dx} \cdot Q\]

\[\Rightarrow \frac{dy}{dx} e^{\int P \, dx} + y \cdot P e^{\int P \, dx} = Q e^{\int P \, dx}\]

\[\Rightarrow \frac{d}{dx} \left( y e^{\int P \, dx} \right) = Q e^{\int P \, dx}\]

\[\therefore y e^{\int P \, dx} = \int Q \cdot e^{\int P \, dx} \, dx + C\]

is the general solution of \(\frac{dy}{dx} + P \cdot y = Q\)

(OR)

2.3.2 : Given equation is \(\frac{dy}{dx} + P \cdot y = Q \quad \text{} (1)\)

Consider \(\frac{dy}{dx} + P \cdot y = 0\)

By applying variables separable method, we get

\[
\int \frac{dy}{y} = -\int P \, dx
\]

\[\Rightarrow \log y + \log C = -\int P \, dx\]
Differential Equation, 
Abstract Algebra...

2.3

Linear and Bernoulli’s
Form

\[ \Rightarrow \log C y = - \int P \, dx \]

\[ \Rightarrow C y = e^{-\int P \, dx} \]

This can be written as \( y \cdot e^{\int P \, dx} = K \)

Now L.H.S. of (1) can be written as \( \frac{1}{e^{\int P \, dx}} \frac{d}{dx} \left( y e^{\int P \, dx} \right) \)

\[ \therefore \frac{1}{e^{\int P \, dx}} \frac{d}{dx} \left( y e^{\int P \, dx} \right) = Q \]

\[ \Rightarrow \frac{d}{dx} \left( y e^{\int P \, dx} \right) = Q \cdot e^{\int P \, dx} \, dx \]

\[ \Rightarrow \int \frac{d}{dx} \left( y e^{\int P \, dx} \right) = \int Q \cdot e^{\int P \, dx} \, dx \]

\[ \Rightarrow y e^{\int P \, dx} = \int Q \cdot e^{\int P \, dx} \, dx \]

2.3.3 Bernoulli’s equation :

Definition : An equation of the form \( \frac{dy}{dx} + P \, y = Q \, y^n \), where P and Q are functions of \( x \) only and \( n \) is a real no.

Case 1 : If \( n = 1 \) then equation (1) can be written as

\[ \frac{dy}{dx} + (P - Q) \, y = 0 \quad \text{-------- (2)} \]

General solution of (2) by variables separable method is

\[ \int \frac{dy}{y} + \int (P - Q) \, dx = C \]

Case 2 : \( n \neq 1 \)

Multiplying the equation (1) by \( y^{-n} \), we get

\[ y^{-n} \frac{dy}{dx} + P \, y^{1-n} = Q \quad \text{-------- (3)} \]
Put $y^{1-n} = u \quad (4)$

Differentiate with respect to 'x'

$$(1-n) y^{-n} \frac{dy}{dx} = \frac{du}{dx} \quad (5)$$

substituting (4) and (5) in (3)

We get $\frac{du}{dx} + (1-n) P \cdot u = (1-n) Q \quad (6)$

(6) is a linear equation in 'u'

Integrating factor $= e^{\int (1-n) P \, dx}$

:. General solution of (6) is

$$ue^{\int (1-n) P \, dx} = \int (1-n) Q \cdot e^{\int (1-n) P \, dx} \quad (7)$$

substituting $u = y^{1-n}$ in (7)

We get general solution of (1)

**Eg. 2.3.5 :** Solve $x^2 \frac{dy}{dx} + (x-2) y = x^2 e^{-2/x}$

**Solution :** Given equation is $x^2 \frac{dy}{dx} + (x-2) y = x^2 e^{-2/x} \quad (1)$

$$\Rightarrow \frac{dy}{dx} + \left( \frac{x-2}{x^2} \right) y = e^{-2/x} \quad (2)$$

(2) is linear in $y$.

$$P = \frac{x - 2}{x^2} = \frac{1}{x} - \frac{2}{x^2} \quad \quad Q = e^{-2/x}$$

$$\int P \, dx = \int \frac{1}{x} - \frac{2}{x^2} \, dx = \log |x| + \frac{2}{x}$$

$$e^{\int P \, dx} = e^{\log x + \frac{2}{x}} = e^{\log x \cdot e^{2/x}} = x \cdot e^{2/x}$$
General solution is \( y \cdot x e^{2/x} = \int e^{-2/x} x e^{2/x} \, dx \)

\[ = \int x \, dx = \frac{x^2}{2} + C \]

**Eg. 2.3.6 :** Solve \( (x^2 - 1) \frac{dy}{dx} + 2xy = 1 \)

**Solution :** Given equation is \( (x^2 - 1) \frac{dy}{dx} + 2xy = 1 \) \( -------- (1) \)

\[ \Rightarrow \frac{dy}{dx} + \frac{2x}{x^2 - 1} y = \frac{1}{x^2 - 1} \] \( -------- (2) \)

(2) is linear in \( y \).

\[ P = \frac{2x}{x^2 - 1}, \quad Q = \frac{1}{x^2 - 1} \]

\[ \int P \, dx = \int \frac{2x}{x^2 - 1} \, dx = \log|x^2 - 1| \]

Integrating factor \( = e^{\int P \, dx} = e^{\log|x^2 - 1|} = x^2 - 1 \)

General solution is \( y \cdot e^{\int P \, dx} = \int Q \cdot e^{\int P \, dx} \, dx \)

i.e. \( y \left( x^2 - 1 \right) = \int \frac{1}{x^2 - 1} \left( x^2 - 1 \right) \, dx = \int 1 \, dx = x + C \)

**Eg. 2.3.7 :** \( x \log x \frac{dy}{dx} + y = 2 \log x \)

**Solution :** Given equation is \( x \log x \frac{dy}{dx} + y = 2 \log x \) \( -------- (1) \)

\[ \Rightarrow \frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x} \] \( -------- (2) \)

(2) is linear in \( y \).
Here \( P = \frac{1}{x \log x} \) \( Q = \frac{2}{x} \)

\[
e^\int P \, dx = e^{\int \frac{1}{x \log x} \, dx}
\]

\[
= \int \frac{1}{x} \, dx = e^{\log(x)} = \log x
\]

General solution is \( y \cdot e^{\int P \, dx} = \int Q \cdot e^{\int P \, dx} \, dx \)

\[
i.e., \quad y(\log x) = \int \frac{2}{x} \cdot \log x \, dx
\]

\[
= 2 \int \log x \cdot \frac{1}{x} \, dx
\]

\[
= \frac{2(\log x)^2}{2} + C
\]

\[
\therefore \quad y(\log x) = (\log x)^2 + C
\]

**Eg. 2.3.8**: Solve \( x(x-1) \frac{dy}{dx} - y = x^2 (x-1)^2 \)

**Solution**: Given equation is \( x(x-1) \frac{dy}{dx} - y = x^2 (x-1)^2 \) \( \cdots (1) \)

\[
\Rightarrow \frac{dy}{dx} - \frac{1}{x(x-1)y} = \frac{x(x-1)}{y} \cdots (2)
\]

(2) is linear in \( y \).

Here \( P = \frac{-1}{x(x-1)} \) \( \quad Q = x(x-1) \)

\[
\int P \, dx = \int \frac{-1}{x(x-1)} \, dx
\]
= \int \frac{(x-1)-x}{x(x-1)} \, dx \\
= \int \left(\frac{1}{x} - \frac{1}{x-1}\right) \, dx \\
= \log|x| - \log|x-1| \\
= \log \left|\frac{x}{x-1}\right|

Integrating factor = e^{\int P \, dx} = e^{\log \frac{x}{x-1}} = \frac{x}{x-1}

General solution is \( y \cdot e^{\int P \, dx} = \int Q \cdot e^{\int P \, dx} \, dx \)

i.e. \( y \cdot \frac{x}{x-1} = \int x(x-1) \cdot \frac{x}{x-1} \, dx \)

\[ = \int x^2 \, dx = \frac{x^3}{3} + C \]

\[ \therefore y \cdot \frac{x}{x-1} = \frac{x^3}{3} + C \]

Eg. 2.3.9 : Solve \( \frac{dy}{dx} + y \log x = e^x \cdot x^{-\frac{1}{2}} \log x \)

Solution : Given equation is \( \frac{dy}{dx} + y \log x = e^x \cdot x^{-\frac{1}{2}} \log x \) \[\text{---------- (1)}\]

\[ \Rightarrow \frac{dy}{dx} + \frac{\log x}{x} y = e^x \cdot \frac{1}{x^{\frac{1}{2}} \log x} \] \[\text{---------- (2)}\]

(2) is linear in \( y \).
Here \( P = \frac{\log x}{x} \) \quad \text{and} \quad Q = \frac{e^x}{x^{1/2} \log x}

\[
\int P \, dx = \int \frac{\log x}{x} \, dx = \int \log x \cdot \frac{1}{x} \, dx = \frac{(\log x)^2}{2}
\]

Integrating factor \( = e^{\int P \, dx} = e^{\frac{(\log x)^2}{2}} \)

\[
= (e^{\log x})^{\frac{1}{2} \log x}
\]

\[
= x^{\frac{1}{2} \log x}
\]

General solution is \( y \cdot x^{\frac{1}{2} \log x} = \int Q \cdot e^{\int P \, dx} \, dx \)

i.e. \( y \cdot x^{\frac{1}{2} \log x} = \int \frac{e^x}{x^{\frac{1}{2} \log x}} \cdot x^{\frac{1}{2} \log x} \, dx \)

\[
= \int e^x \, dx
\]

\[
= e^x + C
\]

\[
\therefore y \cdot x^{\frac{1}{2} \log x} = e^x + C
\]

**Eg. 2.3.10:** Solve \( (x + y + 1) \frac{dy}{dx} = 1 \)

**Solution:** Given equation is \( (x + y + 1) \frac{dy}{dx} = 1 \) \( \quad \text{--------(1)} \)

\[
\Rightarrow \frac{dx}{dy} = x + y + 1
\]

\[
\Rightarrow \frac{dx}{dy} - x = y + 1 \quad \text{--------(2)}
\]
(2) is linear in 'x'.

Here \( P = -1 \) \( Q = y + 1 \)

\[
\int P \, dy = \int (-1) \, dy = -y
\]

Integrating factor = \( e^{\int P \, dy} \)

= \( e^{-y} \)

General solution is \( x \cdot e^{\int P \, dy} = \int Q \cdot e^{\int P \, dy} \, dy \)

i.e., \( x \cdot e^{-y} = \int ((y+1)e^{-y}) \, dy \)

\[
= (y+1)\int e^{-y} \, dy - \int (1 \cdot e^{-y}) \, dy
\]

\[
= (y+1)(-e^{-y}) - \int -e^{-y} \, dy
\]

\[
= -(y+1)e^{-y} - e^{-y} + C
\]

\[\therefore x \cdot e^{-y} = -(y+1)e^{-y} - e^{-y} + C\]

**Eg. 2.3.11:** Solve \( \left(1 + y^2\right) + \left(x - e^{\tan^{-1} y}\right) \frac{dy}{dx} = 0 \)

**Solution:** Given equation is \( \left(1 + y^2\right) + \left(x \cdot e^{\tan^{-1} y}\right) \frac{dy}{dx} = 0 \) \[\text{(1)}\]

\[
\Rightarrow \left(1 + y^2\right) \frac{dx}{dy} + \left(x - e^{\tan^{-1} y}\right) = 0
\]

\[
\Rightarrow \left(1 + y^2\right) \frac{dx}{dy} + x = e^{\tan^{-1} y} \quad \text{*********** (2)}
\]

Dividing by \( \left(1 + y^2\right) \)
\[ \frac{dx}{dy} + \frac{1}{1+y^2} x = \frac{e^{\tan^{-1} y}}{1+y^2} \] \hspace{1cm} \text{(3)}

(3) is linear in \( x \).

Here \( P = \frac{1}{1+y^2} \), \( Q = \frac{e^{\tan^{-1} y}}{1+y^2} \)

\[ \int P \, dy = \int \frac{1}{1+y^2} \, dy = \tan^{-1} y \]

Integrating factor \( = e^{\int P \, dy} = e^{\tan^{-1} y} \)

General solution is \( x \cdot e^{\int P \, dy} = \int Q \cdot e^{\int P \, dy} \, dy \)

i.e. \( x \cdot e^{\tan^{-1} y} = \int e^{\tan^{-1} y} \cdot \frac{e^{\tan^{-1} y}}{1+y^2} \, dy \)

\[ = \int e^{2 \tan^{-1} y} \cdot \frac{1}{1+y^2} \, dy \]

\[ = \frac{e^{2 \tan^{-1} y}}{2} + C \]

\[ \therefore x \cdot e^{\tan^{-1} y} = e^{\frac{2 \tan^{-1} y}{2}} + C \]

\textbf{Eg. 2.3.12 :} Solve \( \frac{dy}{dx} + y \cot x = 4x \csc x \). Given that \( y = 0 \), when \( x = \frac{\pi}{2} \).

\textbf{Solution :} Given equation is \( \frac{dy}{dx} + y \cot x = 4x \csc x \) \hspace{1cm} \text{(1)}

(1) is linear in \( y \).
Here $P = \cot x \quad Q = 4x \csc x$

\[ \int P \, dx = \int \cot x \, dx = \log |\sin x| \]

Integrating factor $= e^{\int P \, dx} = e^{\log \sin x} = \sin x$

General solution is $y \cdot e^{\int P \, dx} = \int Q \cdot e^{\int P \, dx} \, dx$

i.e. $y \sin x = \int 4x \cosec x \sin x \, dx$

$= \int 4x \, dx = 2x^2 + C$

$\therefore y \sin x = 2x^2 + C$

If $x = \frac{\pi}{2}$ then $y = 0 \Rightarrow 0 = \frac{2\pi^2}{4} + C$

$\Rightarrow C = -\frac{\pi^2}{2}$

$\therefore$ Particular solution of (1) is $y \sin x = 2x^2 - \frac{\pi^2}{2}$

Eg. 2.3.13 : Solve $\frac{dy}{dx} + y \cot x = y^2 \sin^2 x \cos^2 x$

Solution : Given equation is $\frac{dy}{dx} + y \cot x = y^2 \sin^2 x \cos^2 x$ \quad \text{-------- (1)}

This is a Bernoulli's equation

Dividing (1) by $y^2$, we get

$\frac{1}{y^2} \frac{dy}{dx} + \frac{1}{y} \cot x = \sin^2 x \cos^2 x$ \quad \text{-------- (2)}
Put \( \frac{1}{y} = u \) \( \cdots \) (3)

Differentiating with respect to 'x' \[
-\frac{1}{y^2} \frac{dy}{dx} = \frac{du}{dx} \]

\( \Rightarrow \) \( \frac{1}{y^2} \frac{dy}{dx} = -\frac{du}{dx} \) \( \cdots \) (4)

substituting (3) and (4) in (2)

\[ -\frac{du}{dx} + u \cot x = \sin^2 x \cos^2 x \]

\( \Rightarrow \frac{du}{dx} = -u \cot x = -\sin^2 x \cos^2 x \) \( \cdots \) (5)

Here (5) is a linear equation in 'u'.

Here \( P = -\cot x \) \hspace{1cm} \( Q = -\sin^2 x \cos^2 x \)

\[
\int P \, dx = \int -\cot x \, dx = -\log \sin x = \log (\sin x)^{-1}
\]

\[
= \log \frac{1}{\sin x}
\]

Integrating factor \( = e^{\int P \, dx} = e^{\log \frac{1}{\sin x}} = \frac{1}{\sin x} \)

General solution of (5) is \( u \cdot \frac{1}{\sin x} = \int -\sin^2 x \cos^2 x \frac{1}{\sin x} \, dx \)

\( \Rightarrow \frac{u}{\sin x} = \int (\cos x)^2 (-\sin x \, dx) \)

\[ = \frac{(\cos x)^3}{3} + C \) \( \cdots \) (6)

substitute \( u = \frac{1}{y} \) in (6)
General solution of (1) is \[ \frac{1}{y \sin x} = \frac{(\cos x)^3}{3} + C \]

**Eg. 2.3.14** : Solve \[ \frac{dy}{dx} + \frac{y}{x} = x^2 y^6 \]

**Solution** : Given equation is \[ \frac{dy}{dx} + \frac{y}{x} = x^2 y^6 \quad (1) \]

This is a Bernoulli’s equation

Dividing (1) by \( y^6 \), we get

\[ \frac{1}{y^6} \frac{dy}{dx} + \frac{1}{xy^5} = x^2 \quad (2) \]

Put \( \frac{1}{y^5} = u \quad (3) \)

Differentiate with respect to ‘\( x \)’

\[ -5 \frac{dy}{y^6} = du \frac{dx}{dx} \]

\[ \Rightarrow \frac{1}{y^6} \frac{dy}{dx} = -\frac{1}{5} \frac{dy}{dx} \quad (4) \]

substitute (3) and (4) in (2) we get

\[ -\frac{1}{5} \frac{du}{dx} + \frac{1}{x} u = x^2 \]

\[ \Rightarrow \frac{du}{dx} - \frac{5u}{x} = -5x^2 \quad (5) \]

(5) is a linear equation in \( u \)

Here \( P = -\frac{5}{x} \quad Q = -5x^2 \).
\[ P \frac{dx}{x} = \int -5 \log|x| = \log \frac{1}{x^5} \]

Integrating factor = \( e^{\int P \frac{dx}{x}} = e^{\log \frac{1}{x^5}} = \frac{1}{x^5} \)

General solution of (5) is \( u \cdot \frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5} \, dx \)

\[ = -5 \int \frac{1}{x^3} \, dx \]

\[ = (-5) \left( -\frac{1}{2x^2} \right) + C \]

\[ = \frac{5}{2x^2} + C \]

\[ \therefore \text{ General solution of (1) is } \frac{1}{x^5} y^5 = \frac{5}{2x^2} + C \]

**Eg. 2.3.15 :** Solve \( x \frac{dy}{dx} + y = y^2 \log x \)

**Solution :** Given equation is \( x \frac{dy}{dx} + y = y^2 \log x \) \( \text{--------- (1)} \)

This is Bernoulli’s equation.

Dividing (1) by \( xy^2 \), we get

\[ \frac{1}{y^2} \frac{dy}{dx} + \frac{1}{xy} = \frac{\log x}{x} \] \( \text{--------- (2)} \)

Put \( \frac{1}{y} = u \) \( \text{--------- (3)} \)

Differentiate with respect to \( x \)
\[-\frac{1}{y^2} \frac{dy}{dx} = \frac{du}{dx}\]

\[\Rightarrow \frac{1}{y^2} \frac{dy}{dx} = -\frac{du}{dx} \quad -------- (4)\]

substituting (3) and (4) in (2)

\[-\frac{du}{dx} + \frac{u}{x} = \log x \quad \frac{x}{x}\]

\[\Rightarrow \frac{du}{dx} - \frac{u}{x} = -\log x \quad -------- (5)\]

(5) is linear equation in ‘u’

Here \( p = -\frac{1}{x} \quad Q = \frac{-\log x}{x} \)

\[\int P \, dx = \int \frac{-1}{x} = -\log |x| = \log \frac{1}{x} \]

Integrating factor = \( e^{\int P \, dx} = e^{-\frac{\log x}{x}} = \frac{1}{x} \)

\[= \int \log x \cdot \frac{1}{x^2} \, dx \]

\[= \left[ \log x \int \frac{1}{x} \, dx \right] - \int \left( \frac{1}{x} \int \frac{1}{x^2} \, dx \right) \, dx \]

\[= \left[ \log x \left( -\frac{1}{x} \right) - \int \frac{-1}{x^2} \, dx \right] \]

\[= -\frac{\log x}{x} - \frac{1}{x} + C \]

General solution of (1) is \( \frac{1}{xy} = -\frac{\log x}{x} - \frac{1}{x} + C \)
Eg. 2.3.16 : Solve \( \frac{3\,dy}{dx} + \frac{2\,y}{1 + x} = \frac{x^3}{y^2} \)

Solution : Given equation is \( \frac{3\,dy}{dx} + \frac{2\,y}{1 + x} = \frac{x^3}{y^2} \) \hspace{1cm} (1)

This is a Bernoulli’s equation.

Multiplying (1) with \( y^2 \), we get

\[ 3\,y^2 \frac{dy}{dx} + \frac{2\,y^3}{1 + x} = x^3 \] \hspace{1cm} (2)

Put \( y^3 = u \) \hspace{1cm} (3)

Differentiate with respect to \( x \)

\[ 3\,y^2 \frac{dy}{dx} = \frac{du}{dx} \] \hspace{1cm} (4)

Substitute (3) and (4) in (2)

\[ \frac{du}{dx} + \frac{2}{1 + x} u = x^3 \] \hspace{1cm} (5)

(5) is linear equation in \( u \)

Here \( P = \frac{2}{1 + x} \hspace{1cm} Q = x^3 \)

\[ \int P\,dx = \int \frac{2}{1 + x} \,dx = 2\log|1 + x| = \log(1 + x)^2 \]

Integrating factor \( = e^{\int P\,dx} = e^{\log(1 + x)^2} = (1 + x)^2 \)

General solution of (5) is \( u \cdot (1 + x)^2 = \int x^3 \,dx \)

\[ = \int \left( x^5 + 2x^4 + x^3 \right) \,dx \]
General solution of (1) is \( y^3 (1+x)^2 = \frac{x^6}{6} + \frac{2x^5}{5} + \frac{x^4}{4} + C \)

**Eg. 2.3.17.** Solve \( \frac{dy}{dx} + \frac{xy}{1-x^2} = xy^{1/2} \)

**Solution:** Given equation is \( \frac{dy}{dx} + \frac{x}{1-x^2} y = xy^{1/2} \) \( \quad \text{------- (1)} \)

This is a Bernoulli's equation

Dividing (1) by \( y^{1/2} \) we get

\( \frac{1}{y^{1/2}} \frac{dy}{dx} + \frac{x}{1-x^2} y^{1/2} = x \) \( \quad \text{------- (2)} \)

Put \( y^{1/2} = u \) \( \quad \text{------- (3)} \)

Differentiate with respect to 'x'

\( \frac{1}{2} y^{-1/2} \frac{dy}{dx} = \frac{du}{dx} \)

\( \Rightarrow \frac{1}{y^{1/2}} \frac{dy}{dx} = 2 \frac{du}{dx} \) \( \quad \text{------- (4)} \)

substitute (3) and (4) in (2)

\( 2 \frac{du}{dx} + \frac{x}{1-x^2} u = x \)

\( \Rightarrow \frac{du}{dx} + \frac{x}{2(1-x^2)} u = \frac{x}{2} \) \( \quad \text{------- (5)} \)

This is linear in 'u'
Here \( P = \frac{x}{2(1-x^2)} \) \( Q = \frac{x}{2} \)

\[
\int P\,dx = \int \frac{x}{2(1-x^2)} \, dx
\]

\[
= \frac{1}{2} \cdot \frac{1}{2} \int \frac{-2x}{1-x^2} \, dx
\]

\[
= -\frac{1}{4} \log(1-x^2)
\]

\[
= \log \left(\frac{1}{1-x^2}\right)^{1/4}
\]

\[
\therefore \text{Integrating factor} = e^{\int P\,dx}
\]

\[
\log \left(\frac{1}{1-x^2}\right)^{1/4} = \frac{1}{(1-x^2)^{1/4}}
\]

\[
= e
\]

General solution of (5) is \( u \cdot \frac{1}{(1-x^2)^{1/4}} = \int \frac{x}{2} \cdot \frac{1}{(1-x^2)^{1/4}} \, dx \)

\[
= -\frac{1}{4} \int \frac{-2x}{(1-x^2)^{1/4}} \, dx
\]

\[
= -\frac{1}{4} \times \left(\frac{1-x^2}{3}\right)^{3/4} + C
\]
\[
3 \left( \frac{3}{4} \right) \left( \frac{1}{2} \right) \left( \frac{3}{4} \right) = \frac{3}{3} + C
\]

\[
\therefore \text{ General solution of (1) is } \frac{\sqrt{y}}{\sqrt{3}} = \frac{- \left(1-x^2\right)^{3/4}}{3} + c
\]

**Eg. 2.3.18**: Solve \( \frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y \)

**Solution**: Given equation is \( \frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y \) \( \cdots \cdots \) (1)

This is Bernoulli’s equation.

Dividing (1) by \( \cos^2 y \)

\[
\frac{1}{\cos^2 y} \frac{dy}{dx} + \frac{x \sin 2y}{\cos^2 y} = x^3
\]

\[\Rightarrow \sec^2 y \frac{dy}{dx} + x(2 \tan y) = x^3 \] \( \cdots \cdots \) (2)

Put \( \tan y = u \) \( \cdots \cdots \) (3)

Differentiate with respect to \( 'x' \)

\[
\sec^2 y \frac{dy}{dx} = \frac{du}{dx} \] \( \cdots \cdots \) (4)

Substitute (3) and (4) in (2)

\[
\frac{du}{dx} + 2xu = x^3 \] \( \cdots \cdots \) (5)

(5) is linear in \( 'u' \).

Here \( P = 2x \) \quad \( Q = x^3 \)

\[
\int P \, dx = \int 2x \, dx = x^2
\]
Integrating factor = $e^{\int P \, dx} = e^{x^2}$

General solution of (5) is $u \cdot e^{x^2} = \int x^3 \cdot e^{x^2} \, dx$

$$= \frac{1}{2} \int e^{x^2} \cdot x^2 \cdot 2x \, dx$$

$$= \frac{1}{2} e^{x^2} \left( x^2 - 1 \right) + C$$

\[ \therefore \, \text{General solution of (1) is} \quad \tan y \cdot e^{x^2} = \frac{1}{2} e^{x^2} \left( x^2 - 1 \right) + C \]

**Eg. 2.3.19**: Solve for $x > 0$, solve $\frac{dy}{dx} + \frac{y}{x} = y^2 \sin x$ given that $y = 1$, when $x = \pi$.

**Solution**: Given equation $\frac{dy}{dx} + \frac{y}{x} = y^2 \sin x \quad \text{--------- (1)}$

This is a Bernoulli’s equation

Dividing (1) by $y^2$ we get

$$\frac{1}{y^2} \frac{dy}{dx} + \frac{1}{x y^2} = \sin x \quad \text{--------- (2)}$$

Put $\frac{1}{y} = u \quad \text{--------- (3)}$

Differentiate with respect to $x$,

$$\frac{-1}{y^2} \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} = -\frac{du}{dx} \quad \text{--------- (4)}$$

Substitute (3) and (4) in (2) we get

$$-\frac{du}{dx} + \frac{u}{x} = \sin x$$
\[ \Rightarrow \frac{du}{dx} = \frac{-u}{x} = -x \sin x \quad \text{-------- (5)} \]

(5) is linear equation in \( u \).

Here \( P = -\frac{1}{x} \) \quad Q = -x \sin x

\[
\int P \, dx = \int \frac{-1}{x} \, dx = -\log |x| = \log \frac{1}{x}
\]

Integrating factor \( = e^{\int P \, dx} = e^{\log \frac{1}{x}} = \frac{1}{x} \)

General solution of (1) is \( u \cdot \frac{1}{x} = \int -x \sin x \cdot \frac{1}{x} \, dx \)

\[
= \int -\sin x \, dx = \cos x + C
\]

\[ \therefore \text{General solution of (1) is} \quad \frac{1}{xy} = \cos x + C \]

Given that if \( x = \pi \) then \( y = 1 \)

\[ \Rightarrow \frac{1}{\pi} = \cos \pi + C \]

\[ \Rightarrow \frac{1}{\pi} = -1 + C \Rightarrow C = \frac{1}{\pi} + 1 = \frac{\pi + 1}{\pi} \]

\[ \therefore \text{The particular solution of (1) given that} \quad y = 1 \]

when \( \frac{1}{xy} = \cos x + \frac{\pi + 1}{\pi} \)

\[ \therefore \text{The particular solution of (1) given that} \quad y = 1, \text{ when} \quad x = \pi \text{ is} \quad \frac{1}{xy} = \cos x + \frac{\pi + 1}{\pi} \]

**Eg. 2.3.20:** Solve \( \frac{dy}{dx} \left( x^2 y^3 + xy \right) = 1 \)
Solution: Given equation is \( \frac{dy}{dx} \left( x^2 y^3 + xy \right) = 1 \) \( \quad \text{(1)} \)

\[ \Rightarrow \frac{dx}{dy} = x^2 y^3 + xy \]

\[ \Rightarrow \frac{dx}{dy} - xy = x^2 y^3 \]

Dividing by \( x^2 \)

\[ \frac{1}{x^2} \frac{dx}{dy} - \frac{1}{x^2} y = y^3 \quad \text{(2)} \]

Put \( \frac{1}{x} = u \) \( \quad \text{(3)} \)

Differentiate with respect to 'y'

\[ -\frac{1}{x^2} \frac{dx}{dy} = \frac{du}{dy} \quad \text{(4)} \]

Substitute (3) and (4) in (2)

\[ - \frac{du}{dx} - uy = y^3 \]

\[ \Rightarrow \frac{du}{dy} + uy = -y^3 \quad \text{(5)} \]

(5) is a linear equation in 'u'

Here \( P = y \), \( Q = -y^3 \)

\[ \int P \, dy = \int y \, dy = \frac{y^2}{2} \]

Integrating factor \( = e^{\int P \, dy} = e^{y^2/2} \)

General solution of (5) is \( u \cdot e^{y^2/2} = \int -y^3 \cdot e^{y^2/2} \, dy \)
\[ \frac{y^2}{2} = t \]

\[ \Rightarrow y \, dy = dt \]

\[ \therefore u \cdot e^{y^2/2} = 2 \int e^t \cdot t - dt \]

\[ = -2 \left[ t \int e^t \, dt - \int \left( \int e^t \, dt \right) \, dt \right] \]

\[ = -2te^t + 2e^t + C \]

\[ = -y^2 e^{y^2/2} + 2e^{y^2/2} + C \]

\[ \therefore \text{General solution (1) is} \quad \frac{1}{x} e^{y^2/2} = -y^2 e^{y^2/2} + 2e^{y^2/2} + C \]

**Eg. 2.3.21** : Solve \[ \frac{dy}{dx} + \left( 2x \tan^{-1} y - x^3 \right) \left( 1 + y^2 \right) = 0 \]

**Solution** : Given equation is \[ \frac{dy}{dx} + \left( 2x \tan^{-1} y - x^3 \right) \left( 1 + y^2 \right) = 0 \quad \text{---------- (1)} \]

Dividing by \( 1 + y^2 \)

\[ \frac{1}{1 + y^2} \frac{dy}{dx} + \left( 2x \tan^{-1} y - x^3 \right) = 0 \]

\[ \Rightarrow \frac{1}{1 + y^2} \frac{dy}{dx} + 2x \tan^{-1} y = x^3 \quad \text{----------(2)} \]

Put \( \tan^{-1} y = u \quad \text{----------(3)} \)

Differentiate with respect to \( x \)
\[
\frac{1}{1 + y^2} \frac{dy}{dx} = \frac{du}{dx} \quad \text{(4)}
\]

Substitute (3) and (4) in (2)

\[
\frac{du}{dx} + 2xu = x^3 \quad \text{(5)}
\]

(5) is a linear equation in 'u'.

Here \( P = 2x \quad Q = x^3 \)

\[
\int P \, dx = \int 2x \, dx = x^2
\]

Integrating factor \( e^{\int P \, dx} = e^{x^2} \)

General solution of (5) is \( u \cdot e^{x^2} = \int x^3 \cdot e^{x^2} \, dx \)

\[
= \frac{1}{2} \int e^{x^2} \cdot x^2 \cdot 2x \, dx
\]

Put \( x^2 = t \) then \( 2x \, dx = dt \)

\[
\therefore u \cdot e^{x^2} = \frac{1}{2} \int e^t \cdot t \, dt
\]

\[
= \frac{1}{2} \left[ t \int e^t \, dt - \int \left( \int e^t \, dt \right) dt \right]
\]

\[
= \frac{1}{2} t e^t - \frac{1}{2} e^t + C
\]

\[
= \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C
\]

General solution of (1) is \( \tan^{-1} y \cdot e^{x^2} = \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C \)
Eg. 2.3.22 : Solve \( \frac{dy}{dx} = (\sin x - \sin y) \frac{\cos x}{\cos y} \)

**Solution :** Given equation is \( \frac{dy}{dx} = (\sin x - \sin y) \frac{\cos x}{\cos y} \) \( \text{-------- (1)} \)

\[ \Rightarrow \cos y \cdot \frac{dy}{dx} = \sin x \cos x - \sin y \cos x \]

\[ \Rightarrow \cos y \cdot \frac{dy}{dx} + \sin y \cdot \cos x = \sin x \cos x \text{ \ (2)} \]

Put \( \sin y = u \) \( \text{ \ (3)} \)

Differentiate with respect to \( 'x' \)

\[ \cos y \frac{dy}{dx} = \frac{du}{dx} \text{ \ (4)} \]

Substitute (3) and (4) in (2)

\[ \frac{du}{dx} + u \cos x = \sin x \cos x \text{ \ (5)} \]

(5) is a linear equation in \( 'u' \)

Here \( P = \cos x \quad Q = \sin x \cos x \)

\[ \int P \, dx = \int \cos x \, dx = \sin x \]

Integrating factor \( = e^{\int P \, dx} = e^{\sin x} \)

General solution of (5) is

\[ u \cdot e^{\sin x} = \int \sin x \cos x \, e^{\sin x} \, dx \]

Put \( \sin x = t \) then \( \cos x \, dx = dt \)

\[ \therefore u \, e^{\sin x} = \int e^{t} \, dt \]

\[ = t \int e^{t} \, dt - \left( \int 1 \, dt \right) dt \]

\[ = t e^{t} - e^{t} + C \]
\[ \sin x \cdot e^{\sin x} = e^{\sin x} (\sin x - 1) + C \]

SELF ASSESSMENT QUESTIONS

1. Solve \( (x^2 - 1) \frac{dy}{dx} + 2xy = 1 \)
2. Solve \( \frac{dy}{dx} + 2xy = e^{-x^2} \)
3. Solve \( x \frac{dy}{dx} + 2y - x^2 \log x = 0 \)
4. Solve \( dx + xdy = e^{-y} \sec^2 y \ dy \)

2.5 ANSWERS TO SELF ASSESSMENT QUESTIONS

1. \( \frac{dy}{dx} + \frac{2x}{x^2 - 1} \cdot y = \frac{1}{x^2 - 1} \)

Here \( P = \frac{2x}{x^2 - 1} \) \quad \( Q = \frac{1}{x^2 - 1} \)

Now \( e^{\int P \, dx} = e^{\int \frac{2x}{x^2 - 1} \, dx} = e^{\log(x^2 - 1)} = x^2 - 1 \)

General solution is \( y \cdot e^{\int P \, dx} = \int Q \cdot e^{\int P \, dx} \, dx \)

\[ \Rightarrow y \cdot (x^2 - 1) = \int \frac{1}{x^2 - 1} \cdot x^2 - 1 \, dx = \int 1 \, dx = x + C \]

2. \( P = 2x \) \quad \( Q = e^{-x^2} \)

\( e^{\int P \, dx} = e^{\int 2x \, dx} = e^{x^2} \)
General solution is \( y \cdot e^{x^2} = \int e^{-x^2} \cdot e^{x^2} \, dx = \int 1 \, dx = x + C \)

3.  
\[
\frac{dy}{dx} + \frac{2}{x} y = x \log x
\]

Here \( P = \frac{2}{x} \) \quad \text{and} \quad Q = x \log x

Now \( e^{\int P \, dx} = e^{\int \frac{2}{x} \, dx} = e^{2 \log x} = e^{\log x^2} = x^2 \)

General solution is \( y \cdot x^2 = \int x \log x \cdot x^2 \, dx \)

\[
\begin{align*}
&= \log x \left[ x^3 \, dx - \frac{1}{x} \int x^3 \, dx \right] \\
&= \frac{x^4 \log x}{4} - \frac{1}{x} \int x^4 \, dx \\
&= \frac{x^4 \log x}{4} - \frac{x^4}{16} + C
\end{align*}
\]

4.  
\[
\frac{dx}{dy} + x = e^{-y} \sec^2 y
\]

Here \( P = 1 \) \quad \text{and} \quad Q = e^{-y} \sec^2 y

\( e^{\int P \, dy} = e^{\int 1 \, dy} = e^y \)

General solution is \( x \cdot e^y = \int e^{-y} \sec^2 y \cdot e^y \, dy \)

\[
= \int \sec^2 y \, dy = \tan y + C
\]

2.6 SUMMARY

In this lesson we discussed linear differential equations, Bernoulli's form and related problems are discussed.

2.7 TECHNICAL TERMS

Linear differential equation, Bernoulli's form, Integrating factor.
EXERCISE 2A

1. Solve \((1 + x)\frac{dy}{dx} - xy = 1 - x\)

2. Solve \(\cos^2 x \frac{dy}{dx} + y = \tan x\)

3. Solve \(\sin 2x \frac{dy}{dx} - y = \tan x\)

4. Solve \(\frac{dy}{dx} - \left(\frac{2}{x}\right) y = x + \frac{1}{x} \sin\left(\frac{1}{x^2}\right)\)

5. Solve \((1 - x^2)\frac{dy}{dx} + 2xy = x\sqrt{1 - x^2}\)

6. Solve \(\frac{dy}{dx} + \frac{y}{x \log x} = \frac{\sin 2x}{\log x}\)

7. Solve \(x^3 \frac{dy}{dx} + \left(2 - 3x^2\right) y = x^3\)

8. Solve \(\left(y - e^{\sin^{-1} x}\right)\frac{dx}{dy} + \sqrt{1 - x^2} = 0, \ |x| < 1\)

9. Solve \(x\left(x^2 + 1\right)\frac{dy}{dx} = y\left(1 - x^2\right) + x^3 \log x\)

10. Solve \(y^2 dx + (3xy - 1) dy = 0\)

11. Solve \((x + \tan y) dy = \sin 2y dx\)

12. Solve \(\frac{dy}{dx} + 2y \tan x = \sin x\). Given that \(y = 0\) when \(x = \frac{\pi}{3}\)

13. Solve \((1 + x^2)\frac{dy}{dx} + 2xy = -\frac{1}{1 + x^2}\). Given that \(y = 0\) when \(x = 1\)
14. Solve \( \frac{dy}{dx} + \frac{y}{x} = x^2 \), Given that \( y = 1 \) when \( x = 1 \)

**ANSWERS:**

1. \( ye^{-x} (x+1) = xe^{-x} + C \)
2. \( y \cdot e^{\tan x} = e^{\tan x} (\tan x - 1) + C \)
3. \( y = \tan x + C \sqrt{\tan x} \)
4. \( 2y = x^2 \log x^2 + x^2 \cos \left( \frac{1}{x^2} \right) + Cx^2 \)
5. \( \frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + C \)
6. \( 2y \log x + \cos 2x = C \)
7. \( 2y = x^3 + Cx^3 e^{1/x^2} \)
8. \( y \cdot e^{\sin^{-1}x} = \frac{1}{2} e^{2\sin^{-1}x} + C \)
9. \( \frac{y (x^2+1)}{x} = \frac{x^2 \log x}{2} - \frac{x^2}{4} + C \)
10. \( yx^3 = \frac{y^2}{2} + C \)
11. \( x \cot y = \log \tan y + C \)
12. \( y \cdot \sec^2 x = \sec x - 2 \)
13. \( y \left(1 + x^2\right) = \tan^{-1} x - \frac{\pi}{4} \)
14. \( 4xy = x^4 + 3 \)

**2.8.2 EXERCISE 2B**

1. Solve \( \left(1-x^2\right) \frac{dy}{dx} + xy = xy^2 \)
2. Solve \( \frac{dy}{dx} + \frac{y}{x-1} = xy^{1/3} \)
3. Solve \( \frac{dy}{dx} + \frac{y}{x} = y^2 x \)
4. Solve \( \frac{dy}{dx} - y \tan x = \frac{\sin x \cos^2 x}{y} \)
5. Solve \( 2y \cos y \frac{dy}{dx} - \frac{2}{x+1} \sin y^2 = (x+1)^3 \)

6. Solve \( \frac{dy}{dx} = e^{x-y} \left( e^x - e^y \right) \)

7. Solve \( x \frac{dy}{dx} + y = y^2 x^3 \cos x \)

8. Solve \( x \frac{dy}{dx} + y \log y = xy e^x \)

9. Solve \( xy^2 \frac{dy}{dx} - 2y^3 = 2x^3 \). Given that \( y = 1 \) when \( x = 1 \).

10. Solve \( x y^2 - x^3 \frac{dy}{dx} = y^4 \cos x \). Given that \( x = \pi \) when \( y = \pi \).

**Answers:**

1. \( \sqrt{x^2 - 1} = \frac{-(x^2 - 1)^{3/2}}{3} + C \)

2. \( y^{2/3} (x-1)^{2/3} = \frac{1}{4} (x-1)^{8/3} \)

3. \( \frac{y}{x} = -x + C \)

4. \( y^2 \cos^2 x = -\frac{2}{5} \cos^5 x + C \)

5. \( \frac{\sin y^2}{(x+1)^2} = \frac{1}{2} (x+1)^2 + C \)

6. \( e^y \cdot e^{e^x} = e^{e^x} \left( e^x - 1 \right) + C \)

7. \( \frac{y}{x} = x \sin x - \cos x + C \)

8. \( x \log y = xe^x - e^x + C \)

9. \( \frac{y^3}{x^6} = -\frac{2}{x^3} + 3 \)

10. \( x^3 - y^3 = 3y^3 \sin x \)
2.10 MODEL QUESTIONS

Solve the following Differential equations.

1. \((1 - x^2) \frac{dy}{dx} + xy = ax\)

2. \((1 + y^2) \, dx = (\tan^{-1} y - x) \, dy\)

3. \((x + 2y^3) \frac{dy}{dx} = y\)

4. \(\frac{dy}{dx}(x^2y^3 + xy) = 1\)

5. \(\frac{dy}{dx} = e^{x-y}(e^x - e^y)\)

2.11 REFERENCE BOOKS

Text Book of Mathematics, Vo. I : Deepthi Publications
Introduction to Ordinary Differential Equations : Earl.A. Coddington

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Lesson - 3

TOTAL DIFFERENTIAL EQUATIONS

3.1 OBJECTIVE OF THE LESSON

After studying this lesson, the student will be in a position to know about total differential equations and simultaneous total differential equations and how to solve them.

3.2 STRUCTURE OF THE LESSON

This lesson has the following components.

3.3 Introduction, Definitions and Examples
3.4 Simulteneous total differential equations
3.5 Answers to Self Assessment Questions
3.6 Summary
3.7 Technical Terms
3.8 Exercises
3.9 Answers to Exercises
3.10 Model Examination Questions
3.11 Reference Books

3.3.1 INTRODUCTION

An ordinary differential equation of the first order and first degree involving one independent variable \( x \) and two dependent variables \( y, z \) is of the form.

\[
P + Q \frac{dy}{dx} + R \frac{dz}{dx} = 0
\]

It can be written as \( Pdx + Qdy + Rdz = 0 \)

where \( P, Q, R \) are functions of \( x, y, z \).

3.3.2 Definition : An equation of the form \( Pdx + Qdy + Rdz = 0 \), where \( P, Q, R \) are functions of \( x, y, z \) is called a total differential equation in three variables. Total differential equation is said to be integrable if \( \exists \phi(x, y, z) \) such that \( \frac{\partial \phi}{\partial x} = P \cdot K(x, y, z) \), \( \frac{\partial \phi}{\partial y} = Q \cdot K(x, y, z) \).
\[ \frac{\partial \phi}{\partial z} = R \cdot K(xyz) \text{ for some } K(xyz). \]

### 3.3.3 Condition for Integrability:

**Theorem:** A necessary and sufficient condition for the total differential equation \( Pdx + Qdy + Rdz = 0 \) to be integrable is

\[ P \left( \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = 0 \]

### 3.3.4 Working Rule to Solve total differential equation:

Given total differential equation is

\[ Pdx + Qdy + Rdz = 0 \quad \text{(1)} \]

(i) Verify the condition of integrability for (1)

(ii) If it is satisfied consider one of the variables say \( z \) to be constant without loss of generality suppose \( dz = 0 \).

(iii) Now equation (1) reduces to \( Pdx + Qdy = 0 \) \( \text{(2)} \) and integrating (2) let the solution be \( f(x, y) = \phi(z) - (3) \) where \( \phi(z) \) is arbitrary function of \( z \).

(iv) Now differentiable (3) totally with respect to \( x, y, z \) and compare the result with (1) to determine \( \phi(z) \). Finally substitute \( \phi(z) \) in (3) we get the required solution of (1).

### 3.3.5 Note:

1. Total derivative of \( f(x, y, z) \) w.r.t. \( x, y, z \) is given by \( \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \) and is denoted by \( df \).

2. Sometimes given d.e. can also be solved simply by regrouping the terms or making into exact differential by inspection.

### 3.4.1 Simultaneous total differential equations:

Two total differential equations

\[ P_1dx + Q_1dy + R_1dz = 0 \quad \text{and} \quad P_2dx + Q_2dy + R_2dz = 0 \]

can be put in the form of \( \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \), where \( P, Q, R \) are functions of \( x, y, z \).

**Example:** Two total d.e’s \( xdx + zdy + ydz = 0 \) and \( ydx + xdy + zdz = 0 \) can be put in the form

\[ \frac{dx}{yx - z^2} = \frac{dy}{xz - y^2} = \frac{dz}{yz - x^2} \]
3.4.2 Equations of the form: \[ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \]

Given equations are

\[ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{----------------- (1)} \] where \( P, Q, R \) are functions of \( x, y, z \).

Consider the set of three equations

\[ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{----------------- (2)} \]

In the above equations (2) any one can be obtained from the remaining two equations. So we take any two equations of (2) as equivalent to the system (1).

To find the complete solution of (1), we will discuss two methods.

(i) Method of grouping

(ii) Method of multipliers

3.4.3 Method of Grouping:

Case (i): If any two equations of the set of equations (2) are integrable by the method of variables separable we can find general solutions and that pair of solutions form the complete solution of the system (1).

Case (ii): If only one equation of the set of equations (2) is integrable, by the method of variables separable, we can find general solution and using this solution another equation of (2) can be solved, the two general solutions obtained form the general solution of the system (1).

3.4.4 Method of Multipliers: Given system of equations are

\[ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{------- (1)} \]

If no equation of the set of equations (2) is integrable. We write

\[ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{Z} = \frac{\ell_1 dx + m_1 dy + n_1 dz}{\ell_1 P + m_1 Q + n_1 R} = \frac{\ell_2 dx + m_2 dy + n_2 dz}{\ell_2 P + m_2 Q + n_2 R} \]

where \( \ell_1, m_1, n_1 \) and \( \ell_2, m_2, n_2 \) are real numbers or functions of \( x, y, z \).

(i) If we choose \( \ell_1, m_1, n_1 \) and \( \ell_2, m_2, n_2 \) (called multipliers) such that \( \ell_1 P + m_1 Q + n_1 R = 0 \) and \( \ell_2 P + m_2 Q + n_2 R = 0 \) then we get

\[ \ell_1 dx + m_1 dy + n_1 dz = 0 \quad \text{and} \]
\( \ell_2 dx + m_2 dy + n_2 dz = 0 \)

which on integration gives general solution of (1)

(ii) If we choose \( \ell_1, m_1, n_1 \) and \( \ell_2, m_2, n_2 \) such that \( \ell_1 P + m_1 Q + n_1 R \neq 0 \);

\[
\frac{\ell_1 dx + m_1 dy + n_1 dz}{\ell_1 P + m_1 Q + n_1 R} = d\phi \quad \text{and}
\]

\[
\ell_2 P + m_2 Q + n_2 R \neq 0; \quad \frac{\ell_2 dx + m_2 dy + n_2 dz}{\ell_2 P + m_2 Q + n_2 R} = d\psi \quad \text{then } \phi(x, y, z) = C_1, \psi(x, y, z) = C_2
\]

will give the general solution of the system (1).

**Solved Examples :**

3.4.5 : Solve \( dx + dy + (x + y + z + 1) dz = 0 \)

**Solution :** Given d.e. is

\( dx + dy + (x + y + z + 1) dz = 0 \)  \text{------- (1)}

Here \( P = 1, Q = 1, R = x + y + z + 1 \)

\[
\Rightarrow \frac{\partial P}{\partial z} = \frac{\partial P}{\partial y} = 0, \quad \frac{\partial Q}{\partial z} = \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial R}{\partial x} = 1, \quad \frac{\partial R}{\partial y} = 1
\]

Now, \( P \left( \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \)

\[
= 1(0-1) + 1(1-0) + (x + y + z + 1)(0-0)
\]

\[
= -1 + 1 = 0
\]

\( \therefore \) condition of integrability is satisfied.

\( \therefore \) (1) is integrable.

Consider \( z \) as constant \( \Rightarrow dz = 0 \)

\( \therefore \) (1) reduces to \( dx + dy = 0 \). Integrating

\[
\int dx + \int dy = \int 0
\]

\( \Rightarrow x + y = \phi(z) \)  \text{------- (2)}
where \( \phi(z) \) is arbitrary function of \( z \) only consider (2)

\[
x + y - \phi(z) = 0
\]

Differentiating with respect to \( x, y, z \).

We get, \( dx + dy - \phi'(z)dz = 0 \) 

Comparing (1) and (3)

\[
\Rightarrow 1 = \frac{x + y + z + 1}{-\phi'(z)}
\]

\[
\Rightarrow \phi'(z) = x + y + z + 1 = 0
\]

\[
\Rightarrow \frac{d\phi}{dz} + \phi = -1 - z \quad [\text{using (2)}]
\]

This is linear d.e. in \( \phi \).

\( I.F = e^{\int dz} = e^z \)

\( \therefore \text{G.S. is} \quad e^z \phi = \int (-1-z)e^z \, dz \)

\[
\Rightarrow e^z \phi + e^z z = c
\]

\[
\Rightarrow e^z (x + y + z) = c \quad [\because \text{from (2)}]
\]

which is the general solution of (1)

3.4.6: Solve \( (2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0 \)

**Solution:** Given d.e. is

\[
(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0 \quad \cdots (1)
\]

Here \( P = 2x^2 + 2xy + 2xz^2 + 1 \quad Q = 1 \quad R = 2z \)

\[
\frac{\partial P}{\partial z} = 4xz, \quad \frac{\partial P}{\partial y} = 2x, \quad \frac{\partial Q}{\partial z} = \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial R}{\partial y} = \frac{\partial R}{\partial x} = 0
\]

Now
\[
P \left( \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)
\]

\[
= P(0 - 0) + 1(0 - 4xz) + 2z(2x - 0)
\]

\[
= -4xz + 4xz
\]

\[
= 0
\]

\[\therefore \text{ condition of integrability is satisfied}\]

\[\therefore (1) \text{ is integrable.}\]

Consider \( x \) as constant \( \Rightarrow dx = 0 \)

\[\therefore (1) \text{ reduces to } dy + 2zdz = 0\]

Integrating

\[
\int dy + 2\int z \, dz = \int 0
\]

\[\Rightarrow y + z^2 = \phi(x) \quad \text{------- (2)}\]

where \( \phi(x) \) is arbitrary function of \( x \) only consider (2)

\[y + z^2 - \phi(x) = 0\]

Differentiating totally w.r.t. \( x, y, z \)

\[-\phi'(x)dx + dy + 2zdz = 0 \quad \text{-------- (3)}\]

Comparing (1) and (3)

\[
\frac{-\phi'(x)}{2x^2 + 2xy + 2xz^2 + 1} = \frac{1}{1} = \frac{2z}{2z}
\]

\[\Rightarrow \phi'(x) + 2x^2 + 2xy + 2xz^2 + 1 = 0\]

\[\Rightarrow \frac{d\phi}{dx} + 2x^2 + 2x\left( y + z^2 \right) + 1 = 0\]

\[\Rightarrow \frac{d\phi}{dx} + 2x\phi = -2x^2 - 1 \quad \text{[using (2)]}\]
This is linear in \( \phi \)

\[
I.F. = e^{\int 2x \, dx} = e^{x^2}
\]

\[\therefore \text{ solution is}
\]

\[
e^{x^2} \phi = -\int e^{x^2} \left(2x^2 + 1\right) \, dx
\]

Put

\[x^2 = t\]

D. w.r.x

\[2x \, dx = dt\]

\[
= -\int e^t \left(2t + 1\right) \frac{dt}{2\sqrt{t}}
\]

\[
= -\int e^t \left[\sqrt{t} + \frac{1}{2\sqrt{t}}\right] \, dt
\]

\[\Rightarrow e^{x^2} \phi = -e^t \sqrt{t} + C
\]

\[\Rightarrow e^{x^2} \phi = -e^{x^2} x + C
\]

\[\Rightarrow \phi = -x + Ce^{-x^2}
\]

\[\Rightarrow x + y + z^2 = Ce^{-x^2}
\]

which is the general solution (1)

3.4.9: Solve \( z^2 \, dx + \left(z^2 - 2yz\right) \, dy + \left(2y^2 - 4z - zx\right) \, dz = 0 \)

Solution: Given d.e. is

\[
z^2 \, dx + \left(z^2 - 2yz\right) \, dy + \left(2y^2 - yz - zx\right) \, dz = 0 \quad \text{(1)}
\]

Here \( P = z^2 \quad Q = z^2 - 2yz \quad R = 2y^2 - yz - zx \)

\[
\frac{\partial P}{\partial z} = 2z \quad \frac{\partial P}{\partial y} = 0 \cdot \frac{\partial Q}{\partial z} = 2z - 2y \cdot \frac{\partial Q}{\partial x} = 0
\]

\[
\frac{\partial R}{\partial y} = 4y - Z \quad \text{and} \quad \frac{\partial R}{\partial x} = -Z
\]

Now
\[
P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \\
= z^2(2z - 2y - 4y + z) + \left(z^2 - 2yz\right)(-z - 2z) + \left(2y^2 - yz - zx\right)(0 - 0) \\
= 3z^3 - 6yz^2 - 3z^3 + 6yz^2 + 0 \\
= 0
\]
\[\therefore \text{ Condition of Integrability is satisfied.}\]
\[\therefore (1) \text{ is Integrable.}\]

Consider \( z \) as constant \( \Rightarrow dz = 0 \).
\[\therefore (1) \text{ Reduces to}\]
\[z^2dx + \left(z^2 - 2yz\right)dy = 0\]

Integrating
\[\int z^2dx + \int (z^2 - 2yz)dy = 0\]
\[\Rightarrow z^2x + z^2y - y^2z = \phi(z) \quad \text{(2)}\]

where \( \phi(z) \) is arbitrary function of \( z \).

Consider (2)
\[z^2x + z^2y - y^2z - \phi(z) = 0\]

Differentiating totally w.r.t. \( x, y, z \)

We get
\[z^2dx + \left(z^2 - 2yz\right)dy + \left(2xz + 2yz - y^2 - \phi'(z)\right)dz = 0 \quad \text{(3)}\]

Comparing (1) and (3)
\[
\frac{z^2}{z^2} = \frac{z^2 - 2yz}{z^2 - 2yz} = \frac{2xz + 2yz - y^2 - \phi'(z)}{2y^2 - yz - zx}
\]
\[ -\phi'(z) - y^2 + 2xz + 2yz = 1 \]
\[ \Rightarrow \frac{\phi'(z)}{2y^2 - yz - zx} = 1 \]
\[ \Rightarrow \phi'(z) + 3y^2 - 3yz - 3xz = 0 \]
\[ \Rightarrow \frac{d\phi}{dz} - \frac{3\phi}{z} = 0 \quad \text{(using (2))} \]
\[ \Rightarrow \frac{d\phi}{dz} = \frac{3\phi}{z} \]
\[ \Rightarrow \frac{d\phi}{\phi} = 3\frac{dz}{z} \]

Integrating
\[ \int \frac{d\phi}{\phi} = 3\int \frac{dz}{z} \]
\[ \Rightarrow \log \phi = 3\log 2 + \log C \]
\[ \Rightarrow \phi = z^3C \]

From (2) \[ \Rightarrow z^2x + z^2y - y^2z = z^3C \]
\[ \Rightarrow xz + yz - y^2 = cz^2 \]

Which is the general solution of (1)

**3.4.8 :** Solve \( (z + z^3)\cos x \, dx - (z + z^3)\, dy + (1 - z^2)(y - \sin x) \, dx = 0 \)

**Solution :** Given d.E. is
\[ (z + z^3)\cos x \, dx - (z + z^3) \, dy + (1 - z^2)(y - \sin x) \, dz = 0 \quad \text{---------- (1)} \]
Here \( P = (z + z^3)\cos x \quad Q = -(z + z^3) \quad R = (1 - z^2)(y - \sin x) \)
\[ \frac{\partial P}{\partial z} = \cos x \left( 1 + 3z^2 \right) \quad \frac{\partial P}{\partial y} = 0, \quad \frac{\partial Q}{\partial z} = -(1 + 3z^2) \]
\[ \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial R}{\partial y} = (1 - z^2)(1 - 0) = \left( 1 - z^2 \right) \quad \text{and} \]
\[
\frac{\partial R}{\partial x} = (1 - z^2)(-\cos x)
\]

Now \( P \left( \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \)

\[
= \left( z + z^3 \right) \cos x \left( -1 - 3z^2 - 1 + z^2 \right) - \left( z + z^3 \right) \left( -\cos x + z^2 \cos x - \cos x - 3z^2 \cos x \right) \\
+ \left( 1 - z^2 \right) (y - \sin x) (0 - 0)
\]

\[
= \left( z + z^3 \right) \cos x \left( -2 - 2z^2 \right) - \left( z + z^3 \right) \cos x \left( -2 - 2z^2 \right)
\]

\[= 0.\]

\[\therefore\] Condition of Integrability is satisfied.

\[\therefore\] (1) is Integrable.

Consider \( z \) as constant \( \Rightarrow dz = 0 \)

\[\therefore\] (1) Reduces to

\[
\left( z + z^3 \right) \cos x \, dx - \left( z + z^3 \right) dy = 0
\]

Integrating

\[
\left( z + z^3 \right) \left[ \int \cos x \, dx - \int dy \right] = \int 0
\]

\[\Rightarrow \left( z + z^3 \right) (\sin x - y) = \phi (z) \quad \text{-------- (2)}\]

where \( \phi (z) \) is arbitrary function of \( z \)

Consider (2)

\[
\left( z + z^3 \right) (\sin x - y) - \phi (z) = 0
\]

Differentiating totally w.r.t. to \( x, y, z \)

\[
\left( z + z^3 \right) (\cos x - 0) \, dx + \left( z + z^3 \right) (0 - 1) \, dy + (\sin x - y) (1 + 3z^2) - \phi' (z) dz = 0 \quad \text{-------- (3)}
\]

Comparing (1) and (3)
\[
\begin{align*}
\frac{(z + z^3) \cos x}{(z + z^3) \cos x} &= \frac{-(z + z^3)}{-(z + z^3)} = \frac{(\sin x - y)(1 + 3z^2) - \phi'(z)}{(1 - z^2)(y - \sin x)} \\
&\Rightarrow \frac{(\sin x - y)(1 + 3z^2) - \phi'(z)}{(1 - z^2)(y - \sin x)} = 1 \\
&\Rightarrow \phi'(z) + \left(1 - z^2\right)(y - \sin x) + (y - \sin x)(1 + 3z^2) = 0 \\
&\Rightarrow \frac{d\phi}{dz} + (y - \sin x)(2z^2 + 2) = 0 \\
&\Rightarrow \frac{d\phi}{dz} - \frac{2\phi}{z} = 0 \text{ (using (2))} \\
&\Rightarrow \frac{d\phi}{dz} = \frac{2\phi}{z} \\
&\Rightarrow \frac{d\phi}{\phi} = 2 \frac{dz}{z} \\
\int \frac{d\phi}{\phi} &= 2 \int \frac{dz}{z} \\
&\Rightarrow \log \phi = 2 \log z + \log c \\
&\Rightarrow \log \phi = \log cz^2 \\
&\Rightarrow \phi = cz^2
\end{align*}
\]

Now from (2) \( (z + z^3)(\sin x - y) = cz^2 \)
\[\Rightarrow (z^2 + 1)(\sin x - y) = cz \]

which is the general solution of (1)

**3.4.9:** Solve \( \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \)
Solution: Taking
\[ \frac{dx}{x} = \frac{dy}{y} \]

Integrating
\[ \int \frac{dx}{x} = \int \frac{dy}{y} \]

\[ \Rightarrow \log x = \log y + \log c_1 \]
\[ \Rightarrow x - c_1 y = 0 \quad \text{-------- (1)} \]

Also taking
\[ \frac{dy}{y} = \frac{dz}{z} \]

Integrating
\[ \int \frac{dy}{y} = \int \frac{dz}{z} \]

\[ \Rightarrow \log y = \log z + \log c_2 \]
\[ \Rightarrow y - c_2 z = 0 \quad \text{-------- (2)} \]

\( \therefore \) The general solution of the given system is given by (1) and (2).

i.e. \( x - c_1 y = 0; y - c_2 z = 0 \)

3.4.10: Solve \( \frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2} \)

Solution: Given system of equations are

\[ \frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2} \]

Taking \( \frac{dx}{y^2} = \frac{dy}{x^2} \)

\[ \Rightarrow x^2 dx = y^2 dy \]

Integrating
\[ \int x^2 dx = \int y^2 dy \]

\[ \Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + \frac{c_1}{3} \]

\[ \Rightarrow x^3 - y^3 - c_1 = 0 \quad \text{(1)} \]

Again \[ \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2} \]

\[ \Rightarrow y^2 dy = \frac{dz}{z^2} \]

Integrating

\[ \int y^2 dy = \int \frac{dz}{z^2} \]

\[ \Rightarrow \frac{y^3}{3} = -\frac{1}{z} + c_2 \quad \text{(2)} \]

\[ \therefore \text{General solution of the given system is given by (1) and (2).} \]

3.4.11 : Solve \[ \frac{xdx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2} \]

Solution : Given system of equations are

\[ \frac{xdx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2} \]

Taking \[ \frac{xdx}{y^2 z} = \frac{dy}{xz} \]

\[ \Rightarrow x^2 dx = y^2 dy \]

Integrating

\[ \int x^2 dx = \int y^2 dy \]

\[ \Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + \frac{c_1}{3} \]
\[ x^3 - y^3 = c_1 \] 

also taking \[ \frac{dx}{y^2z} = \frac{dz}{y^2} \]

\[ \Rightarrow x \, dx = z \, dz \]

Integrating

\[ \int x \, dx = \int z \, dz \]

\[ \Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + \frac{c_2}{2} \]

\[ \Rightarrow x^2 - z^2 = c_2 \] 

\[ \therefore \text{ Complete solution of the given system is given by (1) and (2)} \]

\[ \text{i.e. } x^3 - y^3 = c_1; \ x^2 - z^2 = c_2 \]

3.4.12: Solve \[ \frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z} \]

\textbf{Solution}: Given system of equations are

\[ \frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z} \]

Taking \[ \frac{dx}{y} = \frac{dy}{x} \]

\[ \Rightarrow x \, dx = y \, dy \]

Integrating

\[ \int x \, dx = \int y \, dy \]

\[ \Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + \frac{c_1^2}{2} \Rightarrow x^2 - y^2 = c_1^2 \] 

\[ \text{Also taking } \frac{dy}{x} = \frac{dz}{z} \]
\[ \Rightarrow \frac{dy}{\sqrt{c_1^2 + y^2}} = \frac{dz}{z} \] [using (1)]

Integrating

\[ \int \frac{dy}{\sqrt{c_1^2 + y^2}} = \frac{dz}{z} \]

\[ \Rightarrow \log \left( y + \sqrt{c_1^2 + y^2} \right) = \log z + \log c_2 \]

\[ \Rightarrow \left( y + \sqrt{c_1^2 + y^2} \right) = c_2 z \Rightarrow y + x = c_2 z \text{-------- (2)} \]

\[ \therefore \text{General solution of the given system is given by (1) and (2)} \]

\[ \text{i.e. } x^2 - y^2 = c_1^2, \quad (y + x) = c_2 z \]

3.4.13 : Solve \[ \frac{dx}{-y^2 - z^2} = \frac{dy}{xy} = \frac{dz}{zx} \]

**Solution**: Given system of equations are

\[ \frac{dx}{-y^2 - z^2} = \frac{dy}{xy} = \frac{dz}{zx} \]

Taking \[ \frac{dy}{xy} = \frac{dz}{zx} \]

\[ \Rightarrow \frac{dy}{y} = \frac{dz}{z} \]

Integrating

\[ \log y = \log z + \log c_1 \]

\[ \Rightarrow y = c_1 z \text{-------- (1)} \]

Also taking \[ \frac{dx}{-y^2 - z^2} = \frac{dz}{zx} \]

\[ \Rightarrow \frac{dx}{-c_1^2 z^2 - z^2} = \frac{dz}{zx} \]
\[ xdx = \frac{-z^2\left(c_1^2 + 1\right)}{z} \, dz \]

Integrating

\[ \int x \, dx = -\left(c_1^2 + 1\right) \int z \, dz \]

\[ \frac{x^2}{2} = -\left(c_1^2 + 1\right) \frac{z^2}{2} + \frac{c_2}{2} \]

\[ \Rightarrow x^2 + \left(c_1^2 + 1\right)z^2 = c_2 \]

\[ \Rightarrow x^2 + y^2 + z^2 = c_2 \quad \text{--------- (2) [using (1)]} \]

\[ \therefore \text{General solution of the given system is} \]

\[ y = c_1 z; \ x^2 + y^2 + z^2 = c_2 \]

**3.4.14 : Solve**

\[ \frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y + 2x)} \]

**Solution :** Given system of equations are

\[ \frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y + 2x)} \]

Taking \[ \frac{dx}{1} = \frac{dy}{-2} \]

Integrating

\[ -2 \int dx = \int dy \]

\[ \Rightarrow y + 2x = c_1 \quad \text{--------- (1)} \]

Also taking

\[ \frac{dx}{1} = \frac{dz}{3x^2 \sin(c_1)} \quad [\text{using (1)}] \]

\[ \Rightarrow dx = \frac{dz}{3x^2 \sin c_1} \quad [\text{using (1)}] \]
Integrating

\[3 \sin c_1 \int x^2 \, dx = \int dz\]

\[\Rightarrow 3 (\sin c_1) \frac{x^3}{3} = z + c_2\]

\[\Rightarrow x^3 \sin (y + 2x) - z = c_2 \quad \text{-------- (2)}\]

: General solution of the given system is given by (1) and (2).

3.4.15 : Solve \(\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{2x - 3y}\)

**Solution** : Given system of equations are

\[
\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{2x - 3y}
\]

Taking \(\frac{dx}{y} = \frac{dy}{-x}\)

\[\Rightarrow x \, dx = -y \, dy\]

Integrating

\[
\int x \, dx = -\int y \, dy
\]

\[x^2 + y^2 = c_1^2 \quad \text{-------- (1)}\]

Also taking \(\frac{dx}{y} = \frac{dz}{2x - 3y}\)

\[\Rightarrow \frac{2x - 3y}{y} \, dx = dz\]

\[\Rightarrow \left[ \frac{2x}{\sqrt{c_1^2 - x^2}} - 3 \right] \, dx = dz\]

Integrating
\[-\int \frac{-2x}{\sqrt{c_1^2 - x^2}} \, dx - 3\int dx = \int dz\]

\[\Rightarrow -2\sqrt{c_1^2 - x^2} - 3x = z + c_2\]

\[\Rightarrow z + 3x + 2y + c_2 = 0 \quad \text{(2) [using (1)]}\]

\[\therefore \text{General solution of the given system is given by (1) and (2).}\]

**Note:** Problems 3.4.13 and 3.4.15 can also be solved by the method of multipliers easily.

3.4.16: Solve \( \frac{dx}{x} = \frac{dy}{x + z} = \frac{dz}{-z} \)

**Solution:** Given system of equations are

\[\frac{dx}{x} = \frac{dy}{x + z} = \frac{dz}{-z}\]

Taking \(\frac{dx}{x} = \frac{dz}{-z}\)

Integrating

\[\int \frac{dx}{x} = -\int \frac{dz}{z}\]

\[\Rightarrow x \cdot z = c_1 \quad \text{(1)}\]

using \(-1, 1, 1\) as multipliers, we have

Each fraction of the given system = \(-\frac{dx + dy + dz}{-x + (x + z) - z}\)

\[= \frac{-dx + dy + dz}{0}\]

\[\Rightarrow -dx + dy + dz = 0\]

Integrating

We get \(-x + y + z = c_2 \quad \text{(2)}\)

\[\therefore \text{General solution of the given system is given by (1) and (2)}\]

i.e. \(xz = c_1; \quad -x + y + z = c_2\)
3.4.17: Solve \( \frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z} \)

**Solution:** Given system of equations are
\[
\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z}
\]

Taking \( \frac{dx}{1+y} = \frac{dy}{1+x} \)

\[\Rightarrow (1+x)dx = (1+y)dy\]

Integrating
\[
x + \frac{x^2}{2} = y + \frac{y^2}{2} + \frac{c_1}{2}
\]

\[\Rightarrow 2x - 2y + x^2 - y^2 = c_1 \quad \text{---------------- (1)}\]

using 1,1,0 as multipliers, we have

Each fraction = \( \frac{dx + dy + 0dz}{(1+y)+(1+x)+0\cdot z} \)

\[= \frac{dx + dy}{2+x+y}\]

\[\Rightarrow \frac{dz}{z} = \frac{dx + dy}{2+x+y}\]

Integrating
\[\Rightarrow \log z = \log (2 + x + y) + \log c_2\]

\[\Rightarrow z = (2 + x + y)c_2 \quad \text{---------------- (2)}\]

:\ General solution of the given system is given by (1) and (2)

3.4.18: Solve \( \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \)

**Solution:** Given system of equations are
\[ \frac{dx}{x^2} = \frac{dy}{y^2} \]

Taking \( \frac{dx}{x^2} = \frac{dy}{y^2} \)

Integrating

\[ \int \frac{dx}{x^2} = \int \frac{dy}{y^2} \]

\[ \Rightarrow -\frac{1}{x} = -\frac{1}{y} + c_1 \]

\[ \Rightarrow -\frac{1}{y} - \frac{1}{x} = c_1 \]

\[ \Rightarrow x - y = c_1 xy \quad ------- (1) \]

Using 1, -1, 0 as multipliers, we have

Each fraction \[ \frac{dx - dy + 0 \, dz}{x^2 + (-1) \, y^2 + 0 \,(x + y) \, z} \]

\[ \Rightarrow \frac{dz}{(x + y) \, z} = \frac{dx - dy}{x^2 - y^2} \]

\[ \Rightarrow \frac{dz}{z} = \frac{dx - dy}{x - y} \]

Integrating

\[ \int \frac{dz}{z} = \int \frac{dx - dy}{x - y} \]

\[ \Rightarrow \log z = \log (x - y) + \log c_2 \]

\[ \Rightarrow z = c_2 (x - y) \quad ------- (2) \]

: General solution of the given system is given by (1) and (2).

\[ 3.4.19 : \text{Solve} \quad \frac{dx}{-y^2 - z^2} = \frac{dy}{xy} = \frac{dz}{zx} \]
Solution : Given system of Equations are

\[\frac{dx}{-y^2 - z^2} = \frac{dy}{xy} = \frac{dz}{zx}\]

Taking \(\frac{dy}{xy} = \frac{dz}{zx}\)

Integrating \(y = c_1 z \quad \text{-------- (1)}\)

Using \(x, y, z\) as multipliers, we have

Each fraction \(\frac{xdx + ydy + zdz}{x(-y^2 - z^2) + y \cdot xy + z \cdot zx}\)

\[= \frac{xdx + ydy + zdz}{0}\]

\(\Rightarrow xdx + ydy + zdz = 0\)

Integrating

\[x^2 + y^2 + z^2 = c_2 \quad \text{-------- (2)}\]

\(\therefore\) General solution of the system is given by (1) and (2)

3.4.20 : Solve \(\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}\)

Solution : Given system of equations are

\[\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}\]

Taking \(\frac{dy}{2xy} = \frac{dz}{2xz}\)

Integrating \(\int \frac{dy}{y} = \int \frac{dz}{z}\)

\(\Rightarrow y = c_1 z \quad \text{-------- (1)}\)

using \(x, y, z\) as multipliers, we have
Each fraction \[\frac{x\,dx + y\,dy + z\,dz}{x\left(x^2 - y^2 - z^2\right) + y \cdot 2xy + z \cdot 2xz}\]

\[\Rightarrow \frac{dy}{2xy} = \frac{x\,dx + y\,dy + z\,dz}{x\left(x^2 + y^2 + z^2\right)}\]

Integrating

\[\int \frac{dy}{y} = \int d\left[\log\left(x^2 + y^2 + z^2\right)\right]\]

\[\Rightarrow \log y = \log\left(x^2 + y^2 + z^2\right) + \log c_2\]

\[\Rightarrow y = c_2\left(x^2 + y^2 + z^2\right) \quad \text{------------ (2)}\]

\[\therefore \text{ General solution of the system is given by (1) and (2).}\]

3.4.21 : Solve \[\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}\]

**Solution** : Given system of equations are

\[\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}\]

using 1, 1, 1 as multipliers

Each fraction \[= \frac{dx + dy + dz}{x(y - z) + y(z - x) + z(x - y)}\]

\[= \frac{dx + dy + dz}{0}\]

\[\Rightarrow dx + dy + dz = 0\]

Integrating

\[x + y + z = c_1 \quad \text{------------ (1)}\]

Again using \(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\) as multipliers, then
3.23 Total Differential Equations

Each fraction \( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x(y - z)} + \frac{1}{y(z - x)} + \frac{1}{z(x - y)} \)

\[ \Rightarrow \frac{1}{x} \, dx + \frac{1}{y} \, dy + \frac{1}{z} \, dz = 0 \]

Integrating

\[ \int \frac{1}{x} \, dx + \int \frac{1}{y} \, dy + \int \frac{1}{z} \, dz = \int 0 \]

\[ \log x + \log y + \log z = \log c_2 \]

\[ xyz = c_2 \quad (2) \]

\( \therefore \) General solution of the given system is given by (1) and (2).

3.4.22: Solve \( \frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + xz} = \frac{dz}{xy - xz} \)

**Solution:** Given system of equations are

\[ \frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + xz} = \frac{dz}{xy - xz} \]

Taking \( \frac{dy}{xy + xz} = \frac{dz}{xy - xz} \)

\[ \Rightarrow \frac{dy}{y + z} = \frac{dz}{y - z} \]

\[ \Rightarrow ydy - zdy = ydz + zdz \]

\[ \Rightarrow ydy - zdy = ydz + zdy \]

\[ \Rightarrow ydy - zdz = d(yz) \]

Integrating

\[ \int ydy - \int zdz = \int d(yz) \]
\[ \frac{y^2}{2} - \frac{z^2}{2} = yz + \frac{c_1}{2} \]

\Rightarrow y^2 - z^2 - 2yz = c_1 \quad (1)

using \( x, y, z \) as multipliers, then

Each fraction \(~x \frac{dx}{y} + y \frac{dy}{z} + z \frac{dz}{x}\) = \( \frac{x \left( z^2 - 2yz - y^2 \right) + y (xy + xz) + z (xy - xz)}{x \left( z^2 - 2yz - y^2 \right) + y (xy + xz) + z (xy - xz)} \)

\Rightarrow x \frac{dx}{y} + y \frac{dy}{z} + z \frac{dz}{x} = 0

Integrating

\[ \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2} \]

\Rightarrow x^2 + y^2 + z^2 = c_2 \quad (2)

\therefore \text{ General solution of given system is given by (1) and (2)}

3.4.23 : Solve \( \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \)

\textbf{Solution} : Given system of equations are

\[ \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \]

Clearly

Each fraction \(~\frac{dx-dy+0 \cdot dz}{(y+z)-(z+x)+0(x+y)}\) = \( \frac{0dx+dy-dz}{0(y+z)+(z+x)-(x+y)} \) = \( \frac{dx+dy+dz}{(y+z)+(z+x)+(x+y)} \)

[By choosing 1, -1, 0; 0, 1, -1 and 1, 1, 1 as multipliers respectively]

\[ \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)} \]

Taking \( \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} \)
3.25 Total Differential Equations

\[ \frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} \]

Integrating \( \log(x - y) = \log(y - z) + \log c_1 \)

\[ \Rightarrow (x - y) = c_1(y - z) \quad \ldots \ldots \quad (1) \]

Also taking \( \frac{dx - dy}{y - x} = \frac{dx + dy + dz}{2(x + y + z)} \)

Integrating

\[ 2 \log(x - y) + \log(x + y + z) = \log c_2 \]

\[ \Rightarrow (x - y)^2(x + y + z) = c_2 \quad \ldots \ldots \quad (2) \]

Therefore, General solution of the given system is given by (1) & (2).

3.5 SHORT ANSWER QUESTIONS (SAQ)

1. Find whether the differential equation.
   \( (y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0 \) is integrable or not.

2. Solve \( yz dx = xdy + y^2 dz \)

3. Solve \( (y + z)dx + (z + x)dy + (x + y)dz = 0 \)

4. Solve \( yz(1 + x)dx + xz(1 + y)dy + xy(1 + z)dz = 0 \)

5. Solve \( xdy - ydx - 2x^2zdz = 0 \)

6. Solve \( (x - y)dx - xdy + zdz = 0 \)

7. Solve \( 2yzdx - 3zxdy - 4xydz = 0 \)

8. Solve \( (yz + 2x)dx + (xz + 2y)dy + (xy + 2z)dz = 0 \)

9. Solve \( (y + z)dx + dy + dz = 0 \)

10. Solve \( (x + z)^2dy + y^2(dx + dz) = 0 \)
3.5.1 Solution to SAQ’s :

3.5.1. Give d.e. is \( (y^2 + z^2 - x^2) \, dx - 2xy \, dy - 2xz \, dz = 0 \)

Here \( P = y^2 + z^2 - x^2 \) \( Q = -2xy \) \( R = -2xz \)

\[
\frac{\partial Q}{\partial z} = 0 \quad \frac{\partial R}{\partial y} = 0 \quad 2R \frac{\partial}{\partial x} = -2z
\]

\[
\frac{\partial P}{\partial y} = 2y \quad \frac{\partial P}{\partial z} = 2z \quad \text{and} \quad \frac{\partial Q}{\partial x} = -2y
\]

Now, \( P \left( \frac{\partial Q}{\partial z} - \frac{2R}{2y} \right) + Q \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \)

\[
= P(0 - 0) - 2xy(-2z - 2z) - 2xz(2y + 2y)
\]

\[
= 0 + 8xyz - 8xyz
\]

\[
= 0
\]

\( \therefore \) condition of integrability is satisfied.

\( \therefore \) given d.e. is integrable.

3.5.2. Given d.e. is \( yz \, dx = xz \, dy + y^2 \, dz \)

Regrouping the terms

\[
z(y \, dx - x \, dy) = y^2 \, dz
\]

\[
\Rightarrow \frac{y \, dx - x \, dy}{y^2} = \frac{dz}{z}
\]

Integrating

\[
\int d\left( \frac{x}{y} \right) = \int \frac{dz}{z}
\]

\[
\Rightarrow \frac{x}{y} = \log z + \log c
\]

\[
\Rightarrow \frac{x}{y} = \log cz
\]
which is general solution of given d.e.

3.5.3. \((y + z)dx + (z + x)dy + (x + y)dz = 0\)

\[\Rightarrow ydx + zdx + zdy + xdy + xdz + ydz = 0\]

Regrouping the terms we get

\[\left((ydx + xdy) + (yzd + zdy) + (zdx + xdz) = 0\right)\]

\[\Rightarrow d(xy) + d(yz) + d(zx) = 0\]

Integrating

\[\int d(xy) + \int d(yz) + \int d(zx) = \int 0\]

\[\Rightarrow xy + yz + zx = c \text{ which is general solution of given differential equation.}\]

3.5.4. Given d.e. is \(yz(1 + x)dx + xz(1 + y)dy + xy(1 + z)dz = 0\)

By inspection, Divide by \(xyz\)

\[
\left(\frac{1}{x} + 1\right)dx + \left(\frac{1}{y} + 1\right)dy + \left(\frac{1}{z} + 1\right)dz = 0
\]

Integrating

\[
\int\left(\frac{1}{x} + 1\right)dx + \int\left(\frac{1}{y} + 1\right)dy + \int\left(\frac{1}{z} + 1\right)dz = \int 0
\]

\[\Rightarrow \log x + \log y + \log z = x + y + z = c\]

\[\Rightarrow \log xyz + (x + y + z) = c \text{ is the general solution of given d.e.}\]

3.5.5. Given d.e. is

\(xdy - ydx - 2x^2zdz = 0\)

By Inspection divide by \(x^2\)

\[\Rightarrow \frac{xdy - ydx}{x^2} - 2zdz = 0\]
\[ \Rightarrow d\left(\frac{y}{x}\right) - \int 2z\,dz = \int 0 \]

\[ \Rightarrow \frac{y}{x} - z^2 = c \text{ is the general solution of given d.e.} \]

3.5.6. Given d.e. is \( xdx - ydx - xdy + zdz = 0 \)

\[ \Rightarrow xdx - (ydx + xdy) + zdz = 0 \] (Regrouping the terms)

\[ \Rightarrow xdx - d(xy) + zdz = 0 \]

Integrating

\[ \int xdx - \int d(xy) + \int zdz = \int 0 \]

\[ \Rightarrow \frac{x^2}{2} - xy + \frac{z^2}{2} = c \text{ is the general solution of given d.e.} \]

3.5.7. Given d.e. is \( 2yzdx - 3zxdy - 4xydz = 0 \)

By inspection divide by \( xyz \)

\[ \Rightarrow \frac{2}{x}dx - \frac{3}{y}dy - \frac{4}{z}dz = 0 \]

Integrating

\[ 2\int \frac{1}{x}dx - 3\int \frac{1}{y}dy - 4\int \frac{1}{z}dz = \int 0 \]

\[ \Rightarrow 2\log x - 3\log y - 4\log z = \log c \]

\[ \Rightarrow \log \left[ \frac{x^2}{y^3z^4} \right] = \log c \]

\[ \Rightarrow \frac{x^2}{y^3z^4} = c \]

\[ \Rightarrow x^2 = cy^3z^4 \]

which is the general solution of given d.e.
3.5.8. Given d.e. is \(yzdx + 2xdx + xzdy + 2ydy + xyz + 2zdz = 0\)

\[\Rightarrow d(\text{xyz}) + 2xdx + 2ydy + 2zdz = 0\]

Integrating

\[\Rightarrow \int d(\text{xyz}) + \int 2xdx + \int 2ydy + \int 2zdz = \int 0\]

\[\Rightarrow xyz + x^2 + y^2 + z^2 = c\]

which is the general solution of given d.e.

3.5.9. Given d.e. is

\[(z + y)dx + dy + dz = 0\]

By Inspection

Divide by \(y + z\)

\[\Rightarrow dx + \frac{dy + dz}{y + z} = 0\]

\[\Rightarrow dx + d\left[\log(y + z)\right] = 0\]

Integrating

\[\int dx + \int d\left[\log(y + z)\right] = \int 0\]

\[\Rightarrow x + \log(y + z) = c\]

\[\Rightarrow \log(y + z) = c - x\]

\[\Rightarrow (y + z) = ce^{-x}\]

Which is the general solution of given d.e.

3.5.10. Given d.e. is \((x + z)^2 dy + y^2 (dx + dz) = 0\)

By inspection

divide by \((x + z)^2 y^2\)

\[\Rightarrow \frac{dy}{y^2} + \frac{dx + dz}{(x + z)^2} = 0\]
\[ \Rightarrow \frac{dy}{y^2} + d\left(\frac{-1}{x+z}\right) = 0 \]

Integrating

\[ \int\frac{dy}{y^2} + \int d\left(\frac{-1}{x+z}\right) = \int 0 \]

\[ \Rightarrow \frac{-1}{y} - \frac{1}{x+z} = c \]

\[ \Rightarrow (x+y+z) + cy(x+z) = 0 \]

which is the general solution of given d.e.

### 3.6 SUMMARY

In this lesson we discussed total differential equations and simultaneous total differential equations and related problems.

### 3.7 TECHNICAL TERMS

Total differential equations, simultaneous total differential equations, total derivative, Integrability, Method of grouping, Method of multipliers.

### 3.8 EXERCISE

Solve the following total differential equations if integrable.

1) \((yz - 2x)dx + (xz - 2y)dy + (xy - 2z)dz = 0\)

2) \((yz + 2x)dx + (xz - 2z)dy + (xy - 2y)dz = 0\)

3) \((a-z)(ydx + xdy) + yxdz = 0\)

4) \((x-3y-z)dx + (2y-3x)dy + (z-x)dz = 0\)

5) \(zydx + (x^2y - z)dy + (x^2z - xy)dz = 0\)

6) \(3x^2dx + 3y^2dy - (x^3 + y^3 + c^2z)dz = 0\)

7) \((y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0\)

8) \(xz^3dx - zdy + 2ydz = 0\)
9) \[(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0\]

10) \[2yzdx + 3xdy - xy(1+z)dz = 0\]

11) \[(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0\]

Solve the following simultaneous total differential equations.

12) \[\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}\]

13) \[\frac{dx}{z^2y} = \frac{dy}{z^2x} = \frac{dz}{y^2x}\]

14) \[\frac{xdx}{y^3} = \frac{dy}{x^2z} = \frac{dz}{y^3}\]

15) \[\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{x(yz - 2x)}\]

16) \[\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{x(yz - zx)}\]

17) \[\frac{dx}{x + 4y - 3z} = \frac{dy}{3} = \frac{dz}{4}\]

18) \[\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}\]

19) \[\frac{dx}{y-xz} = \frac{dy}{yz+x} = \frac{dz}{x^2 + y^2}\]

20) \[\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}\]

21) \[\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2 + y^2}\]

22) \[\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z}\]
\[
\frac{dx}{y^2 + yz + z^2} = \frac{dy}{z^2 + zx + x^2} = \frac{dz}{x^2 + xy + y^2}
\]

**ANSWERS TO EXERCISE**

1) \(xyz = c + x^2 + y^2 + z^2\)
2) \(xyz = c + 2yz - x^2\)
3) \(xy = c(z - a)\)
4) \(x^2 + 2y^2 + z^2 - 6xy - 2xz = c\)
5) \(x(y^2 + z^2 - 2c) = 2yz\)
6) \(x^3 + y^3 = e^{2x} + ce^z\)
7) \(y(x + z) = c(y + z)\)
8) \(2y = x^2z^2 + 2cz^2\)
9) \(x^2 + y^2 + z^2 = cx\)
10) \(x^2y = cze^z\)
11) \(x^2z + y^3 = cx\)
12) \(x - y = c_1xy; y - z = c_2yz\)
13) \(x^2 - y^2 = c_1; y^3 - z^3 = c_2\)
14) \(x^4 - y^4 = c_1; x^2 - y^2 = c_2\)
15) \(x + y = c_1; e^{2x} = (c_1^2 + z^2)c_2\)
16) \(x = c_1y; e^x = (z - 2c_1)c_2\)
17) \(4y - 3z = c_1; (x + 4y - 3z)^3 = e^yc_2\)
18) \(x^2 + y^2 + z^2 = c_1; xyz = c_2\)
19) \( xy - z = c_1; \ x^2 - y^2 + z^2 = c_2 \)

20) \( x^2 + y^2 + z^2 = c_1; \ yz = c_2 x \)

21) \( x^2 - y^2 - z^2 = c_1; \ 2xy - z^2 = c_2 \)

22) \( x + y = c_1 z; \ 2y = c_2 (x^2 - y^2) \)

23) \( (y - x) = c_1 (z - x); \ (y - x) = c_2 (z - y) \)

### 3.10 MODEL QUESTIONS

1. Solve the total differential equation \( x \, dx + z \, dy + (y + 2z) \, dz = 0 \) if integrable, solve the following simultaneous total differential equations.

2. \( \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \)

3. \( \frac{dx}{x} = \frac{dy}{z} = \frac{dz}{y} \)

4. \( \frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{2x - 3y} \)

5. \( \frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y} \)

6. \( \frac{dx}{mz - ny} = \frac{dy}{nx - \ell z} = \frac{dz}{\ell y - mx} \)

### 3.11 REFERENCE BOOKS


2) Vol. II - S.Chand - A Text Book of Mathematics

3) Differential Equations by N.Ch.S.N. Iyengar

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Lesson - 4

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

4.1 OBJECTIVE OF THE LESSON

After studying this lesson, the student will be in a position to know about differential equations of first order and higher degree and how to solve them.

4.2 STRUCTURE OF THE LESSON

This lesson has the following components.

4.3 Introduction
4.4 Solvable for p
4.5 Solvable for y
4.6 Solvable for x
4.7 Clairaut's Equation
4.8 Answers to Self Assessment Questions
4.9 Summary
4.10 Technical Terms
4.11 Exercises
4.12 Answers to Exercises
4.13 Model Examination Questions
4.14 Reference Books

4.3.1 INTRODUCTION

In this lesson, for convenience we denote \( \frac{dy}{dx} \) by \( p \). An equation of the form \( f(x, y, p) = 0 \), which is not of first degree, is called a differential equation of first order and higher degree, the general form of first order and \( n \)th degree differential equation is

\[
p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots + A_{n-1} p + A_n = 0 \quad (n > 1)
\]

where \( A_1, A_2, \cdots, A_n \) are functions in \( x \) and \( y \). Now we shall discuss the solution of the
above differential equation in the following four contents.

(i) Solvable for $p$
(ii) Solvable for $y$
(iii) Solvable for $x$
(iv) Clairauts Equation

4.4 SOLVABLE FOR $p$

Let $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_{n-1} p + A_n = 0 \ (n > 1)$ \hspace{1cm} (1) be given differential equation of first order and $n$th degree. If it can be solved for $p$ then (1) can be resolved into $n$ linear factors in $p$.

They may be written in the form

$$\left[ p - f_1(x, y) \right] \left[ p - f_2(x, y) \right] \ldots \left[ p - f_n(x, y) \right] = 0$$

$$\Rightarrow p - f_1(x, y) = 0, \ p - f_2(x, y) = 0 \ldots \ p - f_n(x, y) = 0$$

$$\Rightarrow \frac{dy}{dx} = f_1(x, y) \frac{dy}{dx} = f_2(x, y) \ldots \frac{dy}{dx} = f_n(x, y)$$

Solving each of $n$ differential equations, we get $n$ solutions. Let them be

$$F_1(x, y, c_1) = 0, \ F_2(x, y, c_2) = 0 \ldots \ F_n(x, y, c_n) = 0$$

:. the solution of (1) is

$$F_1(x, y, c_1) \times F_2(x, y, c_2) \times \ldots \times F_n(x, y, c_n) = 0$$

But (1) is of first order differential equation.

:. Number of arbitrary constants in the general solution of (1) is only one.

:. General solution of (1) is

$$F_1(x, y, c) \times F_2(x, y, c) \times \ldots \times F_n(x, y, c) = 0 \hspace{1cm} \text{by taking} \hspace{0.5cm} c_1 = c_2 = \ldots = c_n = c$$

4.5.1 Solvable for $y$ : Let $f(x, y, p) = 0$ \hspace{1cm} (1) be given differential equation. If (1) cannot be resolved into linear factors in '$p$' and if it can be put in the form $y = F(x, p)$ \hspace{1cm} (2) then we say that (1) can be solved for $y$.

Differentiating (2) w.r.t. $x$ we get a differential equation into two variables $x$ and $p$.

Let the solution be $\phi(x, p, c) = 0$ \hspace{1cm} (3)
Now eliminating $p$ from (2) and (3), we get the required general solution of (1) in the form

$$\psi(x, y, c) = 0$$

where $c$ is arbitrary constant.

4.5.2 Note:

1. If it is not possible to eliminate $p$ from (2) and (3) then general solution of (1) is given by

$$f(x, y, p) = 0, \quad \psi(x, p, c) = 0.$$

2. The general solution of (1) can also be expressed in the form $x = f_1(p, c), y = f_2(p, c)$ with parameter $p$ which is called a solution in the parametric form.

3. The solution which does not contain an arbitrary constant is called singular solution (see the definition in lesson 1).

4. If the given differential equation does not contain $x$ i.e. in the form of $f(y, p) = 0$ and if it is solvable for $p$ we get $p = \phi(y)$ which can be solved by using variables separable method. If it is solvable for $y$' it can be written in the form which can also be solved by using variables separable method.

5. If given differential equation is homogeneous in $x$ and $y$ then it can be written in the form $f\left(p, \frac{y}{x}\right) = 0$, this equation can be solved using method of solving homogeneous equation.

4.6.1 Solvable for 'x'

Let $f(x, y, p) = 0$ ------- (1) be given differential equation. If it can be put in the form of $x = F(y, p)$ ------- (2). Then we say that (1) can be solved for $x$.

Differentiating (2) w.r.t. 'y' gives a differential equation in two variables $y$ and $p$. This can be written as

$$\frac{1}{p} = g\left(y, p, \frac{dp}{dy}\right).$$

Let the solution be $\psi(y, p, c) = 0$ ------- (3)

Now eliminating $p$ from (2) and (3) we get the required general solution of (1) in the form

$$\psi(x, y, c) = 0$$

where $c$ is an arbitrary constant.

4.6.2 Note: If it is not possible to eliminate $p$ from (2) and (3) then together represent the general solution of (1).

Suppose that the given differential equation does not contain 'y' i.e. in the form $f(x, p) = 0$.

If it is solvable for 'p' we may written as $p = \phi(x)$ which can be solved by variables separable method. If it is solvable for $x$ we may written as $x = \psi(p)$ which can also be solved as explained above.
4.7.1 Clairaut's Equation: A differential equation of the form \( y = px + \phi(p) \) is called Clairaut's equation.

Clairaut's equation is solvable for 'y'.

Consider \( y = px + \phi(p) \) \( \text{(1)} \)

Differentiating w.r.t. 'x'

\[
\frac{dy}{dx} = \frac{dp}{dx} x + p + \phi'(p) \frac{dp}{dx}
\]

\( \Rightarrow \frac{dp}{dx} (x + \phi'(p)) = 0 \)

\( \Rightarrow \frac{dp}{dx} = 0 \) \( \text{(2)} \)

\( x + \phi'(p) = 0 \) \( \text{(3)} \)

(3) is discarded as it gives singular solution. To find general solution solving (2).

\[
\frac{dp}{dx} = 0
\]

\( \Rightarrow dp = 0 \)

Integrating

\[
\int dp = \int 0
\]

\( \Rightarrow p = c \) where \( c \) is arbitrary constant.

\( \therefore \) eliminating 'p' from (1)

general solution of (1) is \( y = cx + \phi(c) \)

4.7.2 Note: The general solution of the clairauts equation \( y = px + \phi(p) \) will be obtained by replacing \( p \) with \( c \).

4.7.3 Solved Problems: Solve \( p^2 + 2pcotx = y^2 \)

Solution: Given \( p^2 + 2pcotx = y^2 \) \( \text{(1)} \)

This is quadratic in \( p \)

\[
p = \frac{-2pcotx \pm \sqrt{4y^2 cot^2x + 4y^2}}{2}
\]
\[ p = -y \cot x \pm y \sqrt{\cot^2 x + 1} \]
\[ \frac{dy}{dx} = y(-\cot x \pm \csc x) \]

Solving \( \frac{dy}{y} = (-\cot x \pm \csc x) \, dx \)

Integrating
\[ \int \frac{dy}{y} = -\int \cot x \, dx \pm \int \csc x \, dx \]
\[ \Rightarrow \log c + \log y = -\log \sin x \pm \log \tan \frac{x}{2} \]
\[ \Rightarrow \log |cy \sin x| = \log |\tan \frac{x}{2}| \]
\[ \Rightarrow |cy \sin x| = |\tan \frac{x}{2}| \]
\[ \Rightarrow c^2 y^2 \sin^2 x - \tan^2 \frac{x}{2} = 0 \text{ which is the general solution of (1)} \]

4.7.4: Solve \( xy(p^2 + 1) = (x^2 + y^2)p \)

**Solution**: Given d.e. is \( xyp^2 - x^2p - y^2p + xy = 0 \) ------- (1)

\[ \Rightarrow xp(yp - x) - y(yp - x) = 0 \]
\[ \Rightarrow (yp - x)(xp - y) = 0 \]

\[ \Rightarrow yp - x = 0; \quad xp - y = 0 \]
\[ \Rightarrow y \frac{dy}{dx} = x \]
\[ \Rightarrow \frac{dy}{dx} = \frac{y}{x} \]
Solving
\[ ydy = xdx \]
Integrating
\[ \int y \, dy = \int x \, dx \]
\[ \frac{y^2}{2} = \frac{x^2}{2} + c \]
\[ \Rightarrow y = cx \]
General solution of (1) is \( (y^2 - x^2 - c)(y - cx) = 0 \)

4.7.5: Solve \( xy^2 + (x^2 + xy + y^2)p + (x^2 + xy) = 0 \)

Solution: Given d.e. is

\[ xy^2 + x^2p + xyp + x^2 + y^2p + xy = 0 \quad \text{(1)} \]

\[ \Rightarrow xp(yp + x) + x(yp + x) + y(yp + x) = 0 \]

\[ \Rightarrow (yp + x)(xp + x + y) = 0 \]

\[ \Rightarrow yp + x = 0; \quad xp + x + y = 0 \]

\[ \frac{dy}{dx} + y + x = 0 \]

\[ \Rightarrow \frac{dy}{dx} = -x \]

\[ \Rightarrow ydy = -xdx \]

Integrating

\[ \int ydy = - \int xdx \]

\[ \Rightarrow \frac{y^2}{2} = - \frac{x^2}{2} + \frac{c}{2} \]

\[ \Rightarrow \left( y^2 + x^2 - c \right) = 0 \]

\[ \therefore \text{General solution of (1) is } \left( y^2 + x^2 - c \right)\left( 2xy + x^2 - c \right) = 0 \]

4.7.6: Solve \( p^3 + (2x - y^2)p^2 = 2xy^2p \)

Solution: Given d.e. is

\[ p^3 + (2x - y^2)p^2 - 2xy^2p = 0 \quad \text{(1)} \]

\[ \Rightarrow p[p^2 + 2xp - y^2p - 2xy^2] = 0 \]
\[ p\left[p + 2x\right] - y^2 (p + 2x) = 0 \]

\[ p(p + 2x)(p - y^2) = 0 \]

\[ p = 0; \quad p + 2x = 0; \quad p - y^2 = 0 \]

Suppose \( p = 0 \)

\[ \Rightarrow \frac{dy}{dx} = 0 \Rightarrow dy = 0 \]

Integrating we get \( y = c \)

\[ \Rightarrow y - c = 0 \]

Suppose \( p + 2x = 0 \)

Hence \( \frac{dy}{dx} + 2x = 0 \)

\[ \Rightarrow dy + 2xdx = 0 \]

Integrating

\[ \Rightarrow \int dy + \int 2xdx = \int 0 \]

\[ \Rightarrow \left( y + x^2 - c \right) = 0 \]

Suppose \( p - y^2 = 0 \)

Hence \( \frac{dy}{dx} - y^2 = 0 \)

\[ \Rightarrow \frac{dy}{y^2} = dx \]

Integrating

\[ \int y^{-2}dy = \int dx \]

\[ \Rightarrow \frac{-1}{y} = x + c \Rightarrow (xy + cy + 1) = 0 \]

\[ \therefore \text{General solution of (1) is } (y - c)(y + x^2 - c)(xy + cy + 1) = 0 \]
4.7.7: Solve $xy^2(p^2 + 2) = 2py^3 + x^3$

**Solution:** Given d.e. is $xy^2p^2 - 2py^3 + 2xy^2 - x^3 = 0$ \(-------- (1)\)

This is a quadratic equation in $p$.

\[
\therefore p = \frac{2y^3 \pm \sqrt{4y^6 - 4 \cdot xy^2 (2xy^2 - x^3)}}{2xy^2}
\]

\[
\Rightarrow p = \frac{2y^3 \pm 2y\sqrt{y^4 - 2x^2 y^2 + x^4}}{2xy^2}
\]

\[
\Rightarrow p = \frac{y^2 \pm \sqrt{(y^2 - x^2)^2}}{xy}
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{y^2 \pm (y^2 - x^2)}{xy}
\]

Now suppose \(\frac{dy}{dx} = \frac{2y^2 - x^2}{xy}\)

This is a homogeneous differential equation

put \(\frac{y}{x} = v\)

Differentiating w.r.t. (x)

\[
\frac{dy}{dx} = v + \frac{x}{dx}\frac{dv}{dx}
\]

Now \(v + x\frac{dv}{dx} = \frac{2v^2 - 1}{v}\)

\[
x \frac{dv}{dx} = \frac{2v^2 - 1 - v^2}{v}
\]

\[
\Rightarrow \frac{vdv}{v^2 - 1} = \frac{dx}{x}
\]
Differential Equations of First Order and Higher Degree

4.9

Integrating

\[ \frac{1}{2} \int \frac{2v}{v^2 - 1} dv = \int \frac{dx}{x} \]

\[ \Rightarrow \log (v^2 - 1) = 2\log x + \log c \]

\[ \Rightarrow v^2 - 1 = x^2c \Rightarrow y^2 - x^2 - x^4c = 0 \]

Now suppose

\[ \frac{dy}{dx} = \frac{x^2}{xy} \]

\[ ydy = xdx \]

Integrating

\[ \int ydy = \int xdx \]

\[ \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2} \]

\[ \Rightarrow (y^2 - x^2 - c) = 0 \]

\[ \therefore \text{General solution of (1) is} \]

\[ (y^2 - x^2 - x^4c)(y^2 - x^2 - c) = 0 \]

4.7.8: Solve \( xp^2 - 2yp + x = 0 \)

Solution: Given d.e. is \( xp^2 - 2yp + x = 0 \) \( \cdots \cdots \) (1)

This is a quadratic equation in p

\[ \therefore p = \frac{2y \pm \sqrt{4y^2 - 4x^2}}{2x} \]

\[ \frac{dy}{dx} = \frac{y \pm \sqrt{y^2 - x^2}}{x} \]

This is homogeneous d.e.

Solving put \( \frac{y}{x} = v \)
Differentiate w.r.t. 'x' 

\[\frac{dy}{dx} = v + x \frac{dv}{dx}\]

\[v + x \frac{dv}{dx} = v \pm \sqrt{v^2 - 1}\]

\[\Rightarrow x \frac{dv}{dx} = \pm \sqrt{v^2 - 1}\]

Integrating

\[\int \frac{dv}{\sqrt{v^2 - 1}} = \pm \int \frac{dx}{x}\]

\[\log(v + \sqrt{v^2 - 1}) = \pm \log x + \log c\]

\[\Rightarrow (v + \sqrt{v^2 - 1})x = c; \quad \frac{v + \sqrt{v^2 - 1}}{x} = c\]

\[\Rightarrow y + \sqrt{y^2 - x^2} - c = 0; \quad y + \sqrt{y^2 - x^2} - cx^2 = 0\]

General solution of (1) is \[(y + \sqrt{y^2 - x^2} - c)(y + \sqrt{y^2 - x^2} - cx^2) = 0\]

4.7.9 : Solve \(px + y - p^2x^4 = 0\)

Solution : This is solvable for \(y\)

Given d.e. is \(y = p^2x^4 - px \quad \text{(1)}\)

Differentiate w.r.t. 'x'

\[\frac{dy}{dx} = \left[2p \frac{dp}{dx}x^4 + p^2 \cdot 4x^3\right] - \left[\frac{dp}{dx}x + p\right]\]

\[\Rightarrow p - 4p^2x^3 + p + x \frac{dp}{dx} - 2px^4 \frac{dp}{dx} = 0\]

\[\Rightarrow 2p(1 - 2px^3) + x \frac{dp}{dx}(1 - 2px^3) = 0\]
\[ (1 - 2px^3) \left( 2p + x \frac{dp}{dx} \right) = 0 \]

\[ \Rightarrow l - 2px^3 = 0 \quad (2) \quad 2p + x \frac{dp}{dx} = 0 \quad (3) \]

Here (2) is discarded as it does not contain \( \frac{dp}{dx} \) and it gives singular solution.

From (3) we get \( 2p = -x \frac{dp}{dx} \)

\[ \Rightarrow \frac{2dx}{x} = \frac{-dp}{p} \]

Integrating

\[ 2 \int \frac{dx}{x} = - \int \frac{dp}{p} \]

\[ 2 \log x + \log p = \log c \Rightarrow p = \frac{c}{x^2} \quad (4) \]

Eliminating \( p \) from (1) and (4) general solution of (1) is

\[ y = c^2 x^2 - \frac{c}{x}. \]

\( c \) is arbitrary constant.

4.7.10: Solve \( xp^2 - 2yp + x = 0 \)

**Solution**: Given d.e. is \( 2yp = xp^2 + x \quad (1) \)

\[ \Rightarrow y = \frac{1}{2} \left[ \frac{x p^2 + x}{p} \right] \]

Differentiating w.r.t. 'x' \[ \frac{dy}{dx} = \frac{1}{2} \left[ p + x \frac{dp}{dx} + \frac{p - x \frac{dp}{dx}}{p^2} \right] \]

Multiply with \( 2p^2 \)
\[2p^3 = p^2 \left( p + x \frac{dp}{dx} \right) + p - x \frac{dp}{dx}\]

\[x \frac{dp}{dx} (p^2 - 1) - p(p^2 - 1) = 0\]

\[(p^2 - 1) \left( x \frac{dp}{dx} - p \right) = 0\]

\[p^2 - 1 = 0 \quad (2) : \quad x \frac{dp}{dx} - p = 0 \quad (3)\]

(2) is discarded as it gives singular solution.

From (3) we get

\[x \frac{dp}{dx} = p\]

\[\Rightarrow \frac{dp}{p} = \frac{dx}{x}\]

Integrating \[\int \frac{dp}{p} = \int \frac{dx}{x}\]

\[\Rightarrow \log p = \log x + \log c \Rightarrow p = cx \quad (4)\]

Eliminating p from (1) and (4)

General solution of (1) is

\[2ycx = x (cx)^2 + x\]

i.e. \[2cxy = c^2 x^3 + x\]

4.7.11 : \[xp^3 - 2yp^2 + 4x^2 = 0\]

**Solution** : Given d.e. is \[2yp^2 = xp^3 + 4x^2 \quad (1)\]

\[\Rightarrow y = \frac{1}{2} \left[ xp + 4 \frac{x^2}{p^2} \right]\]

Differentiating w.r.t. x
Differential Equations of First Order and Higher Degree

4.13

Differential Equation, Abstract Algebra...

\begin{align*}
\frac{dy}{dx} &= \frac{1}{2} \left[ p + x \frac{dp}{dx} + 4 \left( 2xp^2 - x^2 \frac{dp}{dx} \right) \right] \\
\end{align*}

Multiply by 2p^4

\begin{align*}
\Rightarrow 2p^5 &= p^4 \left( p + x \frac{dp}{dx} \right) + 4 \left( 2xp^2 - 2x^2 p \frac{dp}{dx} \right) \\
\Rightarrow xp \frac{dp}{dx} (p^3 - 8x) - p^2 (p^3 - 8x) &= 0 \\
\Rightarrow xp \frac{dp}{dx} (p^3 - 8x) - p^2 (p^3 - 8x) &= 0 \\
\Rightarrow (p^3 - 8x) \left( xp \frac{dp}{dx} - p^2 \right) &= 0 \\
\Rightarrow p^3 - 8x &= 0 \quad \text{(2)} \\
\Rightarrow xp \frac{dp}{dx} - p^2 &= 0 \quad \text{(3)}
\end{align*}

Here (2) is discarded as it gives singular solution

Solving (3) \( xp \frac{dp}{dx} = p^2 \)

\begin{align*}
\Rightarrow \frac{dp}{p} &= \frac{dx}{x} \\
\end{align*}

Integrating

\begin{align*}
\int \frac{dp}{p} &= \int \frac{dx}{x} \\
\Rightarrow \log p &= \log x + \log c \\
\Rightarrow p &= cx \quad \text{(4)}
\end{align*}

Eliminating \( p \) from (1) and (4)

\begin{align*}
2y(cx)^2 &= x(cx)^3 + 4x^2 \\
\Rightarrow 2c^2 y &= c^3 x^2 + 4 \quad \text{which is general solution of (1)}
\end{align*}
4.7.12 : Solve \( (8p^3 - 27)x = 12p^2y \)

**Solution** : Given d.e. is

\[
y = \frac{(8p^3 - 27)x}{12p^2} \quad \text{-------- (1)}
\]

\[
\Rightarrow y = \frac{2px}{3} - \frac{9x}{4p^2}
\]

Differentiating w.r.t. \( x \)

\[
\frac{dy}{dx} = 2\left[ p + x \frac{dp}{dx} \right] - \frac{9}{4} \left[ \frac{p^2 - 2xp \frac{dp}{dx}}{p^4} \right]
\]

Multiply by 12p^4 \( \Rightarrow 12p^5 = 8p^4 \left( p + x \frac{dp}{dx} \right) - 27 \left( p^2 - 2xp \frac{dp}{dx} \right) \)

\[
\Rightarrow 2xp \frac{dp}{dx} \left( 4p^3 + 27 \right) - p^2 \left( 4p^3 + 27 \right) = 0
\]

\[
\Rightarrow (4p^3 + 27) \left( 2xp \frac{dp}{dx} - p^2 \right) = 0
\]

\[
\Rightarrow 4p^3 + 27 = 0 \quad \text{-------- (2)} \quad 2xp \frac{dp}{dx} - p^2 = 0 \quad \text{-------- (3)}
\]

(2) is discarded as it gives singular solution solving (3) \( 2xp \frac{dp}{dx} = p^2 \)

Separating the variables and integrating

\[
2 \int \frac{dp}{p} = \int \frac{dx}{x}
\]

\[
2\log p = \log x + \log c
\]

\[
\Rightarrow p^2 = cx \quad \text{-------- (4)}
\]

Eliminating \( p \) from (1) & (4)

\[
y = \frac{2\sqrt{c}}{3} x^{3/2} - \frac{9}{4c} \quad \text{which is general solution of (1)}
\]
4.7.13: \( y = x p^2 + p \)

**Solution:** Given d.e. is

\[
y = x p^2 + p \quad \text{(1)}
\]

Differentiating with respect to 'x',

\[
\frac{dy}{dx} = p^2 + 2xp \frac{dp}{dx} + \frac{dp}{dx}
\]

\[
\Rightarrow (2xp + 1) \frac{dp}{dx} + p^2 - p = 0
\]

\[
\Rightarrow (p^2 - p) \frac{dx}{dp} + 2xp = -1 \quad \text{(2)}
\]

This is linear d.e. in x.

\[
J.F. = e^{\int \frac{2dp}{p-1}} = e^{2 \log(p-1)} = (p - 1)^2
\]

General solution of (2) is

\[
x(p-1)^2 = \int \frac{-(p-1)^2}{p(p-1)} dp
\]

\[
\Rightarrow x(p-1)^2 = \int \left(-1 + \frac{1}{p}\right) dp
\]

\[
\Rightarrow x(p-1)^2 = -p + \log p + \log c \Rightarrow e^{x(p-1)^2} + p = pc \quad \text{(3)}
\]

\[
\therefore \ (1) \text{ and } (3) \text{ together represent the general solution of (1)}.
\]

i.e. the solutions given by \( y = x p^2 + p \) and \( e^{x(p-1)^2} + p = pc \)

4.7.14: Solve \( e^y = p^3 + p \)

**Solution:** Given d.e. is \( e^y = p^3 + p \quad \text{(1)} \)

Differentiate w.r.t. \( x \)

\[
e^y \frac{dy}{dx} = (3p^2 + 1) \frac{dp}{dx}
\]
\[ (p^3 + p)p = (3p^2 + 1) \frac{dp}{dx} \]

\[ \Rightarrow \ dx = \frac{3p^2 + 1}{p^2(p^2 + 1)} \ dp \]

Integrating

\[ \int dx = \int \left[ \frac{3}{p^2+1} + \frac{1}{p^2(p^2 + 1)} \right] dp \]

\[ \Rightarrow x = 3 \int \frac{dp}{p^2 + 1} + \int \left( \frac{p^2+1}{p^2(p^2 + 1)} - \frac{p^2}{p^2 + 1} \right) dp \]

\[ \Rightarrow x = 3 \tan^{-1}p + \int \frac{dp}{p^2} - \int \frac{dp}{p^2 + 1} \]

\[ \Rightarrow x = 2 \tan^{-1}p - \frac{1}{p} + c \]

\[ \therefore e^y = p^3 + p \text{ and } x = 2 \tan^{-1}p - \frac{1}{p} + c \text{ together represent general solution of (1)} \]

4.7.15: Solve \( y = yp^2 + 2px \)

**Solution:** Given d.e. is \( y(1-p^2) = 2px \)

i.e. \( y = \frac{2px}{1-p^2} \quad ----- (1) \)

Differentiate w.r.t. 'x'

\[ \frac{dy}{dx} = 2 \left[ \frac{(p+x \frac{dp}{dx})(1-p^2) + 2p \frac{dp}{dx} px}{(1-p^2)^2} \right] \]

\[ \Rightarrow (1-p^2)p = 2(1-p^2)p + 2x(1-p^2) \frac{dp}{dx} + 4xp^2 \frac{dp}{dx} \]
\[ 2x \frac{dp}{dx} (2p^2 + 1 - p^2) + p \left(1 - p^2\right)(2 - 1 + p^2) = 0 \]

\[ \Rightarrow \left(p^2 + 1\right) \left(2x \frac{dp}{dx} + p \left(1 - p^2\right)\right) = 0 \]

\[ \Rightarrow \left(p^2 + 1\right) = 0 \quad 2x \frac{dp}{dx} + p \left(1 - p^2\right) = 0 \quad \text{--------- (3)} \]

As we want general solution we ignore (2)

solving (3)

\[ 2x \frac{dp}{dx} = p \left(1 - p^2\right) \]

\[ \Rightarrow \frac{2dp}{p(1 - p^2)} = \frac{dx}{x} \]

Integrating

\[ 2 \int \frac{dp}{p(1 - p^2)} = \int \frac{dx}{x} \]

\[ \Rightarrow 2 \int \frac{1 - p^2 + p^2}{p(1 - p^2)} \, dp = \log x + \log c \]

\[ \Rightarrow 2 \int \frac{dp}{p} + 2 \int \frac{p \, dp}{1 - p^2} = \log cx \]

\[ \Rightarrow 2 \log p - \log \left(1 - p^2\right) = \log cx \]

\[ \Rightarrow \frac{p^2}{1 - p^2} = cx \]

\[ \therefore \text{ General solution of (1) is given by } y = \frac{2px}{1 - p^2} \text{ and } \frac{p^2}{1 - p^2} = cx \]

4.7.16 : Solve \( y = a + bp + cp^2 \)

\[ \text{Solution : Given d.e. } y = a + bp + cp^2 \quad \text{--------- (1)} \]
Differentiate w.r.t. $x$

\[
\frac{dy}{dx} = b \frac{dp}{dx} + 2cp \frac{dp}{dx}
\]

\[
\Rightarrow (b + 2cp) \frac{dp}{dx} - p = 0
\]

\[
\Rightarrow (b + 2cp) \frac{dp}{dx} = p
\]

Integrating

\[
\int \frac{(b + 2cp)}{p} \frac{dp}{dx} = \int dx
\]

\[
\Rightarrow b \log p + 2cp = x + c
\]

\[
\Rightarrow x = b \log p + 2cp - c
\]

\[
\therefore \text{ General solution of (1) is given by } y = a + bp + cp^2 \text{ and } x = b \log p + 2cp - c.
\]

4.7.17: Solve $p^3 - 4xyp + 8y^2 = 0$

**Solution:** Given d.e. is $p^3 - 4xyp + 8y^2 = 0$

This is an equation which is solvable for ' $x$ '.

\[
\Rightarrow x = \frac{1}{4} \left[ \frac{p^2}{y} + \frac{8y}{p} \right]
\]

Differentiate w.r.t. $y$

\[
\frac{dx}{dy} = \frac{1}{4} \left[ \frac{2p \frac{dp}{dy} - p^2}{y^2} + 8 \frac{p - y \frac{dp}{dy}}{p^2} \right]
\]

Multiply with $4y^2p^2$

\[
\Rightarrow 4y^2p = p^2 \left( 2py \frac{dp}{dy} - p^2 \right) + 8y^2 \left( p - y \frac{dp}{dy} \right)
\]

\[
\Rightarrow 2y \frac{dp}{dy} \left( p^3 - 4y^2 \right) + p \left( 4y^2 - p^3 \right) = 0
\]
\[ 2y \frac{dp}{dy} (p^3 - 4y^2) + p (4y^2 - py^3) = 0 \]

\[ \Rightarrow (p^3 - 4y^2) \left( 2y \frac{dp}{dy} - p \right) = 0 \]

\[ \Rightarrow p^3 - 4y^2 = 0 \quad \text{(2)} \quad 2y \frac{dp}{dy} - p = 0 \quad \text{(3)} \]

Here (2) is discarded as it does not contain \( \frac{dp}{dy} \) and it gives singular solution.

To find general solution, consider (3)

\[ 2y \frac{dp}{dy} = p \]

Separating the variables and integrating

\[ 2 \int \frac{dp}{p} = \int \frac{dy}{y} \]

\[ \Rightarrow 2 \log p = \log y + \log c \]

\[ \Rightarrow p^2 = cy \quad \text{(4)} \]

Eliminating p from (1) and (4)

\[ \therefore \text{General solution of (1) is } 4xy \sqrt{cy} = cy \sqrt{cy} + 8y^2 \]

\[ \Rightarrow 4\sqrt{c} (x - c) = 8\sqrt{y} \]

squaring we get \( c(x - c)^2 = 4y \)

4.7.18: Solve \( p^3 - (y + 3)p + x = 0 \)

Solution: Given d.e. is \( x = (y + 3)p - p^3 \quad \text{(1)} \)

We solve this equation for \( x \)

Differentiate (1) w.r.t. \( 'y' \)

\[ \frac{dx}{dy} = p + (y + 3) \frac{dp}{dy} - 3p^2 \frac{dp}{dy} \]
Multiply with \( p \)

\[
1 = p^2 + p(y + 3) \frac{dp}{dy} - 3p^3 \frac{dp}{dy}
\]

\[
\Rightarrow p \frac{dp}{dy} (y + 3 - 3p^2) + p^2 - 1 = 0
\]

\[
\Rightarrow p(y + 3 - 3p^2) + (p^2 - 1) \frac{dy}{dp} = 0
\]

\[
\Rightarrow \frac{dy}{dp} + \frac{p}{p^2 - 1} y = \frac{3p^3 - 3p}{p^2 - 1}
\]

\[
\Rightarrow \frac{dy}{dp} + \frac{p}{p^2 - 1} y = 3p \quad \cdots (2)
\]

This is linear d.e. in \( y \).

The I.F. is

\[
I.F. = e^{\int \frac{dp}{p^2 - 1}} = e^{\frac{\log \sqrt{p^2 - 1}}{p^2}} = \sqrt{p^2 - 1}
\]

\[
\therefore \text{ solution of (2) is } y \sqrt{p^2 - 1} = \int 3p \sqrt{p^2 - 1} dp
\]

\[
= 3 \int q^2 dq \quad [\text{Integral is evaluated by the method of substitution and substitute } p^2 - 1 = q^2]
\]

\[
= q^3 + c
\]

\[
\Rightarrow y \sqrt{p^2 - 1} = (p^2 - 1)^{3/2} + c
\]

\[
\Rightarrow y = (p^2 - 1) + c(p^2 - 1)^{-1/2}
\]

\[
\therefore \text{ General solution of (1) is } x = (y + 3)p - p^3 \text{ and } y = (p^2 - 1) + c(p^2 - 1)^{-1/2}
\]

**4.7.19:** Solve \((x - \tan^{-1} p)(1 + p^2) = p\)

**Solution:** Given d.e. is
x - \tan^{-1} \frac{p}{1 + p^2}

\Rightarrow x = \tan^{-1} \frac{p}{1 + p^2} \quad \text{-------- (1)}

Differentiate w.r.t 'y'

\frac{dx}{dy} = \frac{1}{1 + p^2} \frac{dp}{dy} \left( \frac{1 + p^2}{1 + p^2} \right) - 2p^2 \frac{dp}{dy} \left( \frac{1 + p^2}{1 + p^2} \right) - \frac{dp}{dy} \left( \frac{1 + p^2}{1 + p^2} \right)

Multiply by \ p(1 + p^2)^2

\Rightarrow (1 + p^2)^2 = p(1 + p^2) \frac{dp}{dy} + p(1 + p^2 - 2p^2) \frac{dp}{dy}

\Rightarrow 2p \frac{dp}{dy} = (1 + p^2)^2

Separating the variables and integrating

\int \frac{2pdp}{(1 + p^2)^2} = \int dy

\Rightarrow c - (1 + p^2)^{-1} = y

\therefore \text{ General solution of (1) is } x = \tan^{-1} \frac{p}{1 + p^2} \text{ and } y = c - (p^2 + 1)^{-1}

4.7.20: Solve \ 4xp^2 + 4yp - y^4 = 0

Solution: Given d.e. is

\begin{align*}
4p^2x &= y^4 - 4yp \\
x &= \frac{1}{4} \left[ \frac{y^4}{p^2} - \frac{4y}{p} \right] \\
\end{align*}
Differentiate w.r.t. 'y'

\[
\frac{dx}{dy} = \frac{1}{4} \left[ \frac{4y^3 p^2 - 2p \frac{dp}{dy} y^4}{p^4} - \frac{4p - 4y \frac{dp}{dy}}{p^2} \right]
\]

Multiply by \(4p^4\).

\[
4p^3 = 4y^3 p^2 - 2py^4 \frac{dp}{dy} - p^2 \left(4p - 4y \frac{dp}{dy}\right)
\]

\[
\Rightarrow 2yp \frac{dp}{dy} \left(2p - y^3\right) - 4p^2 \left(2p - y^3\right) = 0
\]

\[
\Rightarrow \left(2p - y^3\right) \left(2yp \frac{dp}{dy} - 4p^2\right) = 0
\]

\[
\Rightarrow 2p - y^3 = 0 \quad \text{-------- (4)} \quad 2yp \frac{dp}{dy} - 4p^2 = 0 \quad \text{-------- (3)}
\]

Solving (3)

\[
\frac{dp}{p} = \frac{2dy}{y}
\]

Integrating \(\int \frac{dp}{p} = 2 \int \frac{dy}{y}\)

\(\Rightarrow \log p = 2 \log y + \log c\)

\(\Rightarrow p = cy^2 \quad \text{-------- (4)}\)

Eliminating \(p\) from (1) and (4)

\[
4x \left(cy^2\right)^2 + 4y \left(cy^2\right) - y^4 = 0
\]

\(\Rightarrow 4c(1 + cxy) - y = 0\) which is general solution of (1)

4.7.21: \(xp^3 - bp = a\)

Solution: Given d.e. is

\(xp^3 = a + bp\)
\[ \Rightarrow x = \frac{a}{p^3} + \frac{b}{p^2} \quad \text{(1)} \]

Differentiate w.r.t. 'y'

\[ \frac{dx}{dy} = \left[ \frac{-3a}{p^4} - \frac{2b}{p^3} \right] \frac{dp}{dy} \]

Multiply by \( p \)

\[ 1 = \left( \frac{-3a}{p^3} - \frac{2b}{p^2} \right) \frac{dp}{dy} \]

Separating the variables and integrating

\[ \int dy = -3a \int p^{-3} dp - 2b \int p^{-2} dp \]

\[ \Rightarrow y = \frac{3a}{2p^2} + \frac{2b}{p} + c \]

\[ \therefore \text{General solution of (1) is } x = \frac{a}{p^3} + \frac{b}{p^2} \text{ and } y = \frac{3a}{2p^2} + \frac{2b}{p} + c \]

**4.7.22 :** Solve \((px - y)(py + x) = 2p\)

**Solution :** Given d.e. is

\((px - y)(py + x) = 2p \quad \text{---------- (1)}\)

Put \( x^2 = X, \quad y^2 = Y \Rightarrow 2xdx = dX, \quad 2ydy = dY \)

Write \( \frac{dY}{dX} = P \) and \( \frac{dy}{dx} = p \)

\[ \therefore \frac{dY}{dX} = \frac{2ydy}{2xdx} \Rightarrow P = \frac{yp}{x} \quad (\because X = x^2) \]

Multiply (1) by \( xy, \quad y(px - y)(py + x)x = 2pxy \)

\[ \Rightarrow (pxy - y^2)(pxy + x^2) = 2pxy \]

\[ \Rightarrow (PX - Y)(PX + X) = 2PX \]
\[ PX - Y = \frac{2P}{P + 1} \]

\[ Y = PX - \frac{2P}{P + 1} \]

This is the cliarauts form

\[ \because \text{ General solution of (1) is } Y = CX - \frac{2C}{C + 1} \]

\[ \Rightarrow y^2 = cx^2 - \frac{2c}{c + 1}, \text{ c is arbitrary constant.} \]

4.7.23: Solve \( y = 2px + p^2y \)

Solution: Given d.e. is \( y = 2px + p^2y \) \---------- (1)

We reduce (1) into cliarauts form by substituting

\[ x = X, \ y^2 = Y, \ \frac{dy}{dx} = p, \ \frac{dY}{dX} = P \]

\[ \Rightarrow dx = dX, \ 2ydy = dY \]

\[ \frac{dY}{dX} = \frac{2ydy}{dx} \Rightarrow P = 2yp \Rightarrow PX = 2pxy \]

Multiply (1) by \( y' \) \[ y^2 = 2pxy + p^2y^2 \]

\[ Y = PX + \left(\frac{P}{2}\right)^2 \]

This is in the Clierauts form

\[ \because \text{ General solution of (1) is } Y = CX + \frac{c^2}{4}, \text{ i.e. } y^2 = cx + \frac{c^2}{4} \]

4.8 SELF ASSESSMENT QUESTIONS (SAQ)

Solve the following

1. \( \left(\frac{dy}{dx}\right)^2 - 10\left(\frac{dy}{dx}\right) + 21 = 0 \)
Differential Equation, Abstract Algebra...

2. \[ 6 \left( \frac{dy}{dx} \right)^2 - 7 \left( \frac{dy}{dx} \right) - 3 = 0 \]

3. \[ 4x p^2 = (3x - a)^2 \]

4. \[ y^2 - xp - x^2 p^2 = 0 \]

5. \[ x^2 p^2 + xp - 6y^2 = 0 \]

6. \[ x + yp^2 = (1 + xy)p \]

7. \[ y - x = xp + p^2 \]

8. \[ y = p \cos p - \sin p \]

9. \[ y = px + \sqrt{1 + p^2} \]

10. \[ \sin (y - px) = p \]

11. \[ y^2 - 2pxy + p^2 x^2 - p^2 = a^2 \]

4.8.1 Solutions to SAQ’s:

4.8.1: Given differential equation is

\[ p^2 - 10p + 21 = 0 \quad \text{--------} \quad (1) \]

\[ \Rightarrow p^2 - 3p - 7p + 21 = 0 \]

\[ \Rightarrow p(p - 3) - 7(p - 3) = 0 \]

\[ \Rightarrow (p - 3)(p - 7) = 0 \]

\[ \Rightarrow p - 3 = 0, \quad p - 7 = 0 \]

\[ \Rightarrow \frac{dy}{dx} = 3, \quad \frac{dy}{dx} = 7 \]

Solving each \[ dy = 3dx \quad \text{integrating} \quad \int dy = 3 \int dx \]

\[ dy = 7dx \quad \text{integrating} \quad \int dy = 7 \int dx \]
\[ \Rightarrow y = 3x + c \quad \Rightarrow y = 7x + c \]

\[ \therefore \text{General solution of given d.e. (1) is} \]

\[ (y - 3x - c)(y - 7x - c) = 0 \]

**4.8.2:** Given d.e. is \( 6p^2 - 7p - 3 = 0 \)

\[ \Rightarrow 6p^2 - 9p + 2p - 3 = 0 \]

\[ \Rightarrow (2p - 3)(3p + 1) = 0 \]

\[ \Rightarrow 2p - 3 = 0, \quad 3p + 1 = 0 \]

\[ \Rightarrow 2 \frac{dy}{dx} = 3 \quad \frac{dy}{dx} = -1 \]

Solving each

\[ 2dy = 3dx \quad 3dy = -dx \]

Integrating

We get

\[ 2y = 3x + c \quad 3y = -x + c \]

\[ \therefore \text{General solution of given d.e. is} \]

\[ (2y - 3x - c)(3y + x - c) = 0 \]

**4.8.3:** Given d.e. is

\[ 4xp^2 = (3x - a)^2 \quad \text{---------- (1)} \]

\[ \Rightarrow p^2 = \left( \frac{3x - a}{2\sqrt{x}} \right)^2 \]

\[ \Rightarrow p = \pm \frac{3x - a}{2\sqrt{x}} \]

Solving \[ dy = \pm \frac{1}{2} \left( \frac{3x - a}{\sqrt{x}} \right) dx \]

Integrating
\[\int dy = \pm \frac{1}{2} \int \left( \frac{1}{3x^2} - \frac{1}{ax^2} \right) dx\]

\[\Rightarrow y + c = \pm \frac{1}{2} \left( \frac{3}{2x^2} - \frac{1}{2ax^2} \right)\]

Squaring

\[\Rightarrow (y + c)^2 = x(x - a)^2\] which is general solution of (1)

**4.8.4**: Given d.e. is

\[y^2 - xyp - x^2p^2 = 0\]

\[\Rightarrow x^2p^2 + xyp - y^2 = 0\]

This is a quadratic equation in \(p\)

\[\therefore p = \frac{-xy \pm \sqrt{x^2y^2 + 4x^2y^2}}{2x^2}\]

\[\Rightarrow \frac{dy}{dx} = \frac{-xy(1 \pm \sqrt{5})}{2x^2}\]

Separating the variables and integrating

\[2 \int \frac{dy}{y} = -(1 \pm \sqrt{5}) \int \frac{dx}{x}\]

\[\Rightarrow 2 \log y + (1 \pm \sqrt{5}) \log x = \log c\]

\[\Rightarrow \log y^2 x^{(1 \pm \sqrt{5})} = \log c\]

\[\Rightarrow y^2 x^{(1 \pm \sqrt{5})} = c\]

\[\therefore\] General solution the given d.e. is

\[(y^2 x^{1+\sqrt{5}} - c)(y^2 x^{1-\sqrt{5}} - c) = 0\]

**4.8.5**: Given d.e. is \(x^2p^2 + xyp - 6y^2 = 0\)

\[\Rightarrow x^2p^2 + 3xyp - 2xyp - 6y^2 = 0\]
\[ \Rightarrow x(x + 3y) - 2y(x + 3y) = 0 \]
\[ \Rightarrow (x + 3y)(x - 2y) = 0 \]
\[ \Rightarrow x \frac{dy}{dx} + 3y = 0 \quad ; \quad x \frac{dy}{dx} - 2y = 0 \]

Solving each, separating the variables and integrating

\[ \int \frac{dy}{y} + 3 \int \frac{dx}{x} = \int 0 \quad \int \frac{dy}{y} - 2 \int \frac{dx}{x} = \int 0 \]
\[ \Rightarrow \log y + 3 \log x = \log c \quad \log y - 2 \log x = \log c \]
\[ yx^3 = c \quad \frac{y}{x^2} = c \]

\[ \therefore \] General solution of the given d.e. is

\[ (x^3y - c)(y - cx^2) = 0 \]

4.8.6: Given d.e. is

\[ yp^2 - p - xyp + x = 0 \quad ------ (1) \]
\[ \Rightarrow p(yp - 1) - x(yp - 1) = 0 \]
\[ \Rightarrow (yp - 1)(p - x) = 0 \]
\[ \Rightarrow yp - 1 = 0 \quad p - x = 0 \]
\[ \Rightarrow y \frac{dy}{dx} = 1 \quad \frac{dy}{dx} = x \]

Solving each, separating the variables and integrating

\[ \int ydy = \int dx \quad \int dy = \int x \, dx \]
\[ \frac{y^2}{2} = x + c \quad \quad y = \frac{x^2}{2} + c \]

\[ \therefore \] General solution of (1) is \( (y^2 - 2x - 2c)(2y - x^2 - 2c) = 0 \)
General solution of (1) is \((y^2 - 2x - 2c)(2y - x^2 - 2c) = 0\)

4.8.7: Give d.e is

\[ y = x + xp + p^2 \] ------ (1)

Differentiating w.r.t. \(x\)

\[
\frac{dy}{dx} = 1 + p + x \frac{dp}{dx} + 2p \frac{dp}{dx}
\]

\[ \Rightarrow (x + 2p) \frac{dp}{dx} + 1 = 0 \]

Multiply by \(\frac{dx}{dp} \Rightarrow \frac{dx}{dp} + x = -2p \] ------ (2)

This is linear in \(x\)

I.F. \(e^{\int dp} = e^p\) and solution of (2) is \(xe^p = -2\int e^p dp \Rightarrow e^p (x + 2p - 2) = c \) ------ (3)

\(\therefore\) (1) and (3) together represent the general solution of (1)

4.8.8: Given d.e is

\[ y = p \cos p - \sin p \] ------ (1)

Differentiating w.r.t. 'x'

\[
\frac{dy}{dx} = \frac{dp}{dx} \cos p - p \sin p \frac{dp}{dx} - \cos p \frac{dp}{dx}
\]

\[ \Rightarrow dx = -\sin pdp \]

Integrating we get \(x = \cos p + c\) ------ (2)

\(\therefore\) (1) and (2) together represent the general solution of (1)

4.8.9: Given d.e is

\[ y = px + \sqrt{1+p^2} \]

This is in the Clairaut's form

4.8.10: Given d.e is

\[ \sin (y - px) = p \]
\[ \Rightarrow y - px = \sin^{-1} p \]
\[ \Rightarrow y = px + \sin^{-1} p \]

This is in the clairaut's form.

General solution of given d.e. is \[ y = cx + \sin^{-1} c \]

4.8.11: Given d.e. is
\[ y^2 - 2pxy + p^2x^2 - p^2 = a^2 \]
\[ \Rightarrow (y - px)^2 = a^2 + p^2 \]
\[ \Rightarrow (y - px) = \pm \sqrt{a^2 + p^2} \]
\[ \Rightarrow y = px \pm \sqrt{a^2 + p^2} \]

This is in the Clairaut's form

\[ \therefore \text{General solution is } y = cx \pm \sqrt{a^2 + c^2} \]

4.9 SUMMARY

In this lesson we discussed differential equations of first order and higher degree and related problems.

4.10 TECHNICAL TERMS

Equations solvable for p, equations solvable for \( y \), equations solvable for \( x \), Clairaut's form.

4.11 EXERCISES

Solve the following

1. \[ p^2x^2 = y^2 \]
2. \[ 6p^2 + 11p - 10 = 0 \]
3. \[ p^2 + 4px - 5x^2 = 0 \]
4. \[ x^2p^2 + 3xyp + 2y^2 = 0 \]
5. \[ yp^2 + (x - y)p - x = 0 \]
6. \(4y^2p^2 + 2p xy(3x + 1) + 3x^3 = 0\)
7. \(p^2 + x^3y - x^3p - yp = 0\)
8. \(p^2 - (x + y)p + xy = 0\)
9. \(xp^2 - (x^2 - y^2)p - xy = 0\)
10. \((x + 2y)p^3 + 3(x + y)p^2 + (y + 2x)p = 0\)
11. \(xp^2 + p(3x^2 - 2y^2) - 6xy = 0\)
12. \(x^2p^2 - 2xyp + 2y^2 - x^2 = 0\)
13. \(2xp^3 - 6yp^2 + x^4 = 0\)
14. \(x^3p^2 + x^2yp + 4 = 0\)
15. \(x - yp = ap^2\)
16. \(y = 3xp + 4p^3\)
17. \(y - 2px = Tan^{-1}(xp^2)\)
18. \(y = 2px - xp^2\)
19. \(y = 2px + p^3\)
20. \(y = x + p^2\)
21. \(y^2 \log y = xpy + p^2\)
22. \(2px = 2Tany + p^3 \cos^2 y\)
23. \(p^2 - 2xp + 1 = 0\)
24. \(\sin px \cos y = \cos px \sin y + p\)
25. \(x^2(y - px) = yp^2 \quad (\text{put } x^2 = X, \ y^2 = Y)\)
26. \(y = 2px + y^2p^3 \quad (\text{put } x = X, \ y^2 = Y)\)
27. \(xy(y - px) = x + yp \quad (\text{put } x^2 = X, \ y^2 = Y)\)
4.12 ANSWERS TO EXERCISE

1. \((y - cx)(xy - c) = 0\) where \(c\) is arbitrary constant.
2. \((3y - 2x - 3c)(2y + 5x - 2c) = 0\)
3. \((2y + 5x^2 - 2c)(2y - x^2 - 2c) = 0\)
4. \((xy - c)(x^2 y - c) = 0\)
5. \((y - x - c)(x^2 + y^2 - c) = 0\)
6. \((y^2 + x^3 - c)(2y^2 + x^2 - c) = 0\)
7. \((y - ce^x)(y - ce^{-x} + x - 1) = 0\)
8. \((2y - x^2 - 2c)(y - ce^x) = 0\)
9. \((xy - c)(y^2 - x^2 - 2c) = 0\)
10. \((y - c)(x + y - c)(xy + x^2 + y^2 - c) = 0\)
11. \((y - cx^2)(y^2 + 3x^2 - c) = 0\)
12. \([-\log cx][\log cx + \arcsin \left( \frac{y}{x} \right)] = 0\)
13. \(6c^2 y - 2e^3 x^3 - 1 = 0\)
14. \(c^2 + cxy + 4x = 0\)
15. \(x - yp - ap^2 = 0; \ x\sqrt{p^2 - 1} + ap\cosh^{-1} p - cp = 0\)
16. \(x = cp^2 - \frac{12}{7} p^2; \ y = cp^2 - \frac{1}{7} p^3\)
17. \(y = 2\sqrt{cx} + \tan^{-1} c\)
18. \((y + c)^2 = 4cx\)
19. \(y - 2px - p^3 = 0; 4xp^2 + 3p^4 - c = 0\)
20. \( x = 2p + \log (p - 1) - c \); \( y = p^2 + 2p + 2\log (p - 1) - c \)
21. \( \log y = cx + c^2 \)
22. \( 2cx = 2\sin y + c^2 \)
23. \( y = \frac{p}{4} \log \frac{p}{2} + c; \ p^2 - 2xp + l = 0 \)
24. \( y = cx - \sin^{-1} c \)
25. \( y^2 = cx^2 + c^2 \)
26. \( y^2 = 2cx + c^3 \)
27. \( y^2 = cx^2 + c + 1 \)

4.13 **MODEL QUESTIONS:**

1. \( xp^2 = (x - a)^2 \)
2. \( p^2 - 7p + 10 = 0 \)
3. \( (p - xy)(p - x^2)(p - y^2) = 0 \)
4. \( y = 2xp + x^2p^4 \)
5. \( y = a\sqrt{1 + p^2} \)
6. \( y = 2px - p^2 \)
7. \( x + \frac{p}{\sqrt{1 + p^2}} = a \)
8. \( x = y + a\log p \)
9. \( x = 4p + 4p^3 \)
10. \( y = px + \frac{a}{p} \)
11. \( y = (x - a)p - p^2 \)
12. \( y - x \frac{dy}{dx} = e^{\frac{dy}{dx}} \)

13. \( x = 2p^3 + \frac{y}{p} \)

14. \( p = \log(p x - y) \)

15. \( p = \tan(px - y) \)

### 4.14 REFERENCE BOOKS

2. Vol-I - S.Chand (Text Book of Mathematics)
3. Differential Equations by N.Ch.S.N. Iyengar

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Lesson - 5

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS OF HIGHER ORDER

5.1 OBJECTIVE OF THE LESSON

In the previous lessons, we had learnt how to solve linear differential equations of first order and first degree. In this lesson we will learn how to solve homogeneous linear differential equations of second and higher order with constant coefficients which are very useful in various physics and Engineering application.

5.2 STRUCTURE OF THE LESSON

This lesson has the following components.

5.3 Introduction
5.4 Linear Differential Equations
5.5 Solution of the homogeneous linear differential equation
5.6 Rules for finding the complementary function
5.7 A.E. has real and distinct roots
5.8 A.E. has Equal roots
5.9 A.E. has imaginary roots
5.10 A.E. have two pairs of equal imaginary roots
5.11 Answers to SAQ's
5.12 Summary
5.13 Technical Terms
5.14 Exercises
5.15 Answers to Exercises
5.16 Model Examination Questions
5.17 Reference Books

5.3 INTRODUCTION

The ordinary differential equations may be divided into two large classes, namely, linear equations and non-linear equations. Whereas non-linear equations are difficult in general, linear equations are much simpler because their solutions have general properties that facilitate working with them, and there are standard methods for solving many particularly important linear differential
equations. Second order linear differential equations have important applications in mechanics and in electric circuits theory.

5.4 LINEAR DIFFERENTIAL EQUATION

The ordinary differential equations may be divided into two large classes, namely, linear equations and non-linear equations. Whereas non-linear equations are difficult in general, linear equations are much simpler because then solutions have general properties that facilitate working with them, and there are standard methods for solving many particularly important linear differential equations. Second order linear differential equations have important applications in mechanics and in electric circuits theory.

5.4 LINEAR DIFFERENTIAL EQUATIONS

5.4.1 Definition: A linear differential equation of order \( n \) with constant coefficients is an equation of the form

\[
\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = Q(x)
\]

where \( a_0 \neq 0, a_1, a_2, \ldots, a_n \) are real constants, and \( Q(x) \) is some real valued continuous functions of \( x \) on an interval \( I \) and nonlinear if it cannot be written in the above formula. By dividing by \( a_0 \) we can arrive at an equation of the same form with \( a_0 \) replaced by 1. Therefore we can always assume \( a_0 = 1 \), and an equation becomes

\[
\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = Q(x) \quad \text{---------- (1)}
\]

If \( Q(x) = 0 \) in (1). Then the equation

\[
\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = 0 \quad \text{---------- (2)}
\]

is called homogeneous linear differential equation with constant coefficients, otherwise (1) is called a non-homogeneous equation.

5.4 DIFFERENTIAL OPERATOR

Let \( D \) represent the differentiation with respect to \( x \) \( \left( D = \frac{d}{dx} \right) \). Then,

\[
\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \ldots, \frac{d^n y}{dx^n}, \ldots \text{ are denoted by } D y, D^2 y, \ldots, D^n y, \ldots \text{ respectively.}
\]

If \( s \) and \( t \) are differentiable functions of \( x \) and \( c \) is any constant then
Thus the equation (1) can be written as
\[(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = Q.\]
If \(L\) denotes the operator \(D^n + a_1D^{n-1} + \cdots + a_n\), then the equation \(L(y)\) or may be written as \(L(y) = Q\), where \(L\) is called a linear differential operator.

A linear differential operator has the following two basic properties.

(i) \(L(cy) = cL(y)\)  
(ii) \(L(y_1 + y_2) = L(y_1) + L(y_2)\)

where \(y, y_1, y_2\) are functions of \(x\) and \(c\) is a constant.

**Note:** A corollary of properties (i) & (ii) is
\[L\left(\sum_{i=1}^{n} c_i y_i\right) = \sum_{i=1}^{n} c_i L(y_i)\]
where \(c_i\) are constant.

### 5.5 SOLUTION OF THE HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

Now, we shall discuss the nature of solution of the homogenous linear differential equation \((z)\).

**5.5.1 Theorem:** If \(y_1\) and \(y_2\) are solutions of the equation \(L[y] = 0\), then \(y_1 + y_2\) and \(cy_1\), where \(c\) is an arbitrary constant are also solutions of \(L[y] = 0\).

**Proof:** Given \(L[y_1] = 0\) and \(L[y_2] = 0\). It is required to prove that \(L(y_1 + y_2) = 0\) and \(L[cy_1] = 0\).

By using property (i) and (ii) of linear operator \(L\), we get
\[L[y_1 + y_2] = L[y_1] + L[y_2].\]
\[= 0 + 0 = 0\]
and \(L[cy_1] = cL[y_1] = c \cdot 0 = 0\)

**5.5.2 Definition:** Linear combination: A linear combination of \(n\) solutions \(y_1, y_2, \ldots, y_n\) is an expression of the form
\[c_1y_1 + c_2y_2 + \cdots + c_n y_n\]
where \(c_1, c_2, \ldots, c_n\) are constants.

From Theorem 5.5.1, we get the following corollary.
5.5.3 Corollary: A linear combination $\sum_{i=1}^{n} c_i y_i$ of $n$ solutions $y_1, y_2, \ldots, y_n$ of a homogenous linear equation $L[y] = 0$ is a solution of that equation.

5.5.4 Caution: Always remember this highly important Theorem (5.5.1), but don’t forget that it does not hold for non-homogeneous linear equations or non-linear equations, as the following two examples illustrated.

(i) Substitution shows that the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions of the non-homogeneous linear differential equation.

$$y'' + y = 1.$$ 

But the following functions are not solutions of this differential equations:

$$2(1 + \cos x) \text{ and } (1 + \cos x) + (1 + \sin x)$$

(ii) Substitution shows that the functions $y = x^2$ and $y = 1$ are solutions of the non-linear differential equation.

$$y'' - xy' = 0$$

But the following functions are not solutions of this differential equations:

$$-x^2 \text{ and } x^2 + 1$$

Now to obtain the form of the general solution of $L[y] = 0$. We need the following concepts.

5.5.5 Definition: The functions $y_1(x), y_2(x), \ldots, y_n(x)$ are called linearly dependent over a certain interval $I$, if there exist constants $c_1, c_2, \ldots, c_n$ (not all zero) such that, on that interval $I$.

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

5.5.6 Definition: The functions $y_1(x), y_2(x), \ldots, y_n(x)$ are called linearly independent over a certain interval $I$, if $c_1, \ldots, c_n$ are constants and

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0 \text{ on } I$$

$$c_1 = c_2 = \ldots = c_n = 0$$

5.5.7 Definition: If $y_1, y_2, \ldots, y_n$ are real functions having $(n-1)^{th}$ derivatives on an interval $I$, then the determinant
5.5 Differential Equation

Abstract Algebra

Method of Undetermined...

\[
\begin{vmatrix}
  y_1 & y_2 & \cdots & y_n \\
  y'_1 & y'_2 & \cdots & y'_n \\
  \vdots & \vdots & \ddots & \vdots \\
  y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix}
\]

is called the Wronskian of the given functions.

We state the following theorems 5.5.9, 5.5.10, … without which are valid for equations with constant coefficients also, since constant can be treated as a constant function and all constant functions are continuous.

5.5.8 Theorem: The functions \( y_1, y_2, \ldots, y_n \) are linearly dependent on the interval \([a, b]\) if and only if the Wronskian of \( y_1, y_2, \ldots, y_n \) on this interval is identically zero.

5.5.9 Theorem: Let \( y_1, y_2, \ldots, y_n \) be \( n \) solutions of the homogeneous linear equation.

\[
L[y] = \left[ D^n + P_1(x)D^{n-1} + \cdots + P_{n-1}(x)D + P_n(x) \right] y = 0
\]

with continuous coefficients \( P_i(x) \) on the interval \([a, b]\). Then \( y_1, y_2, \ldots, y_n \) are linearly independent if and only if the Wronskian

\[
\begin{vmatrix}
  y_1 & y_2 & \cdots & y_n \\
  y'_1 & y'_2 & \cdots & y'_n \\
  \vdots & \vdots & \ddots & \vdots \\
  y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix}
\]

does not vanish at any point of the interval \([a, b]\).

5.5.10 Theorem: If \( y_1, y_2, \ldots, y_n \) are linearly independent solutions of the homogeneous equations

\[
L[y] = \left[ D^n + P_1(x)D^{n-1} + P_2(x)D^{n-2} + \cdots + P_{n-1}(x)D + P_n(x) \right] y = 0 \quad \text{(1)}
\]

on the interval \([a, b]\), then

\[
y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = \sum_{i=1}^{n} c_i y_i \quad \text{(2)}
\]

is the general solution of (1) on \([a, b]\) in the sense that every solution of (1) on this interval can be obtained from (2) by suitable choice of the arbitrary constants \( c_1, c_2, \ldots, c_n \).
5.5.11 General solution: In view of the above theorem, if \( y_1, y_2, \ldots, y_n \) are any \( n \) linearly independent solutions of the homogeneous linear differential equation \( L[y] = 0 \). Then all solutions of homogenous linear differential equations \( L(y) = 0 \) are given by
\[
y = c_1y_1 + c_2y_2 + \cdots + c_ny_n
\]
where \( c_1, c_2, \ldots, c_n \) are arbitrary constants. Thus \( y = c_1y_1 + c_2y_2 + \cdots + c_ny_n \) is the general solution.

Note: This solution \( y = c_1y_1 + c_2y_2 + \cdots + c_ny_n \) is also called complementary function (CF) of the differential equation \( L(y) = Q(x) \).

5.5.12 Auxiliary Equation: Consider the homogeneous linear differential equation
\[
L[y] \equiv (D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = 0 \quad \text{(1)}
\]
We try \( y = e^{mx} \) for a solution of (1), then
\[
Dy = me^{mx}, \quad D^2y = m^2e^{mx}, \ldots, D^ny = m^ne^{mx}
\]
substituting these values in (1) we get
\[
e^{mx}(m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_n) = 0
\]
\[
\Leftrightarrow m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_n = 0 \quad \text{(2) \ (\because e^{mx} \neq 0)}
\]
Thus, \( y = e^{mx} \) is a solution of (1) iff \( m \) satisfies the equation (2).

Equation (2) is called the Auxiliary equation for the differential equation (1).

The auxiliary equation of the differential equation \( f(D)y = 0 \), where \( f(D) = D^n + a_1D^{n-1} + \cdots + a_n \) is \( f(m) = 0 \). i.e. \( m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_n = 0 \).

5.5.13 Theorem: If \( \alpha \) is a root of the auxiliary equation of the homogeneous linear differential equation with constant coefficients
\[
L[y] \equiv (D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = 0
\]
Then \( e^{\alpha x} \) is a solution of the differential equation \( L(y) = 0 \).

Let \( f(m) = 0 \) be the auxiliary equation of \( L(y) = 0 \).
5.7 Method of Undetermined Coefficients

Proof: If $K > 0$ then \[
\frac{d}{dx^k} (e^{\alpha x}) = \alpha^k e^{\alpha x}
\]

\[
L(e^{\alpha x}) = \sum_{k=0}^{n} a_k D^k (e^{\alpha x}) = \sum_{k=0}^{n} a_k \alpha^k e^{\alpha x} = f(\alpha) e^{\alpha x}
\]

\[
\therefore L(e^{\alpha x}) = 0 \iff f(\alpha) = 0
\]

5.5.14 Theorem: If the coefficients of the auxiliary equation of the differential equation $L(y) = 0$ are real and $\alpha = a + ib, a, b \in \mathbb{R}$ is a root of the auxiliary equation then $e^{ax} \sin bx, e^{ax} \cos bx$ are solutions of $L[y] = 0$.

Proof: Let $f(m) = 0$ be the auxiliary equation of the differential equation $L[y] = 0$.

Since coefficients of $f(m) = 0$ are real, $\alpha$ is a root of $f(m) = 0$.

\[
\therefore \bar{\alpha} \text{ is a root of the equation } f(m) = 0.
\]

By the above theorem (5.5.13), $e^{\alpha x}, e^{\bar{\alpha} x}$ are solutions of the differential equation $L[y] = 0$.

By Theorem (5.5.10), $e^{ax} \cos bx = \frac{e^{\alpha x} + e^{\bar{\alpha} x}}{2}, e^{ax} \sin bx = \frac{e^{\alpha x} - e^{\bar{\alpha} x}}{2}$ are solutions of $L[y] = 0$.

5.5.15 Definition: Let $\alpha$ be a root of the equation $f(x) = 0$ and $\alpha$ repeats $m$ times. Then $m$ is said to be multiplicity of $\alpha$.

5.5.16 Theorem: If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct roots of the auxiliary equation $f(m) = 0$ of the linear differential equation $L[y] = 0$, with multiplicity of $\alpha_i$ is $m_i$ then $e^{\alpha_1 x}, x e^{\alpha_1 x}, \ldots, x^{m_1-1} e^{\alpha_1 x}$ are solutions of the differential equations $L(y) = 0$.

We obtain a set of $n$-linearly independent solutions of $L(y) = 0$ in the following way

(a) For each real root $\alpha$ of the A.E. $f(m) = 0$ with multiplicity $r$, we include the functions $e^{\alpha x}, x e^{\alpha x}, \ldots, x^{r-1} e^{\alpha x}$.

(b) For each complex root $\alpha = a + ib$ with multiplicity $r$, we include three functions $e^{ax} \cos bx, xe^{ax} \cos bx, \ldots, x^{r-1} e^{ax} \cos bx, e^{ax} \sin bx, \ldots, x^{r-1} e^{ax} \sin bx$. 

5.6 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

To solve the differential equation

\[
\left(D^n + a_1D^{n-1} + \cdots + a_n\right)y = 0 \quad \text{where}
\]

\[a_1, a_2, \cdots, a_n\] are constants.

We write it as \(f(D)y = 0\), where \(f(D) = D^n + a_1D^{n-1} + \cdots + a_n\).

The Auxiliary Equation (A.E) of the above differential equation \(f(D)y = 0\) is

\[
m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_n = 0
\]

Let \(m_1, m_2, m_3, \cdots, m_n\) be its roots

We discuss the nature of the roots in the following cases.

5.7

Case 1 : A.E. has real and distinct roots :

The A.E. of \(f(D)y = 0\) is \(f(m) = 0\).

Let \(m_1, m_2, \cdots, m_n\) be \(n\) real and distinct roots. Then by Theorem (5.5.16), the general solution of \(f(D)y = 0\) is

\[
y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}
\]

5.7.1 Example : Solve \(\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0\)

Solution : Given equation can be written as

\[
\left(D^3 + 6D^2 + 11D + 6\right)y = 0
\]

A.E. is \(m^3 + 6m^2 + 11m + 6 = 0\) \(\Rightarrow (m+1)(m+2)(m+3) = 0\)

\(\Rightarrow m = -1, -2, -3\)

\therefore\ The general solution is

\[
y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}
\]

5.7.2 SAQ : Find the solution of \(\left(D^2 - 5D + 4\right)y = 0\)
5.7.3 Example: Solve \( \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \) with \( y = 0 \) and \( \frac{dy}{dx} = 0 \) when \( x = 0 \).

Solution: The given equation can be written as \((D^2 - 3D + 2)y = 0\)

A.E. is \( m^2 - 3m + 2 = 0 \) \( \Rightarrow \) \((m-1)(m-2) = 0 \) \( \Rightarrow m = 1, 2\)

The general solution is \( y = c_1e^x + c_2e^{2x} \) \( \text{------- (1)} \)

Given that \( y = 0 \) when \( x = 0 \).

\( \therefore \) equation (1) yields

\( 0 = c_1 + c_2 \) \( \text{------- (2)} \)

Also, from (1) \( \frac{dy}{dx} = c_1e^x + 2c_2e^{2x} \) using \( \frac{dy}{dx} = 0 \) when \( x = 0 \). we have

\( 0 = c_1 + 2c_2 \) \( \text{---------- (3)} \)

Solving (2) & (3) we get \( c_1 = c_2 = 0 \). Thus the solution of (1) becomes \( y = 0 \).

5.7.2 Example: Solve \( (D^4 - 2D^3 - 13D^2 + 38D - 24)y = 0 \)

Solution: A.E. is \( m^4 - 2m^3 - 13m^2 + 38m - 24 = 0 \)

\( \Rightarrow (m-1)(m-2)(m-3)(m+4) = 0 \)

\( \Rightarrow m = 1, 2, 3, -4 \)

The general solution is

\( y = c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{-4x} \)

5.8 CASE 2

All the roots of the A.E. are real and some roots are repeated.

1. Let \( f(m) = 0 \) have two equal roots \( m_1 = m_2 = m \) (say) and all other distinct roots \( m_3, m_4, \ldots, m_n \). Then the general solution of \( f(D)y = 0 \) is

\( y = (c_1 + c_2x)e^{mx} + c_3e^{m_3x} + c_4e^{m_4x} + \cdots + c_ne^{m_nx} \)

2. Let \( f(m) = 0 \) have \( K \) equal roots \( m_1 = m_2 = \cdots = m_k = m \) (say) and all other distinct
roots $m_{k+1}, m_{k+2}, \ldots, m_k$. Then the general solution of $f(D)y = 0$ is

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_k x^{k-1})e^{mx} + \ldots + c_ne^{mnx}$$

(3) If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are distinct roots with multiplicity of $\alpha_i$ as $m_i = n$. Then the general solution of $f(D)y = 0$ is

$$y = (c_1 + c_2 x + c_3 x^3 + \ldots + c_{m_1x^{m_1-1}})e^{\alpha_1x} + (c_{m_1+1} + c_{m_1+2} x + \ldots + c_{m_2} x^{m_2-1})e^{\alpha_2x} + \ldots$$

$$+ (c_{m_k-1} + c_{m_k+2} x + \ldots + c_{m_k} x^{m_k-1})e^{\alpha_kx}$$

5.8.1 Example: Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

**Solution:** The given equation can be written as $(D^3 - 3D + 2)y = 0$

$\Rightarrow$ A.E. is $m^3 - 3m^2 + 2 = 0 \Rightarrow m = 1, 1, -2$

$\therefore$ The general solution is

$$y = (c_1 + c_2x)e^x + c_3e^{-2x}$$

5.8.2 SAQ: Find the C.F. of $(D^2 - 2D + 1)y = 0$

**Solution:** The given equation can be written as

$$f(D)y = (D^2 - 2D + 1)y = 0$$

$\Rightarrow$ A.E. is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

$\therefore$ The general solution is

$$y = (c_1 + c_2x)e^x$$

5.8.3 Example: Solve $16\frac{d^2y}{dx^2} + 24\frac{dy}{dx} + 9y = 0$

**Solution:** The given equation can be written as

$$\left(16D^2 + 24D + 9\right)y = 0$$
5.11 Method of Undetermined Coefficients

A.E. is $16m^2 + 24m + 9 = 0 \Rightarrow (4m + 3)^2 = 0$

$\Rightarrow m = -\frac{3}{4}, -\frac{3}{4}$

$\therefore$ The general solution is

$$y = (c_1 + c_2x)e^{-\frac{3}{4}x}$$

5.9 CASE III

If the A.E. has one pair of roots be imaginary, i.e. $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the part of the C.F. corresponding to this root $\alpha + i\beta$ is $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

If any real root is repeated for their real root include the functions as in case II.

5.9.1 Example : Solve $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$

Solution : The given equation can be written as

$$\left(2D^2 - 5D + 4\right)y = 0$$

$\therefore$ A.E. is $2m^2 - 5m + 4 = 0$

$\Rightarrow m = \frac{5 \pm i\sqrt{7}}{4}$

$\therefore$ The general solution is

$$y = e^{\frac{5}{4}x} \left[c_1 \cos \frac{\sqrt{7}}{4}x + c_2 \sin \frac{\sqrt{7}}{4}x\right]$$

where $c_1$ and $c_2$ are constants

5.9.2 Example : Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$

Solution : The given equation can be written as

$$\left(D^2 - 2D + 10\right)y = 0$$

$\therefore$ A.E. is $m^2 - 2m + 10 = 0 \Rightarrow m = 1 \pm i3$

$\therefore$ The general solution is
\[ y = e^x \left[ c_1 \cos 3x + c_2 \sin 3x \right] \text{ where } c_1 \text{ and } c_2 \text{ are constants.} \]

5.10 CASE IV

If the A.E. have two pairs of imaginary roots be equal i.e. \( m_1 = m_2 = \alpha + i\beta \),

\[ m_3 = m_4 = \alpha - i\beta . \]
Then the part of the C.F. corresponding the root \( \alpha + i\beta \) is

\[ e^{\alpha x} \left[ (c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x \right], \text{ where } c_1, c_2, c_3, c_4 \text{ are constants.} \]

If A.E. has a real complex root \( m = \alpha + i\beta \) with multiplicity \( r \) then the corresponding to this root the part of the complementary function has the form

\[ e^{\alpha x} \left[ (c_0 + c_1 x + \cdots + c_{r-1} x^{r-1}) \cos \beta x + (\alpha_0 + \alpha_1 x + \cdots + \alpha_{r-1} x^{r-1}) \sin \beta x \right] \]

where \( c_0, c_1, \ldots, c_{r-1}, \alpha_0, \alpha_1, \ldots, \alpha_{r-1} \) are arbitrary constants.

5.10.1 Example : Solve \( (D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0 \)

Solution : A.E. is \( m^4 - 4m^3 + 8m^2 - 8m + 4 = 0 \)

\[ (m^2 - 2m + 2)^2 = 0 \]
\[ \Rightarrow m^2 - 2m + 2 = 0, \quad m^2 - 2m + 2 = 0 \]
\[ \Rightarrow m = \frac{2 \pm \sqrt{4 - 8}}{2}, \quad m = \frac{2 \pm \sqrt{4 - 8}}{2} \]
\[ m = 1 \pm i, 1 \pm i \]

Hence the imaginary roots are equal i.e. \( m_1 = m_2 = 1 + i \)

\[ m_3 = m_4 = 1 - i \]

\[ \therefore \text{ The general solution of the given equation is} \]

\[ y = e^x \left[ (c_1 x + c_2 x) \cos x + (c_3 + c_4 x) \sin x \right] \]

5.10.2 Example : Solve \( \frac{d^4 y}{dx^4} - 2 \frac{d^3 y}{dx^3} + 3 \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 1 = 0 \)

Solution : The given equation can be written as

\[ (D^4 - 2D^3 + 3D^2 - 2D + 1)y = 0 \]

\[ \therefore \text{ A.E. is } m^4 - 2m^3 + 3m^2 - 2m + 1 = 0 \Rightarrow (m^2 - 2m + 1)^2 = 0 \]
\[ m = \frac{1 \pm i\sqrt{3}}{2}, \frac{1 + i\sqrt{3}}{2} \]

Hence the imaginary roots are equal i.e. \( m_1 = m_2 = \frac{1 + i\sqrt{3}}{2} \). \( m_3 = m_4 = \frac{1 - i\sqrt{3}}{2} \)

\[ \therefore \text{The general solution is} \ y = \frac{e^{\sqrt{3}}}{2} \left[ (c_1 + c_2 x) \cos \frac{\sqrt{3}x}{2} + (c_3 + c_4 x) \sin \frac{\sqrt{3}x}{2} \right] \]

5.11 ANSWERS TO SAQ

5.7.2 SAQ: A.E. is \( m^2 - 5m + 4 = 0 \) \( \Rightarrow m = 1, 4 \)

\[ \therefore \text{The general solution is} \ y = c_1 e^x + c_2 e^{4x} \]

5.8.2 SAQ: The A.E. is \( (m^2 - 2m + 1) = 0 \)

\[ \Rightarrow (m - 1)^2 = 0 \Rightarrow m = 1, 1 \]

\[ \therefore \text{The general solution is} \ y = (c_1 + c_2 x)e^x \]

5.12 SUMMARY

To find the general solutions of \( f(D)y = 0 \) or complementary function of \( f(D)y = Q \) as follows.

(i) Write the A.E. \( f(m) = 0 \)
(ii) Solve the A.E. to get the roots \( m_1, m_2, \ldots, m_n \)
(iii) Corresponding to these roots write the terms in the general solution using the following table.

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
Sl.No. & Nature of the roots & Terms in C.F. \\
\hline
1. & Real and Distinct roots \( m_1, m_2, \ldots, m_k \) & \( c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_k e^{m_k x} \) \\
2. & If all the roots are real and one root \( m_1 = m_2 = \cdots = m_r = m \) (say) & Then the part of the C.F. of the corresponding root is \( (c_1 + c_2 x)e^{mx} \) \\
3. & If all the roots are real and any one root \( m \) has multiplicity \( r \) \( (i.e. m_1 = m_2 = \cdots = m_r = m) \) & Then the part of the C.F. corresponding to this root \( m \) shall be \( (c_0 + c_1 x + c_2 x^2 + \cdots + c_{r-1} x^{r-1})e^{mx} \) \\
4. & If A.E. has a complex root say \( m = \alpha + i\beta \) which is not repeated & Then the part of the C.F. corresponding to this root \( m \) shall be
\end{tabular}
\end{center}
5. If A.E. has a complex root \( m = \alpha + i\beta \) with multiplicity \( r \)

Then the part of the C.F. corresponding to this root \( m \) shall be

\[
e^{\alpha x} \left( c_1 \cos \beta x + c_2 \sin \beta x \right)
\]

\[
+ \left( d_0 + d_1 x + d_2 x^2 + \ldots + d_{r-1} x^{r-1} \right) \sin \beta x
\]

5.13 TECHNICAL TERMS

- Homogeneous linear differential equations
- Differential operator \( D \)
- Linear independent
- Complementary function
- Particular integral (P.I.)
- Auxiliary Equation (A.E)
- Complex Conjugate

5.14 EXERCISE

Solve the following differential equations

1. \[ \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + y = 0 \]
2. \[ \frac{d^2 y}{dx^2} - a^2 y = 0, \ a \neq 0 \]
3. \[ \frac{d^3 y}{dx^3} - 4 \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} - 2y = 0 \]
4. \( (D^3 + 6D^2 + 2D + 8)y = 0 \)
5. \( (D^2 - 7D + 44)y = 0 \)
6. \( (D^3 - D^2 - D - 2)y = 0 \)
7. \( (D^2 - 2aD + a^2 + b^2)y = 0 \)
8. \( (D^2 + (a + b)D + ab)y = 0 \)
9. \( (D^4 + 4D^3 - 5D^2 - 36D - 36)y = 0 \)
10. \( (D^2 + D + 1)^2 y = 0 \)
5.15 ANSWER TO EXERCISE

(1) \[ y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x} \] 

(or) \[ y = e^{2x} \left[ c_1 \cosh \sqrt{3}x + c_2 \sinh \sqrt{3}x \right] \]

(2) \[ y = c_1 e^{ax} + c_2 e^{-ax} \]

(3) \[ y = (c_1 + c_2 x)e^x + c_3 e^{2x} \]

(4) \[ y = (c_1 + c_2 x + c_3 x^2)e^{-2x} \]

(5) \[ y = c_1 e^{-4x} + c_2 e^{11x} \]

(6) \[ y = c_1 e^{2x} + e^{2x} \left[ c_1 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x \right] \]

(7) \[ y = e^{ax} \left[ c_1 \cos bx + c_2 \sin bx \right] \]

(8) \[ y = c_1 e^{-ax} + c_2 e^{-bx} \]

(9) \[ y = c_1 e^{-3x} + c_2 e^{3x} + (c_3 + c_1 x)e^{-2x} \]

(10) \[ y = e^{-x/2} \left[ (c_1 + c_2 x) \cos \frac{\sqrt{3}}{2}x + (c_3 + c_4 x) \sin \frac{\sqrt{3}}{2}x \right] \]

5.16 MODEL QUESTIONS

(i) If \( y_1 \) and \( y_2 \) are two solutions of the differential equation \( (D^2 + a_1 D + a_2)y = 0 \). Then prove that \( c_1 y_1 + c_2 y_2 \) is also a solution.

(ii) If \( \alpha \) is a root of the A.E. \( f(m) = 0 \). Then \( e^{\alpha x} \) is a solution of \( f(D)y = 0 \)

(iii) Solve \( (D^4 - 81)y = 0 \)

5.17 REFERENCE

2. Zafar Ahsan, Differential Equation and their application

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Lesson - 6

NON HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

6.1 OBJECTIVE OF THE LESSON

In the previous lesson, we discussed how to solve homogeneous linear differential equation with constant coefficients. In this lesson we discuss how to solve with homogeneous linear differential equations with constant coefficients.

6.2 STRUCTURE OF THE LESSON

This lesson has the following components.

6.3 Introduction
6.4 General solution of non-homogeneous linear equation
6.5 Methods for finding the particular Integral
6.6 Method (1) when \( Q(x) = e^{ax} \)
6.7 Method (2) when \( Q(x) = \cos(ax + b) \) or \( \sin(ax + b) \)
6.8 Method (3) when \( Q(x) = x^m \)
6.9 Method (4) when \( Q(x) = e^{ax} \cdot V \)
6.10 Method (5) when \( Q(x) = xV \)
6.11 Answers to SAQ's
6.12 Summary
6.13 Technical Terms
6.14 Exercises
6.15 Answers to Exercises
6.16 Model Examination Questions
6.17 Reference Books

6.3 INTRODUCTION

In this lesson we will learn the following methods for solving non-homogeneous linear differential equations \( f(D)y = Q(x) \).
General solution of non-homogenious linear differential equation; methods for finding particular integral (when \( Q(x) = e^{ax} \), \( Q(x) = \cos(ax + b) \) or \( \sin(ax + b) \), \( Q(x) = x^m \), \( Q(x) = e^{ax}V \), \( Q(x) = x \cdot V \).

6.4 GENERAL SOLUTION OF NONHOMOGENEOUS LINEAR EQUATION

6.4.1 Definition : A nonhomogeneous linear differential equation with constant coefficients is of the form

\[
(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = Q \quad \text{------- (1)}
\]

where \( a_1, a_2, \ldots, a_n \) are real constants and \( Q \) is a nonzero continued function of \( x \) only.

6.4.2 General solution of Equation (1) : Let \( Q(x) \) be a continuous function on an interval \( I \), and consider the equation

\[
f(D)y = \left(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n\right)y = Q(x)
\]

where \( a_1, a_2, \ldots, a_n \) are constants. If \( y_P \) is a particular solution of \( f(D)y = Q(x) \) and \( y \) is any other solution, then

\[
f(y - y_P) = f(y) - f(y_P) = Q(x) - Q(x) = 0
\]

Thus, \( y - y_P \) is a solution of the homogeneous linear differential equation \( f(D) = 0 \) and is of the form

\[
y - y_P = c_1y_1 + c_2y_2 + \cdots + c_ny_n \quad \text{where, } y_1, y_2, \ldots, y_n \text{ are linearly independent solution of } f(D) = 0 \text{ and } c_1, c_2, \ldots, c_n \text{ are constants.}
\]

\[\Rightarrow \] Any solution \( y \) of \( f(D)y = Q(x) \) can be written in the form

\[
y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_P
\]

i.e. \( y = y_c + y_P \)

where \( y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n \) is a complementary function (C.F.) of \( f(D)y = 0 \) and \( y_P \) is a particular solution or particular integral (P.I.) of \( f(D)y = Q(x) \) and \( c_1, c_2, \ldots, c_n \) are constants.

The solution \( y = y_c + y_P \) is called general solution of the nonhomogeneous linear differential equations with constant coefficients. \( f(D)y = Q(x) \).

Note : Particular solution does not contain any arbitrary constants.
6.4.3 Determination of Particular Integral (P.I.) :

6.4.4 Inverse Operator : \( \frac{1}{f(D)}(Q) \) is the function of \( x \), free from arbitrary constants, which when operated up on by \( f(D) \) gives \( Q \).

Thus \( f(D)\left[\frac{1}{f(D)}Q\right] = Q \)

\( : f(D) \) and \( \frac{1}{f(D)} \) are inverse operators.

6.4.5 Result : \( \frac{1}{f(D)}Q \) is the particular integral of \( f(D)y = Q \).

Proof : The given equation is

\[ f(D)y = Q \] \( ------------ (1) \)

Put \( y = \frac{1}{f(D)}y(Q) \) in (1)

we have \( f(D)\left[\frac{1}{f(D)}Q\right] = Q \)

\[ \Rightarrow Q = Q \]

\( \therefore y = \frac{1}{f(D)}Q \) is a solution of (1)

Since it contains no arbitrary constants, it is the particular integral of \( f(D)y = Q \).

6.4.6 Result : \( \frac{1}{D}Q = \int Q \, dx \)

Proof : Let \( \frac{1}{D}Q = y \)

Operating both sides by \( D \)

we have \( D\left[\frac{1}{D}Q\right] = Dy \)

\[ \Rightarrow Q = Dy \]
\[ Q = \frac{dy}{dx} \]
\[ \Rightarrow dy = Qdx \]
Integrating on both sides, with respect to \( x \), we get
\[ y = \int Qdx \]

\[ \therefore \frac{1}{D}Q = \int Qdx \]

**6.4.7 Theorem**: Consider the equation \((D - a)y = Q\). Particular solution of \((D - a)y = Q\) is given by
\[ \frac{1}{D-a}Q = e^{ax} \int Qe^{-ax}dx \]

**Proof**: Let \( \frac{1}{D-a}Q = y \)

Operating by \((D-a), (D-a)\left(\frac{1}{D-a}Q\right) = (D-a)y \)

(or) \[ Q = Dy - ay \]

(or) \[ \frac{dy}{dx} - ay = Q \]

which is a linear differential equation in \( y \) whose

I.F. =\( e^{\int a \, dx} = e^{-ax} \) and hence the solution is

\[ y \cdot e^{-ax} = \int Qe^{-ax}dx \quad \text{(no constant being added P.I. does not contain any constant)} \]

\[ y = e^{ax} \int Q \cdot e^{-ax}dx \]

Thus \[ \frac{1}{D-a}Q = e^{ax} \int Qe^{-ax}dx \]

**6.4.8 Example**: Solve \((p^2 + a^2)y = \sec ax\), \( a \) is real.

**Solution**: A.E. is \( m^2 + a^2 = 0 \Rightarrow m = \pm ai \)
The complementary function \( y_c = c_1 \cos ax + c_2 \sin ax \)

Particular integral (P.I.) \( y_p = \frac{1}{D^2 + a^2} \sec ax \)

\[
y_p = \frac{1}{2ai} \left( \frac{1}{D - ai} - \frac{1}{D + ai} \right) \sec ax
\]

Now, \( \frac{1}{D - ai} \sec ax = e^{iax} \int \sec ax \cdot e^{-iax} \, dx \)

\[
= e^{iax} \int \left( \cos ax - i \sin ax \right) \frac{1}{\cos ax} \, dx
\Rightarrow \left( e^{i\theta} = \cos \theta + i \sin \theta \right)
\]

\[
= e^{iax} \int \left( 1 - i \tan ax \right) \, dx
\]

\[
= e^{iax} \left[ x + \frac{i}{a} \log \cos ax \right]
\]

Similarly \( \frac{1}{D + ai} \sec ax = e^{-iax} \left[ x - \frac{i}{a} \log \cos ax \right] \)

\[
y_p = \frac{1}{2ai} \left[ \frac{1}{D - ai} \sec ax - \frac{1}{D + ai} \sec ax \right]
\]

\[
= \frac{1}{2ai} \left[ e^{iax} \left( x + \frac{i}{a} \log \cos ax \right) - e^{-iax} \left( x - \frac{i}{a} \log \cos ax \right) \right]
\]

\[
= x \left( \frac{e^{iax} - e^{-iax}}{2ai} \right) + \frac{1}{a^2} \log \cos ax \left[ \frac{e^{ax} - e^{-ax}}{2} \right]
\]

\[
= \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \cdot \log \cos ax
\]

\( \therefore \) the general solution of the given differential equation is \( y = y_c + y_p \)

\[
y = c_1 \cos ax + c_2 \sin ax + x \cos ax + \frac{1}{a^2} \cos ax \cdot \log \cos ax
\]

6.4.9 Definition: If \( \frac{1}{D - \beta}, \frac{1}{D - \alpha} \) are two inverse operators then we define
\[ \frac{1}{(D - \beta)(D - \alpha)} Q = \frac{1}{(D - \beta)} \left[ \frac{1}{(D - \alpha)} Q \right] \]

where \( \alpha, \beta \) are constants and \( Q \) is a function of \( x \).

That is \[ \frac{1}{(D - \beta)(D - \alpha)} Q = \frac{1}{D - \beta} \left[ e^{\alpha x} \int Q e^{-\alpha x} \, dx \right] = e^{\beta x} \int \left[ e^{\alpha x} \int Q e^{-\alpha x} \, dx \right] e^{-\beta x} \, dx \]

**6.4.10 Example:** Find

(a) \( \frac{1}{D} x^2 \)  
(b) \( \frac{1}{D^2} x^2 \)  
(c) \( \frac{1}{D^2} e^{4x} \)  
(d) \( \frac{1}{D - 2} e^{3x} \)  
(e) \( \frac{1}{(D - 2)(D - 3)} e^{2x} \)

**Solution:**

(a) \( \frac{1}{D} x^2 = \int x^2 \, dx = \frac{x^3}{3} \)

(b) \( \frac{1}{D^2} x^2 = \int \frac{1}{D} x^2 \, dx = \frac{1}{D^3} \int x^3 \, dx = \frac{1}{3} \int x^3 \, dx = \frac{1}{3} \cdot \frac{x^4}{4} = \frac{x^4}{12} \)

(c) \( \frac{1}{D^2} e^{4x} = \int \frac{1}{D} e^{4x} \, dx = \frac{1}{D^4} \int e^{4x} \, dx = \frac{1}{4} \int e^{4x} \, dx = \frac{1}{4} \cdot \frac{e^{4x}}{4} = \frac{e^{4x}}{16} \)

(d) \( \frac{1}{D - 2} e^{3x} = e^x \int e^{3x} \, e^{-2x} \, dx = e^x \int e^{x} \, dx = e^x \cdot e^x = e^{2x} \)

(e) \( \frac{1}{(D - 2)(D - 3)} e^{2x} = \frac{1}{(D - 2)} \left[ \frac{1}{(D - 3)} e^{2x} \right] = \frac{1}{(D - 2)} e^{3x} \int e^{2x} \cdot e^{-3x} \, dx \)

\[ = \frac{1}{(D - 2)} e^{3x} \cdot \int e^{-x} \, dx = \frac{1}{(D - 2)} e^{3x} \cdot (e^{-x}) = \frac{-1}{(D - 2)} e^{2x} \]

\[ = -e^{2x} \int e^{2x} \cdot e^{-2x} \, dx = -e^{2x} \cdot 1 \, dx \]

\[ = -e^{2x} x \]

**6.4.11 SAQ:** Find the particular values of \( \frac{1}{D + 4} \cos x \)

**6.4.12 Working rule to solve differential equation** \( f(D) y = Q \):
Step 1: Write the A.E. of $f(D) = 0$

Step 2: Solve the A.E.

Step 3: Write the complementary functions (CF) using the table in summary of the lesson (5).

Step 4: Write the particular integral (P.I.) using the results 6.4.5, 6.4.6, 6.4.7, and 6.4.9

Step 5: Write the general solutions (or complete solution) as $y = C.F. + P.I.$

6.5 METHODS FOR FINDING THE PARTICULAR INTEGRAL

Consider the differential equations

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = Q$$

This equation can be written as

$$\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n\right)y = Q$$

i.e. $f(D)y = Q$ where $f(D) = D^n + a_1, D^{n-1} + a_2 D^{n-2} + \ldots + a_n$

$$\therefore P.I. = \frac{1}{f(D)} Q$$

6.6 METHOD I

P.I. of $f(D)Y = Q(x)$ when $Q(x) = e^{ax}, a$ is constant

Since $D(e^{ax}) = ae^{ax}$

$$D^2(e^{ax}) = a^2 e^{ax}, \ldots, D^n(e^{ax}) = a^n e^{ax}$$

$$\therefore \left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n\right)e^{ax} = \left(a^n + a_1 a^{n-1} + a_2 a^{n-2} + \ldots + a_n\right)e^{ax}$$

i.e. $f(D)e^{ax} = f(a)e^{ax}$

operating on both sides by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)} [f(D)e^{ax}] = \frac{1}{f(D)} [f(a)e^{ax}]$$

$$e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$$
\[
\Rightarrow \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax} \quad \text{provided} \ f(a) \neq 0
\]

If \( f(a) = 0 \), the above method fails and we proceed further. \( f(a) = 0 \Rightarrow a \) is a root of \( f(D) = 0 \).

\( \Rightarrow (D-a) \) is a factor of \( f(D) \).

By division algorithm, \( f(D) = (D-a)\varphi(D) \) (where \( \varphi(a) \neq 0 \))

Then, \[
\frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)\varphi(D)}e^{ax} = \frac{1}{(D-a)} \left( \frac{1}{\varphi(D)}e^{ax} \right)
\]

\[
= \frac{1}{(D-a)\varphi(a)}e^{ax} \quad (\because \varphi(a) \neq 0)
\]

\[
= \frac{1}{\varphi(a)}e^{ax} \int e^{ax} \cdot e^{-ax} \, dx
\]

\[
= \frac{1}{\varphi(a)}e^{ax} \cdot \int 1 \, dx
\]

\[
= \frac{1}{\varphi(a)}e^{ax} \cdot x
\]

\[
\therefore \frac{1}{f(D)}e^{ax} = x \cdot \frac{1}{f'(a)}e^{ax}
\]

\[
\because f(D) = (D-a)\varphi(D)
\]

\[
f'(D) = (D-a)\varphi'(a) + 1 \cdot \varphi(D)
\]

\[
f'(a) = \varphi(a)
\]

If \( f'(a) = 0 \) then applying above result, we get

\[
\frac{1}{f(D)}e^{ax} = x^2 \cdot \frac{1}{f''(a)}e^{ax} \quad \text{provided} \ f''(a) \neq 0 \quad \text{and so on.}
\]

**6.6.1 Result:**

\[
\frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}
\]

**Proof:** We can take this note as a corollary to 6.4.7 by taking \( Q = e^{ax} \) and defining the operator

\[
\frac{1}{(D-a)^{r-1}} \left[ \frac{1}{(D-a)}Q \right]
\]
6.6.2 SAQ: Find the P.I. of \((D - 2)^3 y = e^{2x}\)

6.6.3 Example: Solve \(\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = e^x\)

Solution: The given equation can be written as

\[
(D^2 - 3D + 2)y = e^x \quad \text{i.e.} \quad f(D)y = e^x \quad \text{where} \quad f(D) = D^2 - 3D + 2.
\]

A.E. is \(m^2 - 3m + 2 = 0 \Rightarrow (m - 1)(m - 2) = 0\)
\[\Rightarrow m = 1, 2\]

\[\therefore \text{C.F.} = y_c = c_1 e^x + c_2 e^{2x}\]

P.I. \(y_P = \frac{1}{f(D)}Q = \frac{1}{D^2 - 3D + 2}e^x = \frac{1}{(D - 1)(D - 2)}e^x\)
\[= \frac{1}{(D - 1)}\left(\frac{1}{(D - 2)}e^x\right)\]
\[= \frac{1}{(D - 1)}\left(\frac{1}{1 - 2}e^x\right) \quad \text{(see Method 6.6)}\]
\[= \frac{-1}{(D - 1)}e^x = \frac{-x'}{1!}e^x = -xe^x \quad \text{(see Result 6.6.1)}\]

General solution is \(y = \text{C.F.} + \text{P.I.}\)
\[y = c_1 e^x + c_2 e^{2x} - xe^x\]

6.6.4 Example: Solve \((D^2 - 4)y = e^{2x} + e^{-4x}\)

Solution: Given equation is \((D^2 - 4)y = e^{2x} + e^{-4x}\) i.e. \(f(D)y = Q\) where

\[f(D) = D^2 - 4, \quad Q = e^{2x} + e^{-4x}\]

A.E. is \(m^2 - 4 = 0 \Rightarrow m = \pm 2\)

\[\text{C.F.} = y_c = c_1 e^{2x} + c_2 e^{-2x}\]
P.I. = \( y_p = \frac{1}{f(D)} Q = \frac{1}{D^2 - 4} \left( e^{2x} + e^{-4x} \right) \)

\[
= \frac{1}{(D^2 - 4)} e^{2x} + \left( \frac{1}{4^2 - 4} \right) e^{-4x}
\]

\[
= \frac{1}{(D - 2)(D + 2)} e^{2x} + \frac{1}{12} e^{-4x}
\]

\[
= \left( \frac{1}{D - 2} \right) \left( \frac{1}{D + 2} e^{2x} \right) + \frac{1}{12} e^{-4x}
\]

\[
= \frac{1}{4} \left( \frac{1}{D - 2} e^{2x} \right) + \frac{1}{12} e^{-4x} = \frac{1}{4} \left( \frac{x}{1!} e^{2x} + \frac{1}{12} e^{-4x} \right)
\]

\[
= \frac{x e^{2x}}{4} + \frac{e^{-4x}}{12}
\]

The general solution is \( y = C.F. + P.I. = y_c + y_p \)

\[
y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x e^{2x}}{4} + \frac{e^{-4x}}{12}
\]

### 6.6.5 Example:

Solve \( \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 5y = -2\cosh x \)

**Solution:**

The given equations can be written as

\[
(D^2 + 4D + 5)y = -2\cosh x
\]

i.e. \( f(D)y = Q \) where \( f(D) = D^2 + 4D + 5, \quad Q = -2\cosh x \)

A.E. is \( m^2 + 4m + 5 = 0 \Rightarrow m = -2 \pm i \)

\[
\therefore \text{C.F.} = y_a = e^{-2x} \left[ c_1 \cos x + c_2 \sin x \right]
\]
Non-Homogeneous linear Differential Equations...

P.I. y_p = \frac{1}{f(D)} Q = \frac{1}{(D^2 + 4D + 5)} \left(-2 \cosh x\right)

= \frac{1}{D^2 + 4D + 5} \left[-Z \cdot \left(\frac{e^x + e^{-x}}{Z}\right)\right] \quad \left(\because \cosh x = \frac{e^x + e^{-x}}{2}\right)

= -\left[\frac{1}{D^2 + 4D + 5} e^x + \frac{1}{D^2 + 4D + 5} e^{-x}\right]

= -\left[\frac{1}{1 + 4 + 5} e^x + \frac{1}{1 - 4 + 5} e^{-x}\right]

= -\left[\frac{1}{10} e^x + \frac{1}{2} e^{-x}\right]

\therefore \text{ The general solution is } y = y_c + y_p

y = e^{-2x} \left[c_1 \cos x + c_2 \sin x\right] - \left(\frac{e^x}{10} + \frac{e^{-x}}{2}\right)

6.7 METHOD 2

P.I. of f(D)y = Q \text{ where } Q = \cos(ax + b) \text{ or } \sin(ax + b):

We know that

D \sin(ax + b) = a \cos(ax + b)

D^2 \sin(ax + b) = -a^2 \sin(ax + b)

D^3 \sin(ax + b) = -a^3 \cos(ax + b)

D^4 \sin(ax + b) = a^4 \sin(ax + b)

i.e. D^2 \sin(ax + b) = -a^2 \sin(ax + b)

\left(D^2\right)^2 \sin(ax + b) = (-a^2)^2 \sin(ax + b)

In general \left(D^2\right)^r \sin(ax + b) = (-a^2)^r \sin(ax + b)

\therefore f(D^2) \sin(ax + b) = f\left(-a^2\right) \sin(ax + b)
Operating on both sides \( \frac{1}{f(D^2)} \), we get

\[
\frac{1}{f(D^2)} \left[ f(D^2) \sin(ax + b) \right] = \frac{1}{f(D^2)} f(-a^2) \sin(ax + b)
\]

or \( \sin(ax + b) = f(-a^2) \frac{1}{f(D^2)} \sin(ax + b) \)

If \( f(-a^2) \neq 0 \) dividing by \( f(-a^2) \), we get

\[
\frac{1}{f(D^2)} \sin(ax + b) = \frac{1}{f(-a^2)} \sin(ax + b)
\]

Hence the rule is replace \( D^2 \) by \(-a^2\) provided \( f(-a^2) \neq 0 \)

If \( f(-a^2) = 0 \), the above method fails and we use the following method,

since \( \cos(ax + b) + i \sin(ax + b) = e^{i(ax+b)} \) (Euler’s theorem)

\[
:\therefore \frac{1}{f(D^2)} \sin(ax + b) = \text{Imaginary part of} \ \frac{1}{f(D^2)} e^{i(ax+b)} \quad (:\because f(-a^2) = 0)
\]

\[
\text{Imaginary part of} \ \frac{1}{f^2(D^2)} e^{i(ax+b)} \quad (\text{where} \ D^2 = -a^2)
\]

\[
\therefore \frac{1}{f(D^2)} \sin(ax + b) = x \frac{1}{f'(-a^2)} \sin(ax + b) \quad \text{provided} \ f'(-a^2) \neq 0 \ . \text{Using the procedure explained in 6.6 and 6.6.1 we get}
\]

If \( f'(-a^2) = 0 \), \( \frac{1}{f(D^2)} \sin(ax + b) = x^2 \frac{1}{f''(-a^2)} \sin(ax + b) \quad \text{provided} \ f''(-a^2) \neq 0 \ etc.

Similarly since \( \cos(ax + b) \) is the real part of \( e^{i(ax+b)} \) we get

\[
\frac{1}{f(D^2)} \cos(ax + b) = \frac{1}{f(-a^2)} \cos(ax + b), \quad \text{provided} \ f(-a^2) \neq 0 \ .
\]
If \( f(-a^2) = 0 \), then
\[
\frac{1}{f(D^2)} \cos(ax + b) = x \cdot \frac{1}{f'(-a^2)} \cos(ax + b); \quad \text{provided } f'(a^2) \neq 0 \quad \text{and so on.}
\]

6.7.1 SAQ: Find P.I. of \( (D^2 + 2D + 2)y = \sin x \)

6.7.2 Alternative Method: If \( f(-a^2) = 0 \). Then \( D^2 + a^2 \) is a factor of \( f(D^2) \). For finding \( \frac{1}{D^2 + a^2} \sin(ax + b) \), the following procedure will be adopted.

We know that, \( e^{jox} = \cos ax + j\sin bx \)

Consider
\[
\frac{1}{D^2 + a^2} e^{j(ax+b)} = \frac{1}{(D-ia)(D+ia)} e^{j(ax+b)}
\]
\[
= \frac{1}{(ia+ia)} \frac{1}{D-ia} e^{j(ax+b)}
\]
\[
= \frac{1}{2ia} \cdot \frac{x'}{1!} e^{j(ax+b)}
\]
\[
= \frac{-i}{2a} \cdot x \cdot \left[ \cos(ax + b) + j\sin(ax + b) \right]
\]

i.e.
\[
\frac{1}{D^2 + a^2} \left[ \cos(ax + b) + j\sin(ax + b) \right] = -\frac{x}{2a} \left[ \cos(ax + b) - \sin(ax + b) \right]
\]

Equating real and imaginary parts we get
\[
\left( \frac{1}{D^2 + a^2} \right) \cos(ax + b) = \frac{x}{2a} \sin(ax + b)
\]
and
\[
\left( \frac{1}{D^2 + a^2} \right) \sin(ax + b) = -\frac{x}{2a} \cos(ax + b)
\]

Formula:
\[
\frac{1}{D^2 + a^2} \sin(ax + b) = \frac{x}{2} \left( -\frac{\cos(ax + b)}{a} \right) = \frac{x}{2} \int \sin(ax + b) dx
\]
\[
\frac{1}{D^2 + a^2} \cos(ax + b) = \frac{x}{2} \left( \frac{\sin(ax + b)}{a} \right) = \frac{x}{2} \int \cos(ax + b) dx
\]
6.7.3 Example : Find the P.I. of \( (p^3 + 1)y = \cos(2x - 1) \)

\[
P.I. = \frac{1}{D^3 + 1} \cos(2x - 1) = \frac{1}{D \cdot D^2 + 1} \cos(2x - 1)
\]

\[
= \frac{1}{D(-4) + 1} \cos(2x - 1) \quad \text{(put } D^2 = -2^2 = -4)\]

\[
= \frac{1}{1 - 4D} \cos(2x - 1)
\]

\[
= \frac{(1 + 4D)}{(1 - 4D)(1 + 4D)} \cos(2x - 1)
\]

\[
= \frac{1 + 4D}{1 - 16D^2} \cos(2x - 1)
\]

\[
= \frac{(1 + 4D) \cos(2x - 1)}{1 - 16(-4)} \quad \text{(put } D^2 = -2^2 = -4) \]

\[
= \cos(2x - 1) - 4D \cos(2x - 1)
\]

\[
= \frac{1}{65} \left[ \cos(2x - 1) - 8 \sin(2x - 1) \right]
\]

6.7.4 Example : Find the P.I. of \( \frac{d^3 y}{dx^3} + 4 \frac{dy}{dx} = \sin 2x \)

Given equation may be written as

\[
(D^3 + 4D)y = \sin 2x
\]

\[
P.I. = \frac{1}{D^3 + 4D} \sin 2x = \frac{1}{D(D^2 + 4)} \sin 2x \quad \left( \because D^2 + 4 = 0 \text{ for } D^2 = -2^2 \quad \frac{d}{dD}(D^3 + 4) = 3D^2 + 4 \right) \]

\[
= x \cdot \frac{1}{3D^2 + 4} \sin 2x
\]

\[
= x \cdot \frac{1}{3(-2^2) + 4} \sin 2x \quad \text{Put } D^2 = -2^2 = -4
\]

\[
= \frac{x}{-8} \sin 2x
\]
### 6.7.5 Example:

Solve \((D^2 - 4)y = \sin^2 x\)

A.E. is \(m^2 - 4 = 0 \Rightarrow m = -2, 2\)

C.F. is \(y_c = c_1 e^{-2x} + c_2 e^{2x}\)

P.I. 

\[y_p = \frac{1}{D^2 - 4} \sin^2 x = \frac{1}{D^2 - 4} \left[ \frac{(1 - \cos 2x)}{2} \right] \]

\[= \frac{1}{D^2 - 4} \left[ \frac{1}{2} e^{0x} - \frac{1}{2} \cos 2x \right] = \frac{1}{D^2 - 4} \left[ \frac{1}{2} e^{0x} - \frac{1}{2} \cos 2x \right] \]

\[= \frac{1}{-4} \frac{1}{2} e^{0x} - \frac{1}{4-4} \frac{1}{2} \cos 2x \]  

(Put \(D^2 = -2^2\) in second)

\[= -\frac{1}{8} + \frac{1}{16} \cos 2x \]

Complete solution is \(y = y_c + y_p\)

\[y = c_1 e^{-2x} + c_2 e^{2x} + \frac{1}{16} \cos 2x - \frac{1}{8} \]

### 6.7.6 Example:

Solve \((D^3 + 2D^2 + D)y = e^{2x} + \sin 2x\)

A.E. is \(m^3 + 2m^2 + m = 0 \Rightarrow m(m+1)^2 = 0 \Rightarrow m = 0, -1, -1\)

C.F. is \(y_c = c_1 e^{0x} + (c_2 + c_3 x)e^{-x} = c_1 + (c_2 + c_3 x)e^{-x}\)

P.I. 

\[y_p = \frac{1}{D^3 + 2D^2 + D} [e^{2x} + \sin 2x] \]

\[= \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x \]

\[= \frac{1}{2^3 + 2 \cdot 2^2 + 2} \frac{e^{2x}}{D(-4) + 2(-4) + D} \]

\[= \frac{1}{18} e^{2x} - \frac{1}{3D + 8} \sin 2x \]
\[
\frac{1}{18}e^{2x} - \frac{3D - 8}{(3D + 8)(3D - 8)}\sin 2x
\]

\[
= \frac{1}{18}e^{2x} - (3D - 8) \cdot \frac{1}{9D^2 - 64}\sin 2x
\]

\[
= \frac{1}{18}e^{2x} - (3D - 8) \cdot \frac{1}{9(-4) - 4}\sin 2x \quad \text{(Put } D^2 = -2^2 = -4\text{)}
\]

\[
= \frac{1}{18}e^{2x} - \frac{1}{100}\sin 2x
\]

\[
= \frac{1}{18}e^{2x} + \frac{3D\sin 2x - 8\sin 2x}{100}
\]

\[
= \frac{1}{18}e^{2x} + \frac{1}{50}[3\cos 2x - 4\sin 2x]
\]

Complete solution is \( y = y_c + y_p \)

\[
y = c_1 + (c_2 + c_3x)e^{-x} + \frac{1}{18}e^{2x} + \frac{1}{50}[3\cos 2x - 4\sin 2x]
\]

6.7.7 Example: Solve \( (D^4 + 3D^2 - 4)y = \cos 3x \cdot \cos 2x \)

A.E. is \( m^4 + 3m^2 - 4 = 0 \Rightarrow (m^2 + 4)(m^2 - 1) = 0 \)

\( m = -1, 1, \pm 2i \)

C.F. is \( y_c = c_1e^{-x} + c_2e^x + c_3\cos 2x + c_4\sin 2x \)

P.I. P.I. = \( \frac{1}{D^4 + 3D^2 - 4}\cos 3x \cdot \cos 2x = \frac{1}{D^4 + 3D^2 - 4}\frac{(\cos 5x + \cos x)}{2} \)

\[
= \frac{1}{D^4 + 3D^2 - 4} \cdot \frac{1}{2}\cos 5x + \frac{1}{D^4 + 3D^2 - 4} \cdot \frac{1}{2}\cos x
\]

\[
= \frac{1}{(-5^2)^2 + 3(-5^2) - 4} \cdot \frac{1}{2}\cos 5x + \frac{1}{(-1)^2 + 3(-1)^2 - 4} \cdot \frac{1}{2}\cos x
\]

\[
= \frac{1}{1092}\cos 5x - \frac{1}{12}\cos x
\]
The complete solution is  
\[ y = y_c + y_p \]

\[ y = c_1 e^{-x} + c_2 e^x + c_3 \cos 2x + c_4 \sin 2x + \frac{1}{1092} \cos 5x - \frac{1}{12} \cos x \]

### 6.8 METHOD 3

**P.I. of** \( f(D)y = Q \text{ when } Q = x^m, m \text{ being +ve integer} :**

Here \( P.I. = \frac{1}{f(D)} = x^m \)

Taking out the lowest degree term from \( f(D) \) to make the first term unity. The remaining factor will be of the form \([1 + \phi(D)] \text{ or } [1 - \phi(D)]\).

Take this factor in the numerator. It takes the form \([1 + \phi(D)]^{-1} \text{ or } [1 - \phi(D)]^{-1}\). Expanding it in ascending powers of \(D\) by using Binomial expansion as far as the term containing \(D^m\), since \(D^{m+1}(x^m) = 0, D^{m+2}(x^m) = 0 \) and so on.

Operate on \( x^m \) term by term

**Note:** The following results are use in finding P.I.

1. \( (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \)
2. \( (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots \)
3. \( (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots \)
4. \( (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \)
5. \( (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \)

**6.8.1 Example:** Solve \((D^2 - 4)y = x^2\)

**Solution:** The given equation can be written by

\[ f(D)y = Q \text{ where } f(D) = D^2 - 4, Q = x^2 \]

A.E. is \( m^2 - 4 = 0 \Rightarrow m = \pm 2 \)
C.F. = $y_c = ce^{2x} + c_2e^{-2x}$

P.I. = \[ \frac{1}{f(D)}Q = \frac{1}{D^2 - 4}x^2 = \frac{1}{-4\left[1 - \frac{D^2}{4}\right]}x^2 \]

\[ = -\frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1}x^2 \]

\[ = -\frac{1}{4}\left[1 + \frac{D^2}{4} + \left(\frac{D^2}{4}\right)^2 + \ldots\right]x^2 \]

\[ = -\frac{1}{4}\left(x^2 + \frac{1}{2}\right) \]

∴ The General solution is \( y = C.F. + P.I \)

\[ y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right) \]

**6.8.2 SAQ:** Find P.I. of \( (D^2 + 3)y = x^3 \)

**6.8.3 Example:** Solve \( (D^2 + 2D + 1)y = 2x + x^2 \)

**Solution:** A.E. is \( m^2 + 2m + 1 = 0 \Rightarrow (m + 1)^2 = 0 \Rightarrow m = -1, -1 \)

∴ C.F. = \( y_c = (c_1 + c_2x)e^{-x} \)

P.I. = \[ \frac{1}{D^2 + 2D + 1}(2x + x^2) \]

\[ = \frac{1}{(D + 1)^2}(2x + x^2) \]

\[ = (1 + D)^{-2}(2x + x^2) \]

\[ = (1 - 2D + 3D^2 + \ldots)(2x + x^2) \]

\[ = 2x + x^2 - 2(2 + 2x) + 3(2) \]


\[ x^2 - 2x + 2 \]

Hence, the general solution is \( y = y_c + y_p \)

\[ y = (c_1 + c_2x)e^{-x} + x^2 - 2x + 2 \]

6.8.4 Example: Solve \( (D^3 + 2D^2 + D)y = x^2 + x \)

Solution: A.E. is \( m^3 + 2m^2 + m = 0 \)

\[ \Rightarrow m(m^2 + 2m + 1) = 0 \]

\[ \Rightarrow m(m + 1)^2 = 0 \]

\[ \Rightarrow m = 0, -1, -1 \]

:. C.F. is \( y_c = c_1 + (c_2 + c_3x)e^{-x} \)

P.I. = \( \frac{1}{D^3 + 2D^2 + D}(x^2 + x) \)

\[ = \frac{1}{D[D^2 + 2D + 1]}(x^2 + x) = \frac{1}{D(D + 1)^2}(x^2 + x) \]

\[ = \frac{1}{D}[1 - 2D + 3D^2 - 4D^3 + \ldots ](x^2 + x) \]

\[ = \frac{1}{D}[x^2 + x - 2(2x + 1) + 3(2)] \]

\[ = \frac{1}{D}(x^2 - 3x + 4) \]

\[ = \int(x^2 - 3x + 4)dx = \frac{x^3}{3} - \frac{3x^2}{2} + 4x \]

:. The general solution is given by

\[ y = y_c + y_p \]

\[ = c_1 + (c_2 + c_3x)e^{-x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x \]
6.8.5 Example: Solve \( \frac{d^3y}{dx^3} + 3 \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = x^2 \)

Solution: The given equation can be written as

\[
(D^3 + 3D^2 + 2D)y = x^2
\]

A.E. is \( m^3 + 3m^2 + 2m = 0 \Rightarrow m(m+1)(m+2) = 0 \)

\( \Rightarrow m = 0, -1, -2 \)

\( \therefore \) CF is \( y_c = c_1 + c_2e^{-x} + c_3e^{-2x} \)

P.I. = \( y_p = \frac{1}{D^3 + 3D^2 + 2D} x^2 \)

\[
= \frac{1}{D(D+1)(D+2)} x^2
\]

\[
= \frac{1}{D} \left( \frac{1}{(1+D)^2} \left( \frac{1}{1+D} \right) \right) x^2
= \frac{1}{2D} (1+D)^{-1} \left( 1 + \frac{D}{2} \right)^{-1} x^2
\]

\[
= \frac{1}{2D} (1+D)^{-1} \left( 1 - \frac{D}{2} + \frac{D^2}{4} + \cdots \right) x^2
= \frac{1}{2D} \left( 1 - D + D^2 - D^3 + \cdots \right) \left( x^2 - x + \frac{1}{2} \right)
\]

\[
= \frac{1}{2D} \left( x^2 - x + \frac{1}{2} - 2x + 1 \right)
= \frac{1}{2D} \left( x^2 - 3x + \frac{7}{2} \right)
= \frac{1}{4D} \left( 2x^2 - 6x + 7 \right)
\]

\[
= \frac{1}{4} \int (2x^2 - 6x^2 + 7) \, dx
= \frac{1}{4} \left( \frac{2x^3}{3} - \frac{6x^2}{2} + 7x \right)
= \frac{1}{12} \left( 2x^3 - 9x^2 + 21x \right)
\]

\( \therefore \) The general solution is \( y = y_c + y_p \)

\[
y = c_1 + c_2e^{-x} + c_3e^{-2x} + \frac{1}{12} \left( 2x^3 - 9x^2 + 21x \right)
\]
6.9 METHOD 4

P.I. of \( f(D) y = Q \) when \( Q = e^{ax} V, \ V \) being a function of \( x \):

Here, 
\[
P.I. = \frac{1}{f(D)}\left(e^{ax} V\right) = e^{ax} \cdot \frac{1}{f(D + a)} V
\]

Proof: If \( u \) is a function of \( x \), then
\[
D\left(e^{ax} u\right) = e^{ax} Du + ae^{ax} u = e^{ax} (D + a)u
\]
\[
D^2\left(e^{ax} u\right) = e^{ax} D^2u + 2ae^{ax} Du + a^2e^{ax} u = e^{ax} (D + a)^2 u
\]
and in general \( D^n\left(e^{ax} u\right) = e^{ax} (D + a)^n u \)

\[
\therefore f(D)\left(e^{ax} u\right) = e^{ax} f(D + a) u
\]

operating both sides by \( \frac{1}{f(D)} \), we get
\[
\frac{1}{f(D)}\left(f(D)\left(e^{ax} u\right)\right) = \frac{1}{f(D)}\left[e^{ax} f(D + a) u\right]
\]
\[
e^{ax} u = \frac{1}{f(D)}\left[e^{ax} \cdot f(D + a) u\right]
\]

Now, put \( f(D + a) u = V \) i.e. \( u = \frac{1}{f(D + a)} V \), so that
\[
e^{ax} \frac{1}{f(D + a)} V = \frac{1}{f(D)}\left(e^{ax} \cdot V\right)
\]

i.e. 
\[
\frac{1}{f(D)}\left(e^{ax} V\right) = e^{ax} \cdot \frac{1}{f(D + a)} V
\]

6.9.1 Examples:

(1) Solve \( (D^2 - 4D + 3)y = e^{-x} \sin x \)

Solution: A.E. of the given equation is 
\[
m^2 - 4m + 3 = 0 \Rightarrow (m - 1)(m - 3) = 0
\]
\[ \Rightarrow m = 1, 3 \]

\[ \therefore \text{C.F. is } y_c = c_1 e^x + c_2 e^{3x} \]

P.I. \[ y_p = \frac{1}{D^2 - 4D + 3} (e^{-x} \sin x) \]

\[ = e^{-x} \frac{1}{(D-1)^2 - 4(D-1) + 3} \sin x \]

\[ = e^{-x} \frac{1}{D^2 - 6D + 8} \sin x \]

\[ = e^{-x} \cdot \frac{1}{-1^2 - 6D + 8} \sin x \]

\[ = e^{-x} \cdot \frac{1}{7 - 6D} \sin x \]

\[ = e^{-x} \cdot \frac{(7 + 6D)}{49 - 36D^2} \sin x \quad (\text{rationalising the denominator}) \]

\[ = e^{-x} \cdot \frac{(7 + 6D)}{49 - 36(-1)^2} \sin x \quad (\text{Put } D^2 = -1^2) \]

\[ = e^{-x} \cdot \frac{1}{85} (7 + 6D) \sin x \]

\[ = \frac{e^{-x}}{85} (7 \sin x + 6 \cos x) \]

\[ \therefore \text{The general solution of the given equation is} \]

\[ y = y_c + y_p \]

\[ = c_1 e^x + c_2 e^{3x} + \frac{e^{-x}}{85} (7 \sin x + 6 \cos x) \]

**6.9.2 Example:** Solve \((D^2 + 4)y = x \cdot e^{2x}\)

**Solution:** A.E. of the given equation is

\[ m^2 + 4 = 0 \Rightarrow m = \pm 2i \]
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C.F. is

\[ y_c = c_1 \cos 2x + c_2 \sin 2x \]

P.I. = \( y_p = \frac{1}{D^2 + 4} xe^{2x} = e^{2x} \frac{1}{(D + 2)^2 + 4} x \)

\[ = e^{2x} \cdot \frac{1}{D^2 + 4D + 8} x = e^{2x} \frac{1}{8 \left( D + \frac{D^2 + 4D}{8} \right)} x \]

\[ = e^{2x} \frac{8}{8} \left[ 1 - \left( \frac{D^2 + 4D}{8} \right) + \left( \frac{D^2 + 4D}{8} \right)^2 - \cdots \right] x \]

\[ = e^{2x} \frac{8}{8} \left[ 1 - \frac{D}{2} \right] x = e^{2x} \frac{8}{8} \left( x - \frac{1}{2} \right) = \frac{e^{2x}}{16} (2x - 1) \]

\[ \therefore \text{ The general solution of the given equation is} \]

\[ y = y_c + y_p \]

\[ = c_1 \cos 2x + c_2 \sin 2x + \frac{e^{2x}}{16} (2x - 1) \]

6.9.3 Example: Solve \( (D^3 - 3D^2 + 3D - 1) y = (x^2 + 1)e^x \)

Solution: A.E. of the given equation is

\[ m^3 - 3m^2 + 3m - 1 = 0 \Rightarrow (m - 1)^3 = 0 \]

\[ \Rightarrow m = 1, 1, 1 \]

C.F. is

\[ y_c = \left( c_1 + c_2 x + c_3 x^2 \right) e^x \]

P.I. = \( y_p = \frac{1}{D^3 - 3D^2 + 3D - 1} (x^2 + 1)e^x = \frac{1}{(D - 1)^3} (x^2 + 1)e^x \)

\[ = e^x \cdot \frac{1}{(D + 1 - 1)^3} (x^2 + 1) = e^x \cdot \frac{1}{D^3} (x^2 + 1) \]

\[ = e^x \left[ \frac{x^5}{60} + \frac{x^3}{6} \right] = \frac{e^x \cdot x^3}{60} \left[ x^2 + 10 \right] \]

\[ \therefore \text{ The general solution of the given equation is} \]

\[ y = y_c + y_p \]
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6.9.4 Example: Solve \((D^3 - 2D^2 + 2D)y = e^x \cdot \cos x\)

Solution: A.E. of the given equation is

\[ m^3 - 2m^2 + 2m = 0 \Rightarrow m(m^2 - 2m + 2) = 0 \]

\[ \Rightarrow m = 0, 1 \pm i \]

C.F. is \(y_c = c_1 + e^x (c_2 \cos x + c_3 \sin x)\)

P.I. = \(y_p = \frac{1}{D^3 - 2D^2 + 2D} e^x \cdot \cos x = \frac{1}{D(D^2 - 2D + 2)} e^x \cos x\)

\[ = e^x \frac{1}{(D + 1)[(D + 1)^2 - 2(D + 1) + 2]} \cos x \]

\[ = e^x \frac{1}{(D + 1)[D^2 + 1]} \cos x \]

\[ = e^x \frac{D - 1}{(D^2 - 1)(D^2 + 1)} \cos x \]

\[ = e^x \frac{(D - 1)}{(-1^2 - 1)(D^2 + 1)} \cos x \quad \text{(put } D^2 = -1) \]

\[ = e^x \frac{-\sin x - \cos x}{-2(D^2 + 1)} \]

\[ = e^x \frac{1}{2D^2 + 1} (\sin x + \cos x) \]

\[ = e^x \frac{x}{2} [\sin x + \cos x] = \frac{xe^x}{4} (\sin x - \cos x) \quad \left( \because \frac{1}{D^2 + a^2} \sin x = \frac{x}{2} \int \sin x \, dx \right) \]

\[ \frac{1}{D^2 + a^2} \cos x = \frac{x}{2} \int \cos x \, dx \]

\[ \therefore \text{The general solution is} \]

\[ y^2 = y_c + y_p = c_1 + e^x (c_2 \cos x + c_3 \sin x) + \frac{xe^x}{4} (\sin x - \cos x) \]
6.10 METHOD 5

P.I. of $f(D)y = Q$. When $Q = x \cdot v, v$ is a function of $x$.

Here $P.I. = \frac{1}{f(D)}(x \cdot v) = \left[ x \cdot \frac{1}{f(D)} - \frac{f'(D)}{[f(D)]^2} \right] v$

**Proof**: If $u$ is a function of $x$ then

$$D(ux) = xDu + u = xDu + (D)'u$$

$$D^2(ux) = xD^2u + 2Du = xD^2u + (D^2)'u$$

By using mathematical induction, we can prove that

$$D^n(ux) = xD^n u + (D^n)'u$$

$$\therefore f(D)(ux) = xf(D)u + (f(D))'u$$

Now, put $f(D)u = v$, then

$$f(D)\left[ x \cdot \frac{1}{f(D)}v \right] = x\left[ f(D) \cdot \frac{1}{f(D)}v \right] + (f(D))' \cdot \frac{1}{f(D)^2}v$$

$$= x \cdot v + (f(D))' \cdot \frac{1}{f(D)^2}v$$

$$x \cdot \frac{1}{f(D)}v = \frac{1}{f(D)}(x \cdot v) + \frac{1}{f(D)}(f(D))' \cdot \frac{1}{f(D)}v$$

$$\frac{1}{f(D)}(x \cdot v) = x \cdot \frac{1}{f(D)}v - \frac{(f(D))'}{[f(D)]^2}v$$

$$\therefore \frac{1}{f(D)}(xv) = \left[ x \cdot \frac{1}{f(D)} - \frac{f'(D)}{[f(D)]^2} \right] v$$

6.10.1 Example: Solve $(D^2 + 2D + 1)y = x \sin x$

**Solution**: A.E. is $m^2 + 2m + 1 = 0 \Rightarrow (m + 1)^2 = 0 \Rightarrow m = -1, -1$
\[ y_c = (c_1 + c_2 x)e^{-x} \]

\[ \text{P.I.} = y_p = \frac{1}{D^2 + 2D + 1} \sin x = \left[ \frac{x}{D^2 + 2D + 1} - \frac{2D + 2}{D^2 + 2D + 1} \right] \sin x \]

\[ = x \frac{1}{-1 + 2D + 1} \sin x - \frac{2D + 2}{(-1 + 2D + 1)^2} \sin x \quad \text{(Put } D^2 = -1) \]

\[ = x \frac{1}{2D} \sin x - \frac{2(D + 1)}{4D^2} \sin x = \frac{x}{2} \int \sin x \, dx - \frac{(D + 1)}{2(-1)} \sin x \]

\[ = \frac{x}{2} (-\cos x) + \frac{1}{2} (\cos x + \sin x) = \frac{-x \cos x}{2} + \frac{1}{2} (\cos x + \sin x) \]

\[ \therefore \text{ General solution is } y = y_c + y_p \]

\[ y = (c_1 + c_2 x)e^{-x} - \frac{x}{2} \cos x + \frac{1}{2} (\cos x + \sin x) \]

6.10.2 Example : Solve \((D^2 + 3D + 2)y = xe^x \cos x\)

Solution : The A.E. of the given equation is \(m^2 + 3m + 2 = 0 \Rightarrow (m + 1)(m + 2) = 0 \Rightarrow m = -1, -2\)

\[ \text{C.F. is } y_c = c_1 e^{-x} + c_2 e^{-2x} \]

\[ \text{P.I.} = y_p = \frac{1}{D^2 + 3D + 2} xe^x \cos x = \frac{1}{(D + 1)(D + 2)} xe^x \cos x \]

\[ = e^x \cdot \frac{1}{(D + 1 + 1)(D + 1 + 2)} x \cos x \]

\[ = e^x \cdot \frac{1}{(D + 2)(D + 3)} x \cos x \]

\[ = e^x \cdot \frac{1}{D^2 + 5D + 6} x \cos x \]

\[ = e^x \left[ x \cdot \frac{1}{D^2 + 5D + 6} - \frac{2D + 5}{(D^2 + 5D + 6)^2} \right] \cos x \]
\[
\begin{align*}
&= e^x \left[ x \cdot \frac{1}{-1+5D+6} - \frac{2D+5}{(-1+5D+6)^2} \right] \cos x \\
&= e^x \left[ \frac{x}{5(D+1)} - \frac{2D+5}{25(D+1)^2} \right] \cos x \\
&= e^x \left[ \frac{x(D-1)}{5(D^2-1)} - \frac{2D+5}{25(D^2+2D+1)} \right] \cos x \\
&= e^x \left[ \frac{x(D-1)}{5(-1-1)} - \frac{2D+5}{25(-1+2D+1)} \right] \cos x \\
&= e^x \left[ \frac{x(D-1)}{5(-2)} \cos x - \frac{(2D+5)}{25 \cdot 2D} \cos x \right] \\
&= e^x \left[ \frac{x}{10} (\sin x + \cos x) - \frac{(2D+5)}{50} \sin x \right] \\
&= e^x \left[ \frac{x}{10} (\sin x + \cos x) - \frac{1}{50} (2 \cos x + 5 \sin x) \right] \\
\end{align*}
\]

General solution of the given equation is
\[
\begin{align*}
&= y_c + y_p \\
&= c_1 e^{-x} + c_2 e^{-2x} + e^x \left[ \frac{x}{10} (\sin x + \cos x) - \frac{1}{50} (2 \cos x + 5 \sin x) \right]
\end{align*}
\]

6.10.3 Example: Solve \( (D^2 + 1) y = x \cos^2 x \)

Solution: A.E. of the given equation is
\[
m^2 + 1 = 0 \Rightarrow m = \pm i
\]
C.F. is \( y_c = c_1 \cos x + c_2 \sin x \)

P.I. = \( y_p = \frac{1}{(D^2 + 1)} x \cos^2 x = \frac{1}{D^2 + 1} x \left( \frac{1 + \cos 2x}{2} \right) \)

\[
= \frac{1}{2} \left[ \frac{1}{D^2 + 1} x + \frac{1}{D^2 + 1} x \cos 2x \right]
\]
\[
\begin{align*}
&= \frac{1}{2} \left[ (1 + D^2)^{-1} x + x - \frac{1}{D^2 + 1} \cos 3x - \frac{2D}{(p^2 + 1)^2} \cos 2x \right] \\
&= \frac{1}{2} \left[ x + \frac{x}{-4 + 1} \cos 2x - \frac{2D}{(-4 + 1)^2} \cos 2x \right] \\
&= \frac{1}{2} \left[ x - \frac{x}{3} \cos 2x + \frac{4}{9} \sin 2x \right] \\
\therefore (1 + D^2)^{-1} x = (1 - D^2 + D^4 - \ldots) x = x \\
\therefore \text{cos 2x} = -2 \sin 2x
\end{align*}
\]

\[
\therefore \text{General solution of the given equation is}
\]
\[
y = y_c + y_p
\]
\[
c_1 \cos x + c_2 \sin x + \frac{1}{2} \left[ x - \frac{x}{3} \cos 2x + \frac{4}{9} \sin 2x \right]
\]

6.10.4 Example: Solve \(\frac{d^3 y}{dx^4} + 2 \frac{d^2 y}{dx^2} + y = x^2 \cos x\)

Solution: The given equation in operator form is
\[
(D^4 + 2D^2 + 1)y = x^2 \cos x
\]
\[
(D^2 + 1)^2 y = x^2 \cos x
\]
A.E. is \((m^2 + 1)^2 = 0 \Rightarrow m^2 = -1, m^2 = -1\)
\[
\Rightarrow m = \pm i, \pm i
\]
The C.F. is \(y_c = e^{0x} \{ (c_1 + c_2 x) \cos x +(c_3 + c_4 x) \sin x \}\)
\[
= (c_1 + c_2 x) \cos x +(c_3 + c_4 x) \sin x
\]
P.I. = \(y_p = \frac{1}{(D^2 + 1)^2} x^2 \cos x = \frac{1}{(D^2 + 1)^2} x^2 \) real part of \(e^{ix}\)
\[
= \text{R.P. of} \quad \frac{1}{(D^2 + 1)^2} x^2 e^{ix}
\]
\[
= \text{R.P. of} \quad e^{ix} \cdot \frac{1}{(D + i)^2 + 1} x^2 \quad \text{(R.P. means real part)}
\]
\[ \frac{1}{D^2 + 2iD + i^2 + 1} \]

= R.P. of \( e^{ix} \)

= R.P. of \( \frac{1}{D^2 + 2iD + 1} \)

= Real R.P. of \( \frac{1}{D^2 + 2iD + 1} \)

= R.P. of \( \frac{1}{4i^2D^2[1 + D + \frac{D^2}{2i}]^2} \)

= R.P. of \( \frac{1}{4(-1)D^2[1 - iD + \frac{D^2}{2}]^2} \)

= \(-\frac{1}{4}\) Real P. of \( \frac{1}{D^2[1 + iD + 3i^2D^2/4 - \cdots]} \)

= \(-\frac{1}{4}\) R.P. of \( \frac{1}{D^2[1 + iD + 3i^2D^2/4 - \cdots]} \)

= \(-\frac{1}{4}\) R.P. of \( \frac{1}{D^2[1 + 2iD - 3D/2]} \)

= \(-\frac{1}{4}\) R.P. of \( \frac{1}{D} \left[ (x^2 + 2ix - 3/2) \right] \)

= \(-\frac{1}{4}\) R.P. of \( \frac{1}{D} \left[ (x^3 + ix^2 - 3/2x) \right] \)

= \(-\frac{1}{4}\) R.P. of \( \frac{1}{4} \left[ x^4/12 + ix^3/3 - 3/4x^2 \right] \)

= \(-\frac{1}{4}\) R.P. of \( \left( \cos x + i\sin x \right) \left( \frac{x^4/12 + ix^3/3 - 3/4x^2} \right) \)

= \(-\frac{1}{4}\) \left( \frac{x^4/12}{\cos x - 3/4x^2 \cos x - x^3/3 \sin x} \right)
General solution is \( y = y_c + y_p \)

\[
y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x - \frac{1}{4}\left(\frac{x^4}{12}\cos x - \frac{3}{4}x^2\cos x - \frac{x^3}{3}\sin x\right)
\]

**6.10.5 Example :** Solve \((D^3 - D^2 + 3D + 5)y = e^x \sin 3x\)

**Solution :** A.E. of the given equation is

\[
m^3 - m^2 + 3m + 5 = 0
\]

\[
\Rightarrow (m+1)(m^2 - 2m + 5) = 0
\]

\[
\Rightarrow m = -1, 1 \pm 2i
\]

C.I. is \( y_c = c_1e^{-x} + e^x (c_2 \cos 2x + c_3 \sin 2x) \)

P.I. = \( y_p = \frac{1}{D^3 - D^2 + 3D + 5}e^x \sin 3x \)

\[
= \frac{1}{(D+1)(D^2 - 2D + 5)}e^x \sin 3x
\]

\[
= e^x \cdot \frac{1}{(D+1)(D^2 + 2)} \sin 3x
\]

\[
= e^x \cdot \frac{(D - 2)}{(D^2 - 4)(D^2 + 4)} \sin 3x
\]

\[
= e^x \cdot \frac{(D - 2) \sin 3x}{(-9 - 4)(-9 + 4)}
\]

(put \( D^2 = -3^2 = -9 \))

\[
= \frac{e^x}{65} (3\cos 3x - 2\sin 3x)
\]

\[
\therefore \text{ The general solution is } y = y_c + y_p
\]

\[
y = c_1e^{-x} + e^x (c_2 \cos 2x + c_3 \sin 2x) + \frac{e^x}{65} (3\cos 3x - 2\sin 3x)
\]
6.12 SUMMARY

In this lesson we discussed the various types of finding P.I. of non-homogeneous linear differential equations with constant coefficients, while proving the relevant theorems. Some problems are also discussed.

6.13 TECHNICAL TERMS

Non-homogeneous linear differential equation, particular integral or particular solution, Inverse operator.

6.14 EXERCISE

Solve the following equations

1. \( \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = e^x \)
2. \( (D^3 - 5D^2 + 7D - 3)y = e^{2x} \cosh x \)
3. \( (D^3 - 4D^2)y = 8 \)
4. \( (D^3 - 12D + 16)y = (e^{-2x} + e^x)^2 \)
5. \( (D^2 + D + 1)y = \sin 2x \)
6. \( (D^2 - 4)y = \sin^2 x \)
7. \( (D^3 + 2D^2 + D)y = e^{2x} + \sin 2x \)
8. \( (D^2 - 8D + 9)y = 8 \cos 5x \)
9. \( (D^2 + 1)y = x \)
10. \( (D^2 + 3D + 2)y = x^2 \)
11. \( (D^2 - 4D + 4)y = 8x^2 \)
12. \( (D^2 + 3D + 2)y = e^{-x} + \cos x + x^2 \)
13. \( \frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 13y = 8e^{3x} \sin 2x \)
14. \( (D^2 - 3D + 2)y = xe^{3x} + \sin 2x \)
15. \( \frac{d^2y}{dx^2} - 4y = x \sinh x \)

16. \( (D^2 - 2D + 1)y = xe^x \sin x \)

17. \( (D^2 - 1)y = x^2 e^x + x \sin x \)

18. \( (D^2 + 4)y = x \sin x \)

19. \( (D^3 - 7D + 6)y = (x + 1)e^{2x} \)

20. \( (D^2 - 7D + 6)y = (1 + x)e^{2x} \)

### 6.11 ANSWERS TO SAQ

**6.4.11:**

\[
\frac{1}{D+4} \cos x = \frac{1}{D-(-4)} \cos x
\]

\[
= e^{-4x} \int \cos x \cdot e^{4x} \, dx
\]

\[
= e^{-4x} \left[ \frac{e^{4x} \sin x + 4e^{4x} \cos x}{17} \right]
\]

\[
= \frac{\sin x}{17} + \frac{4 \cos x}{17}
\]

**6.6.1:**

P.I. = \( \frac{1}{(D-2)^3} e^{2x} \)

\[
= \frac{x^3}{3!} e^{2x}
\]

\[
\therefore \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}
\]

\[
= \frac{x^3}{6} e^{2x}
\]

**6.7.1:**

P.I. = \( \frac{1}{D^2 + 2D + 2} \sin x \)

\[
= \frac{1}{-1 + 2D + 2} \sin x \quad \text{(put } D^2 = -1^2 = -1)\]
\[
= \frac{1}{1+2D} \sin x \\
= \frac{1-2D}{(1+2D)(1-2D)} \sin x \\
= \frac{(1-2D)}{1-4D^2} \sin x \\
= \frac{(1-2D) \sin x}{1-4(-1)} = \frac{\sin x - 2 \cos x}{5}
\]

6.8.2 SAQ: P.I. = \[
= \frac{1}{D^2 + 3} x^3 = \frac{1}{3} \left[ \frac{1}{1 + \frac{D^2}{3}} \right] x^3 \\
= \frac{1}{3} \left( 1 + \frac{D^2}{3} \right)^{-1} x^3 \\
= \frac{1}{3} \left( 1 - \frac{D^2}{3} + \frac{D^4}{9} - \cdots \right) x^3 \\
= \frac{1}{3} \left( x^3 - \frac{6x}{3} \right) \\
= \frac{1}{3} \left( x^3 - 2x \right)
\]

6.16 MODEL QUESTIONS

1. Solve \((D^2 - 4D + 3)y = \sin 3x \cos 2x\)
2. Solve \((D^2 + 16)y = e^{-3x} + \cos 4x\)
3. Solve \((D^2 + a^2)y = \cos ax\)
4. Solve \((D^2 + 1)y = x^2 \sin 2x\)
5. Solve \((D^4 - 1)y = e^x \cos x\)
6. Solve \((D^4 - 2D^3 + D^2)y = x^3\)
6.15 ANSWERS TO EXERCISE

1. \[ y = (c_1 + c_2 x) e^x + \frac{x^2}{2} e^x \]

2. \[ y = (c_1 + c_2 x) e^x + c_3 e^{3x} + \frac{x e^{3x}}{8} - \frac{x^2}{8} e^x \]

3. \[ y = c_1 + c_2 x + c_3 e^{4x} - x^2 \]

4. \[ y = (c_1 + c_2 x) e^{2x} + c_3 e^{-4x} + \frac{x^2 e^x}{12} + \frac{x e^{-4x}}{36} + \frac{2e^{-x}}{27} \]

5. \[ y = e^{-x/2} \left[ c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x \right] - \frac{1}{13} (2 \cos 2x + 3 \sin 2x) \]

6. \[ y = c_1 e^{-2x} + c_2 e^{-2x} + \frac{1}{16} \cos 2x - \frac{1}{8} \]

7. \[ y = c_1 + (c_2 + c_3 x) e^{-x} + \frac{e^{2x}}{18} + \frac{1}{50} [3 \cos 2x - 4 \sin 2x] \]

8. \[ y = e^{4x} \left[ c_1 \cos \sqrt{7}x + c_2 \sin \sqrt{7}x \right] - \frac{1}{29} [2 \cos 5x + 5 \sin 5x] \]

9. \[ y = c_1 \cos x + c_2 \sin x + x \]

10. \[ y = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2} \left( x^2 - 3x + \frac{7}{2} \right) \]

11. \[ y = (c_1 + c_2 x) e^{2x} + 2x^2 + 4x + 3 \]

12. \[ y = c_1 e^{-x} + c_2 e^{-2x} + xe^{-x} + \frac{x^2}{2} - \frac{3x}{2} + \frac{7}{4} + \frac{1}{10} (\cos x + 3 \sin x) \]

13. \[ y = e^{3x} (c_1 \cos 2x + c_2 \sin 2x) - 2xe^{3x} \cdot \cos 2x \]

14. \[ y = c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{2} \left( x - \frac{3}{2} \right) + \frac{1}{20} (3 \cos 2x - \sin 2x) \]

15. \[ y = c_1 e^{-2x} + c_2 e^{-2x} - \left( \frac{x}{3} \sinh x + \frac{2}{9} \cosh x \right) \]
16. \[ y = (c_1 x + c_2) e^x + \frac{1}{2} e^x \left[ x \cos x + \cos x - \sin x \right] \]

17. \[ y = c_1 e^x + c_2 e^{-x} + \frac{e^x}{2} \left[ \frac{x^3}{3} - \frac{x^2}{2} + x \right] - \frac{x}{2} \sin x - \frac{x}{2} \cos x \]

18. \[ y = c_1 \cos x + c_2 \sin x + \frac{x}{3} \sin x - \frac{2}{9} \cos x \]

19. \[ y = c_1 e^{-3x} + c_2 e^x + c_3 e^{2x} + \frac{1}{50} e^{2x} \left( 5x^2 - 2x \right) \]

20. \[ y = c_1 e^{2x} + c_2 e^{6x} - \frac{x e^{2x}}{4} - \frac{1}{16} e^{2x} \]

6.17 REFERENCES

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Lesson - 7

LINEAR EQUATIONS OF THE SECOND ORDER WITH VARIABLE COEFFICIENTS

7.1 OBJECTIVE OF THE LESSON

In previous lessons, we learned how to solve linear differential equations with constant coefficients. In this lesson we will learn how to solve linear differential equations of the second order with variable coefficients.

7.2 STRUCTURE OF THE LESSON

This lesson has the following components.

7.3 Introduction
7.4 Cauchy - Euler Equation
7.5 Legendre’s linear differential equation
7.6 General solution of second order linear differential equations with variable coefficients
7.7 Normal form
7.8 Method of variation of parameters
7.9 Answers to Self Assessment Questions (SAQs)
7.10 Summary
7.11 Technical terms
7.12 Exercises
7.13 Model Examination Questions

7.3 INTRODUCTION

In this lesson we will learn the following methods for solving linear equation of the second order with variable coefficients.

\[
\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = R(x) \quad \text{(1)}
\]

where \( P(x), Q(x) \) & \( R(x) \) are function of \( x \).

Some of the equations discussed are given below:
Cauchy-Euler equation:

Legendre's linear equation:

- General solution of (1) when one solution is known.
- General solution of (1) by changing dependent variable and removing the first derivative.
- General solution of (1) by changing independent variable. Method of variation of parameters.

### 7.4 The Cauchy-Euler-Equation

An equation of the form

\[
x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_{n-1} x \frac{dy}{dx} + P_n y = Q(x)
\]

where \(P_1, P_2, \ldots, P_n\) are constants is called a Cauchy Euler equation of order \(n\) and its operator form is

\[
\left[x^n D^n + P_1 x^{n-1} D^{n-1} + \cdots + P_{n-1} x D + P_n \right] y = Q(x)
\]

where \(D \equiv \frac{d}{dx}\). To find the solution of such an equation, we make some suitable substitution so that equation (1) may reduce to an equation for which the methods of solutions are known. The substitution \(x = e^t\) reduces equation (1) to a linear equations with constant coefficients.

Take \(x = e^t\). Then \(\log x = t\).

\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt}
\]

\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{1}{x} \frac{dy}{dt} \right) = \frac{1}{x} \frac{d}{dx} \left( \frac{dy}{dt} \right) + \frac{dy}{dt} \frac{d}{dx} \left( \frac{1}{x} \right)
\]

\[
= \frac{1}{x} \frac{d}{dt} \left( \frac{dy}{dt} \right) \frac{dt}{dx} + \frac{dy}{dt} \left( -\frac{1}{x^2} \right)
\]

\[
= \frac{1}{x^2} \left( \frac{d^2 y}{dt^2} \cdot \frac{dy}{dt} \right)
\]

Thus, \(x \frac{dy}{dx} = \frac{dy}{dt}, x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} = \frac{dy}{dt}\)

Similarly,
\[
\frac{d^3 y}{dx^3} = \frac{1}{x^3} \left[ \frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} \right]
\]

We take \( \frac{d}{dx} \equiv D \). Let the differential operator \( \frac{d}{dt} \) be denoted by \( \theta \) so that \( \frac{d}{dt} \equiv \theta \). Then
\[
\frac{d^2}{dt^2} \equiv \theta^2, \quad \frac{d^3}{dt^3} = \theta^3, \ldots, \quad \frac{d^n}{dt^n} = \theta^n
\]

Hence, \( x \frac{dy}{dx} = xDy = 0y \)
\[
x^2 \frac{d^2 y}{dx^2} = x^2 D^2 y = (\theta^2 y - \theta y) = \theta(\theta - 1)y
\]
\[
x^3 \frac{d^3 y}{dx^3} = x^3 D^3 y = \theta^3 y - 3\theta^2 y + 2\theta y = \theta(\theta - 1)(\theta - 2)y
\]
\[
x^n \frac{d^n y}{dx^n} = x^n D^n y = [\theta(\theta - 1)(\theta - 2) \ldots (\theta - (n - 1))]y
\]

substituting these values in equation (1) we get
\[
\theta(\theta - 1) \ldots (\theta - (n - 1))y + P_1 \theta(\theta - 1) \ldots (\theta - (n - 2))y + \ldots + P_n y = Q(e^t)
\]

or \( \left[ \theta(\theta - 1) \ldots (\theta - n + 1) + P_1 \theta(\theta - 1) \ldots (\theta - n + 2) + \ldots + P_n \right]y = Q(e^t) \)

i.e \( f(\theta)y = R(t) \)

when \( f(\theta) = \theta(\theta - 1) \ldots (\theta - n + 1) + P_1 \theta(\theta - 1) \ldots (\theta - n + 2) + \ldots + P_n \)

and \( R(t) = Q(e^t) \)

This is a linear differential equation with constant coefficients and can be solved by using the methods discussed in lesson 6.

7.4.1 Example: Solve \( x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^5 \)

Solution: Put \( x = e^t \).

Let \( \theta \equiv \frac{d}{dt} \). Then the given equation becomes
\[ \theta(\theta - 1)y - 40y + 6y = (e^t)^5 \]

i.e. \( (\theta^2 - 50 + 6)y = e^{5t} \)

A.E. is \( m^2 - 5m + 6 = 0 \Rightarrow (m - 2)(m - 3) = 0 \Rightarrow m = 2, 3 \)

\[ \therefore \text{CF is } y_c = c_1e^{2t} + c_2e^{3t} = c_1x^2 + c_2x^3 \]

\[ \text{P.I. } y_p = \frac{1}{\theta^2 - 50 + 6}e^{5t} = \frac{1}{5^2 - 5 \cdot 5 + 6}e^{5t} = \frac{1}{6}e^{5t} = \frac{x^5}{6} \]

\[ \therefore \text{The general solution is } y = y_c + y_p \]

\[ y = c_1x^2 + c_2x^3 + \frac{x^5}{6} \]

**7.4.2 Example:** Solve \( x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = 2\log x \)

**Solution:** Put \( x = e^t \) then \( t = \log x \)

Let \( \theta \equiv \frac{d}{dt} \). Then the given equation becomes

\[ (\theta - 1)y - 20y - 4y = 2t \]

\[ (\theta^2 - 30 - 4)y = 2t \]

A.E. is \( m^2 - 3m - 4 = 0 \Rightarrow (m - 4)(m + 1) = 0 \Rightarrow m = 4, -1 \)

C.F is \( y_c = c_1e^{4t} + c_2e^{-t} = c_1x^4 + \frac{c_2}{x} \)

\[ \text{P.I. is } y_p = \frac{1}{\theta^2 - 30 - 4}2t = \frac{1}{-4\left[\frac{\theta^2 - 30}{4}\right]2t} \]

\[ = \frac{2}{-4\left[\frac{\theta^2 - 30}{4}\right]}^{-1}t \]

\[ = \frac{1}{2\left[1 + \frac{\theta^2 - 30}{4} + \cdots\right]}t \]
7.5 Differential Equation, Abstract Algebra...

7.4.3 Example: Solve

\[ \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \cos(\log x) \]

Solution: Put \( x = e^t \) then \( t = \log x \). Let \( \theta = \frac{d}{dt} \) the given equation becomes

\[ \theta(\theta-1)(\theta-2)y + 3\theta(\theta-1)y + \theta y + 8y = 65 \cos t \]

i.e. \( (\theta^3 + 8)y = 65 \cos t \)

A.E. is \( m^3 + 8 = 0 \)

\[ (m + 2)(m^2 - 2m + 4) = 0 \]

\[ m = -2, 1, \pm 2i \]

C.F. is \( y_c = c_1 e^{-2t} + e^t \left[ c_2 \cos(2t) + c_3 \sin(2t) \right] \)

\[ = c_1 \left( \frac{1}{x^2} \right) + x \left[ c_2 \cos(\sqrt{3} \log x) + c_3 \sin(\sqrt{3} \log x) \right] \]

P.I. = \( y_p = \frac{1}{\theta^3 + 8} \cdot 65 \cos t \)

\[ = \frac{1}{\theta(-1)^2 + 8} \cdot 65 \cos t = \frac{1}{\frac{8 + \theta}{64 - \theta^2}} \cdot 65 \cos t \]

\[ = \frac{8 + \theta}{(8 - \theta)(8 + \theta)} \cdot 65 \cos t = \frac{8 + \theta}{64 - \theta^2} \cdot 65 \cos t \]

\[ = \frac{8 + \theta}{64 - (-1)^2} \cdot 65 \cos t = \frac{8 + \theta}{65} \cdot 65 \cos t \]

\[ = 8 \cos t - \sin t \]

\[ = 8 \cos(\log x) - \sin(\log x) \]
The general solution is

\[ y = y_c + y_p \]

\[ y = \frac{c_1}{x^2} + x \left[ c_2 \cos (3 \log x) + c_3 \sin \left( \sqrt{3} \log x \right) + 8 \cos (\log x) - \sin (\log x) \right] \]

7.4.4 Example: Solve \( x^2 D^2 y - xDy - 3y = x^2 \log x \)

Solution: Put \( x = e^t \) then \( t = \log x \)

Let \( \theta = \frac{d}{dt} \). The given equation becomes

\[ \theta(\theta - 1)y - \theta y - 3y = (e^t)^2 t \]

\[ (\theta^2 - 2\theta - 3)y = t e^{2t} \]

A.E. is \( m^2 - 2m - 3 = 0 \Rightarrow (m - 2)(m + 1) = 0 \Rightarrow m = 2, -1 \)

C.F. is \( y_c = c_1 e^{-t} + c_2 e^{2t} = c_1 x^{-1} + c_2 x^2 \)

\[ = \frac{c_1}{x} + c_2 x^2 \]

P.I. = \( y_p = \frac{1}{\theta^2 - 2\theta - 3} t e^{2t} \)

\[ = e^{2t} \cdot \frac{1}{(\theta + 2)^2 - 2(\theta + 2) - 3} t \]

\[ = e^{2t} \cdot \frac{1}{\theta^2 + 4\theta + 4 - 2\theta - 4 - 3} \cdot t \]

\[ = e^{2t} \cdot \frac{1}{\theta^2 + 2\theta - 3} t \]

\[ = e^{2t} \cdot \frac{1}{(-3) \left[ 1 - \frac{2}{3} \theta - \frac{1}{3} \theta^2 \right]} t \]

\[ = e^{2t} \left[ \frac{1}{1 - \left( \frac{2}{3} \theta + \frac{1}{3} \theta^2 \right)} \right]^{-1} t \]
\[ \frac{e^{2t}}{3} \left( 1 + 2 \cdot 0 + \frac{1}{3} \cdot 0^2 + \cdots \right) t \]

\[ \frac{e^{2t}}{3} \left( t + \frac{2}{3} \right) = \frac{x^2}{3} \left( \log x + \frac{2}{3} \right) \]

\[ \therefore \text{The general solution is } y = y_c + y_p \]

\[ y = c_1 \frac{e}{x} + c_2 x^3 - \frac{1}{3} x^2 \left( \log x + \frac{2}{3} \right) \]

**7.4.5 Example:** Solve \( x^3 D^3 y + 3x^2 D^2 y + xDy + y = x + \log x \)

**Solution:** Put \( x = e^t \) then \( t = \log x \)

Let \( \theta = \frac{d}{dt} \) then the given equation becomes

\[ \theta(\theta - 1)(\theta - 2)y + 3\theta(\theta - 1)y + \theta y + y = e^t + t \]

\[ (\theta^3 + 1)y = e^t + t \]

The A.E. is \( m^3 + 1 = 0 \Rightarrow (m + 1)(m^2 - m + 1) = 0 \)

\( m = -1, \frac{1 \pm i\sqrt{3}}{2} \)

Thus C.F. is

\[ y_c = c_1 e^{-t} + e^{\frac{t}{3}} \left[ c_2 \cos \frac{\sqrt{3}}{2} t + c_3 \sin \frac{\sqrt{3}}{2} t \right] \]

\[ = c_1 x^{-1} + \sqrt{x} \left[ c_2 \cos \left( \frac{\sqrt{3}}{2} \log x \right) + c_3 \sin \left( \frac{\sqrt{3}}{2} \log x \right) \right] \]

P.I. \( y_p = \frac{1}{\theta^3 + 1} \left( e^t + t \right) = \frac{1}{\theta^3 + 1} e^t + \frac{1}{\theta^3 + 1} t \)

\[ = \frac{1}{3^3 + 1} e^t + \left( 1 + \theta^3 \right)^{-1} t \]

\[ = \frac{1}{2} e^t + \left( 1 - \theta^3 + \theta^6 - \cdots \right) t \]
\[ y = \frac{1}{2}e^t + t \]
\[ = \frac{1}{2}x + \log x \]

The general solution of the given equation is
\[ y = y_c + y_p = c_1x^{-1} + \sqrt{x} \left[ c_2 \cos \left( \frac{\sqrt{3}}{2} \log x \right) + c_3 \sin \left( \frac{\sqrt{3}}{2} \log x \right) \right] + \frac{1}{2}x + \log x \]

7.7.5 Legendre's Linear differential equation: An equation of the form
\[ (ax + b)^n \frac{d^n y}{dx^n} + P_1(ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = Q(x) \quad \text{-------- (1)} \]

where \( P_1, P_2, \ldots, P_n \) are constants and \( Q(x) \) is a function of \( x \) is called Legendre's linear equation.

Such equation can be reduced to linear equations with constant coefficients by the substitution
\[ (ax + b) = e^t, \quad t = \log (ax + b) \]

Also, if \( \theta = \frac{d}{dt}, \)

\[ Dy = \frac{dy}{dx} = \frac{dt}{dx} \cdot \frac{dy}{dt}, \quad \frac{a}{ax + b} \frac{dy}{dt} - \frac{a}{ax + b} \theta y \quad \text{or} \quad (ax + b)Dy = ay \]

\[ D^2 y = \frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{a}{ax + b} \frac{dy}{dt} \right) \]
\[ = \frac{a}{ax + b} \frac{d}{dx} \left( \frac{dy}{dt} \right) + \frac{dy}{dt} \left( \frac{-a^2}{(ax + b)^2} \right) \]
\[ = \frac{a}{ax + b} \frac{d}{dt} \left( \frac{dy}{dt} \right) + \frac{a^2}{(ax + b)^2} \frac{dy}{dt} \]
\[ = \frac{a}{ax + b} \frac{d^2 y}{dt^2} + \frac{a}{(ax + b)^2} \frac{dy}{dt} \]
\[ = \frac{a^2}{(ax + b)^2} \left[ \frac{d^2 y}{dt^2} - \frac{dy}{dx} \right] = \frac{a^2}{(ax + b)^2} \left( \theta^2 y - \theta y \right) \]
(ax + b)^2 D^2 y = a^2 \left( \theta^2 y - 0y \right) = a^2 \theta(0 - 1)y

Similarly

(ax + b)^3 D^3 y = a^3 \theta(0 - 1)(0 - 2)y

and

(ax + b)^n D^n y = a^n \theta(0 - 1)(0 - 2) \cdots (0 - n + 1)y

By making these replacements in equation (1), we get a linear equation with constant coefficients.

7.5.1 SAQ : (1) Solve \( \frac{d^2 y}{dx^2} + \frac{dy}{dx} = \frac{1}{x} \)

7.5.2 Example : Solve \((1 + x)^2 \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + y = 2\sin \left[ \log (1 + x) \right] \)

Solution : Put \( 1 + x = e^t \), \( t = \log (1 + x) \) and \( \theta = \frac{d}{dt} (1 + x) \frac{dy}{dx} = \theta y \)

\((1 + x)^2 \frac{d^2 y}{dx^2} = \theta^2 \theta(0 - 1)y\)

substituting these values in given equation we get

\[ \theta(0 - 1)y + \theta y + y = 2\sin t \]

\[ (\theta^2 + 1)y = 2\sin t \]

which is a linear equation with constant coefficients. Here A.E. is \( m^2 + 1 = 0 \Rightarrow m = \pm i \)

C.F. is \( y_c = c_1 \cos t + c_2 \sin t \)

\[ = c_1 \cos \left[ \log (1 + x) \right] + c_2 \sin \left[ \log (1 + x) \right] \]

P.I. = \( y_p = \frac{1}{\theta^2 + 1} 2\sin t = \theta \cdot \frac{-t}{\theta^2} \cos t = -t \cos t \)

\[ = -\log (1 + x) \cos \left[ \log (1 + x) \right] \]

\( \therefore \) The general solution of the given equation is \( y = y_c + y_p \)

\[ y = c_1 \cos \left[ \log (1 + x) \right] + c_2 \sin \left[ \log (1 + x) \right] - \log (1 + x) \cos \left[ \log (1 + x) \right] \]
7.6 GENERAL SOLUTION OF SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

General solution of

\[
\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R \quad (1)
\]

When one solution of the homogeneous equation is known

The given equation is

Let \( u \) be a solution of the homogeneous equation of (1)

Let us assume that \( y = uv \) be the general solution of (1) where \( v \) is a function of \( x \) which is to be determined.

Then \( \frac{dy}{dx} = u'v + uv' \)

\[
\frac{d^2y}{dx^2} = u''v + 2u'v' + uv''
\]

substituting these values in (1), we get

\[
(u''v + 2u'v' + uv'') + P(u'v + uv') + Quv = R
\]

i.e. \( (u'' + Pu' + Qu) v + uv'' + (2u' + Pu)v' = R \quad (2) \)

since \( u \) is a solution of homogeneous equation (1)

\[ u'' + Pu' + Qu = 0 \]

\[ \therefore \quad (2) \text{ reduces to} \]

\[ u \frac{d^2v}{dx^2} + \left( 2 \frac{du}{dx} + Pu \right) \frac{dv}{dx} = R \]

Put \( \frac{dv}{dx} = w \). Then

\[ u \cdot \frac{dw}{dx} + \left( 2 \frac{du}{dx} + Pu \right) w = R \]

Then, \( u \cdot \frac{dw}{dx} + \left( 2 \frac{du}{dx} + Pu \right) w = R \)

or \( \frac{dw}{dx} + \left( \frac{2}{u} \frac{du}{dx} + P \right) w = \frac{R}{u} \quad (3) \)

This is a linear equation in \( w \) and \( x \).
Its integrating factor (I.F.) is

\[
I.F. = e^{\int \left( p + \frac{2}{u} \frac{du}{dx} \right) dx} = e^{\int P dx + 2 \log u} = e^{\int P dx} \cdot e^{\log u^2} = e^{\int P dx} \cdot u^2
\]

Solution of (3) is

\[
w \cdot e^{\int P dx} \cdot u^2 = \left( \frac{R}{u^2} u^2 e^{\int P dx} \right) dx + c_1
\]

or

\[
w = \frac{dv}{dx} = \frac{e^{-\int P dx}}{u^2} \left( \int Ru e^{\int P dx} \right) dx + c_1 \frac{e^{-\int P dx}}{u^2} dx + c_2
\]

Integrating, the above differential equation gives

\[
v = \left[ \frac{e^{-\int P dx}}{u^2} \int \left( Ru e^{\int P dx} \right) dx \right] dx + c_1 \left[ \frac{e^{-\int P dx}}{u^2} dx + c_2 \right]
\]

Putting this value of \( v \) in equation \( y = uv \), we get the general solution of the equation (1),

general solution of (1) is \( y = uv \)

\[
y = u \left[ \frac{e^{-\int P dx}}{u^2} \int \left( Ru e^{\int P dx} \right) dx \right] dx + c_1 \left[ \frac{e^{-\int P dx}}{u^2} dx + c_2 \right]
\]

\[
y = c_2u + c_1u \left[ \frac{e^{-\int P dx}}{u^2} dx + u \left[ \frac{e^{-\int P dx}}{u^2} \int Ru e^{\int P dx} \right] dx \right]
\]

7.6.2 Rules for getting an integral belonging to complementary functions (CF) i.e. solution of \( y'' + Py' + Qy = 0 \)

7.6.3 : Rule (1) \( y = e^{ax} \) is a solution of \( y'' + Py' + Qy = 0 \). If \( a^2 + Pa + Q = 0 \)

Proof : \( y = e^{ax} \Rightarrow \frac{dy}{dx} = ae^{ax} \), and \( \frac{d^2y}{dx^2} = a^2 e^{ax} \)

substituting these values in \( y'' + Py' + Qy = 0 \) we get

\[
a^2 e^{ax} + Pae^{ax} + Qe^{ax} = 0
\]

\[
\Rightarrow e^{ax} \left( a^2 + Pa + Q \right) = 0
\]

\[
\Rightarrow a^2 + Pa + Q = 0 \quad \left( \because e^{ax} \neq 0 \right)
\]
Particular case (i) : Take $a = 1$ then $y = e^x$ is a solution if $1 + P + Q = 0$

case (ii) : Take $a = -1$ then $y = e^{-x}$ is a solution if $1 - P + Q = 0$

7.6.4 Rule II : $y = x^m \Rightarrow \frac{dy}{dx} = mx^{m-1}$ and $\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$ is a solution of $y'' + Py' + Qy = 0$ if $m(m-1) + Pmx + Qx^2 = 0$

Proof : $y = x^m$ is a solution of $y'' + Py' + Qy = 0$ if $m(m-1) + Pmx + Qx^2 = 0$.

substituting these values in $y'' + Py' + Qy = 0$, we get

$m(m-1)x^{m-2} + Pmx^{m-1} + Qx^m = 0$

$\Rightarrow x^{m-2}[m(m-1) + Pmx + Qx^2] = 0$

$\Rightarrow m(m-1) + Pmx + Qx^2 = 0$

particular case (i) : Take $m = 1$, then $y = x$ is a solution if $P + Qx = 0$

case (ii) : Take $m = 2$, then $y = x^2$ is a solution if $2 + 2Px + Qx^2 = 0$

7.6.5 : Working rule for finding general solution of $y'' + Py + Q = R$ when an integral of C.F. is known

Step 1 : Put the given equation in standard form

$y'' + Py' + Qy = R$

in which coefficient of $\frac{d^2y}{dx^2}$ is unity.

Step 2 : Find an integral $u$ of C.F. by using the following.

(i) $u = e^x$ if $1 + P + Q = 0$
(ii) $u = e^{-x}$ if $1 - P + Q = 0$
(iii) $u = e^{ax}$ if $a^2 + aP + Q = 0$
(iv) $u = x$ if $P + Q = 0$
(v) $u = x^2$ if $2 + 2Px + Qx^2 = 0$
(vi) $u = x^m$ if $m(m-1) + Pmx + Qx^2 = 0$

If a solution (or integral) $u$ is given in a problem, when this step is omitted.

Step 3 : Assume that the complete solution of given equation is $y = uv$.

Then the given equation reduces to
\[
\frac{d^2v}{dx^2} + \left( P + \frac{12}{u} \frac{du}{dx} \right) \frac{dv}{dx} = \frac{R}{u} \quad \text{(1)}
\]

**Step 4:** Put \( \frac{dv}{dx} = w \) then \( \frac{d^2v}{dx^2} = \frac{dw}{dx} \)

substitute these values in (1) we get

\[
\frac{dw}{dx} + \left( P + \frac{2}{u} \frac{du}{dx} \right) w = \frac{R}{u}
\]

or \( \frac{dw}{dx} + P_1 w = R_1 \) where \( P_1 = P + \frac{2}{u} \frac{du}{dx}, R_1 = \frac{R}{u} \).

This is a linear equation of first order in \( w \) and can be solved for \( w \).

since \( \frac{dv}{dx} = w \), we can obtain \( v \) by integration.

7.6.6 **SAQ:** Find an integral \( u \) of C.F of the equation \( x^2y'' - 2x(1 + x)y' + 2(1 + x)y = x^3 \)

7.6.7 **Example:** Solve \( \sin x \frac{d^2y}{dx^2} = 2y \), given \( y = \cot x \) is a solution.

**Solution:** Standard form of the given equation is

\[ y'' + 0y' - 2(\csc^2 x)y = 0 \]

Comparing this with \( y'' + Py' + Qy = R \), we have

\[ P = 0, Q = -2\csc^2 x, R = 0 \quad \text{---------(1)} \]

Given that \( u = \cot x \) is a solution.

Let \( y = uv \) be the complete solutions of the given equations.

Then \( v \) is given by

\[
\frac{d^2v}{dx^2} + \left( P + \frac{2}{u} \frac{du}{dx} \right) \frac{dv}{dx} = \frac{R}{u} \quad \text{where } u = \cot x.
\]

\[
\frac{d^2v}{dx^2} + \left[ 0 + \frac{2}{\cot x} (-\csc^2 x) \right] \frac{dv}{dx} = 0 \quad \text{---------(2)}
\]

Put \( \frac{dv}{dx} = w \) then \( \frac{d^2v}{dx^2} = \frac{dw}{dx} \)

substituting these values in (2) we get
\[
\frac{dw}{dx} - \frac{2 \sin x}{\cos x} \cdot \frac{1}{\sin^2 x} w = 0
\]

\[
\frac{dw}{dx} - \frac{4}{\sin 2x} w = 0 \quad \text{or} \quad \frac{dw}{dx} = \frac{4}{\sin 2x} w
\]

or \[
\frac{dw}{w} = 4 \csc 2x \cdot dx
\]

Integrating

\[
\int \frac{dw}{w} = 4 \int \csc 2x \, dx
\]

\[
\Rightarrow \log w = 2 \int \csc \theta \, d\theta + \log c_1
\]

\[
\Rightarrow \log w = \log \left( c_1 \tan^2 x \right)
\]

\[
w = \frac{dv}{dx} = c_1 \tan^2 x
\]

\[
\Rightarrow dv = c_1 \tan^2 x \, dx
\]

Integrating

\[
\int dv = c_1 \int \tan^2 dx = c_1 \left( \sec^2 x - 1 \right) dx
\]

\[
v = c_1 (\tan x - x) + c_2
\]

Hence the required complete solution is \( y = uv \)

\[
y = \cot x \left[ c_1 (\tan x - x) + c_2 \right] = c_1 (1 - x \cot x) + c_2 \cot x
\]

7.6.8 Example: Solve \( \frac{d^2 y}{dx^2} - \frac{x}{x-1} \frac{dy}{dx} + \frac{y}{x-1} = x - 1 \)

Solution: Comparing the given equation with \( y'' + Py' + Qy = R \), we have

\[
P = -\frac{x}{x-1}, \quad Q = \frac{1}{x-1}, \quad R = x - 1
\]

Here \( 1 + P + Q = 1 - \frac{x}{x-1} + \frac{1}{x-1} = \frac{x-1-x+1}{x-1} = 0 \)

\[
\therefore u = e^x \quad \text{is a part of C.F.}
\]

Let \( y = uv \) be the complete solution.
Then \( v \) is given by

\[
\frac{d^2v}{dx^2} + \left( \frac{P + 2 \frac{du}{dx}}{u} \right) \frac{dv}{dx} = \frac{R}{u} \quad \text{(1)}
\]

\[
P + 2 \frac{du}{dx} = \frac{-x}{x-1} + \frac{2}{e^x} = \frac{-x + 2x - 2}{x-1} = \frac{x-2}{x-1}
\]

\[
\frac{R}{u} = \frac{x-1}{e^x}
\]

Put \( \frac{dv}{dx} = w \) then \( \frac{d^2v}{dx^2} = \frac{dw}{dx} \)

Equation (1) changes to

\[
\frac{dw}{dx} + \frac{x-2}{x-1}w = \frac{x-1}{e^x} \quad \text{(2)}
\]

This is a linear equation.

\[
\text{I.F.} = e^{\int \frac{x-2}{x-1} dx} = e^{\int \left(1 - \frac{1}{x-1}\right) dx} = e^{x - \log(x-1)} = e^x e^{\log(x-1)-1} = \frac{e^x}{x-1}
\]

Solution of (2) is

\[
w \cdot \text{I.F.} = \int \frac{x-1}{e^x} \text{I.F.} dx + c_1
\]

\[
w \cdot \frac{e^x}{x-1} = x + c_1
\]

\[
w = e^{-x} \left[ x^2 - x + c_1 (x - 1) \right]
\]

\[
w = \frac{dv}{dx} = e^{-x} \left[ x^2 - x + c_1 (x - 1) \right]
\]

\[dv = e^{-x} \left( x^2 - x + c_1 (x - 1) \right) dx\]

\[\therefore v = \int \left[ x^2 - x + c_1 (x - 1) \right] e^{-x} dx\]

\[= \left[ x^2 - x + c_1 (x - 1) \right] \left( e^{-x} \right) - \left( 2x - 1 + c_1 \right) (e^{-x}) + 2( -e^{-x}) + c_2\]

\[= -\left[ x^2 - x + c_1 (x - 1) \right] e^{-x} - ( 2x - 1 + c_1 ) e^{-x} - 2e^{-x} + c_2\]
General solution is  
\[ y = uv = e^x \cdot v \]

\[ = -\left[ x^2 - x + c_1(x - 1) \right] - (2x - 1 + c_1) - 2 + c_2e^x \]

\[ = c_1x + c_2e^x - (x^2 + x + 1) \]

7.6.9 Example: Solve  
\[ x^2 \frac{d^2y}{dx^2} - (x^2 + 2x) \frac{dy}{dx} + (x + 2)y = x^3e^x \]

Solution: Given equation in standard form is

\[ \frac{d^2y}{dx^2} - \left( \frac{x^2 + 2x}{x^2} \right) \frac{dy}{dx} + \left( \frac{x + 2}{x^2} \right)y = xe^x \]

Comparing this with standard form  \( y'' + Py' + Qy = R \) we have

\[ P = -\left( \frac{x^2 + 2x}{x^2} \right), \quad Q = \frac{x + 2}{x^2}, \quad R = xe^x \]

\[ P + Qx = -\left( \frac{x^2 + 2x}{x^2} \right) + \left( \frac{x + 2}{x^2} \right)x = -\frac{x^2 - 2x + x^2 + 2x}{x^2} = 0 \]

\[ \therefore u = x \text{ is a part of C.F.} \]

Let  \( y = uv \) be the complete solutions. Then  \( v \) is given by

\[ \frac{d^2v}{dx^2} + P_1 \frac{dv}{dx} = R_1 \quad \text{-------- (1)} \]

where  \( P_1 = \left( P + \frac{2 \frac{du}{u}}{dx} \right), \quad R_1 = \frac{R}{u} \)

\[ P_1 = \left( P + \frac{2 \frac{du}{u}}{dx} \right) = \left( -\frac{x^2 + 2x}{x^2} \right) + \frac{2}{x} = -\frac{x^2 - 2x + 2x}{x^2} = -1 \]

\[ R_1 = \frac{xe^x}{u} = e^x \]

Let  \( \frac{dv}{dx} = w \), then equation (1) changes to

\[ \frac{dw}{dx} + P_1w = R_1 \text{ or } \frac{dw}{dx} - w = e^x. \]

This is linear equation.
I.F. = $e^\int-1dx = e^{-x}$

Solution of (1) is $we^{-x} = \int e^x \cdot e^{-x} \, dx + c_1$

$we^{-x} = x + c_1$

$w = xe^x + c_1e^x$

$$\frac{dv}{dx} = w = xe^x - c_1e^x$$

$$dv = \left(xe^x + c_1e^x\right)dx$$

Integrating

$$\int dv = \int \left(xe^x + c_1e^x\right)dx$$

$$v = xe^x - e^x + c_1e^x + c_2$$

general solution is $y = uv$

$$y = x \left[ xe^x - e^x + c_1e^x + c_2 \right]$$

$$= c_1xe^x + c_2xe^x + e^x \left(x^2 - x\right)$$

7.6.10 Example: Solve $(x + 2)\frac{d^2y}{dx^2} - (2a + 5)\frac{dy}{dx} + 2y = (x + 1)e^x$

Solution: Given equation in standard form is

$$\frac{d^2y}{dx^2} - \frac{2x + 5}{x + 2} \frac{dy}{dx} + \frac{2}{x + 2} y = \frac{(x + 1)e^x}{x + 2}$$

Comparing this with standard form $y'' + Py' + Qy = R$, we have

$$P = -\frac{2x + 5}{x + 2}, ~ Q = \frac{2}{x + 2}, ~ R = \frac{(x + 1)e^x}{x + 2} ~ \text{------- (1)}$$

$$2^2 + 2P + Q = 4 + 2 \left(\frac{-(2x + 5)}{x + 2}\right) + \frac{2}{x + 2}$$

$$= \frac{4x + 8 - 4x - 10 + 2}{x + 2} = 0$$

$\therefore u = e^{2x}$ is a part of C.F.

Let $y = uv$ be the general solution
Then \( v \) is given by \( \frac{d^2 v}{dx^2} + P_1 \frac{dv}{dx} = R_1 \) where

\[
P_1 = P + \frac{2}{u} \frac{du}{dx}, \quad R_1 = \frac{R}{u} \quad \text{and} \quad u = e^{2x} \quad \text{(2)}
\]

\[
P_1 = \frac{-(2x + 5)}{x + 2} + \frac{2}{e^{2x}} \cdot 2e^{2x} = \frac{-2x - 5 + 4x + 8}{x + 2} = \frac{2x + 3}{x + 2}
\]

\[
R_1 = \frac{(x + 1)e^x}{x + 2} \frac{1}{e^{2x}} = \frac{x + 1}{x + 2} e^{-x}
\]

substitute these values in equation (2) we get

\[
\frac{d^2 u}{dx^2} + \frac{2x + 3}{x + 2} \frac{dv}{dx} = \frac{x + 1}{x + 2} e^{-x} \quad \text{(3)}
\]

Put \( \frac{d\theta}{du} = w \) then \( \frac{d^2 v}{dx^2} = \frac{dw}{dx} \). Substituting these values in the above equation (3) we get

\[
\frac{dw}{dx} + \frac{2x + 3}{x + 2} w = \frac{x + 1}{x + 2} e^{-x}. \quad \text{This is a linear equation}
\]

\[
\text{I.F.} = e^{\int \left( \frac{2x + 3}{x + 2} \right) dx} = e^{\frac{2x}{x + 2} - \frac{1}{x + 2} \log(x + 2)} = e^{2x(x + 2)^{-1}}
\]

\[
= \frac{e^{2x}}{x + 2}
\]

Then w.I.F. = \( \int \left( \frac{x + 1}{x + 2} e^{-x} \right) \frac{dx}{x + 2} + c_1 \)

\[
w \cdot \frac{e^{2x}}{x + 2} = \int \frac{x + 1}{x + 2} e^{-x} \cdot \frac{e^{2x}}{x + 2} dx + c_1
\]

\[
= \int e^{x} \left[ \frac{1}{x + 2} - \frac{1}{(x + 2)^2} \right] dx + c_1 = \frac{e^x}{x + 2} + c_1
\]

\[
\therefore \frac{dv}{dx} = w = e^{-x} + c_1 (x + 2)e^{-2x}
\]

\[
\therefore v = \int \left( e^{-x} + c_1 (x + 2)e^{-2x} \right) dx + c_2 = -e^{-x} + c_1 \left[ (x + 2) \left( \frac{-e^{-2x}}{2} \right) - 1 \cdot \frac{e^{-2x}}{4} \right] + c_2
\]

\[
= -e^{-x} - c_1 e^{-2x} \left( \frac{2x + 5}{4} \right) + c_2
\]
7.6.11 Example: Solve \((1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0\) given that \(u = e^{a \sin^{-1} x}\) is a solution.

Given equation in standard form is

\[
\frac{d^2y}{dx^2} - \frac{x}{1-x^2} \frac{dy}{dx} - \frac{a^2}{1-x^2} y = 0
\]

Comparing this with standard equation \(y'' + Py' + Qy = R\) we have

\[
P = -\frac{x}{1-x^2}, \quad Q = -\frac{a^2}{1-x^2}, \quad R = 0
\]

Given \(u = e^{a \sin^{-1} x}\) is a part of C.F.

Let \(y = uv\) be the general solution. Then \(v\) is given by

\[
\frac{d^2v}{dx^2} + P_1 \frac{dv}{dx} = R_1
\]

where

\[
P_1 = P + \frac{2}{u} \frac{du}{dx}, \quad R_1 = \frac{R}{u}
\]

and \(u = e^{a \sin^{-1} x}\) \hspace{1cm} (2)

\[
P_1 = -\frac{x}{1-x^2} + \frac{2}{e^{a \sin^{-1} x}} e^{a \sin^{-1} x} \cdot \frac{a}{\sqrt{1-x^2}} = -\frac{x}{1-x^2} + \frac{2a}{\sqrt{1-x^2}}
\]

\[
R_1 = 0
\]

\[
\therefore \quad \text{Equation (2) becomes}
\]

\[
\frac{d^2v}{dx^2} + \left( -\frac{x}{1-u^2} + \frac{2a}{\sqrt{1-u^2}} \right) \frac{dv}{dx} = 0
\]

Put \(\frac{dv}{dx} = w\) then \(\frac{d^2v}{dx^2} = \frac{dw}{dx}\). Then (3) becomes

\[
\frac{dw}{dx} + \left( -\frac{x}{1-x^2} + \frac{2a}{\sqrt{1-x^2}} \right) w = 0
\]

This is a linear equation.

\[
\frac{dw}{w} = \left( \frac{x}{1-x^2} - \frac{29}{\sqrt{1-x^2}} \right) dx
\]

\[
\Rightarrow \int \frac{dw}{w} = \left( \frac{x}{1-x^2} - \frac{29}{\sqrt{1-x^2}} \right) dx
\]
\[
\log w = -\frac{1}{2} \log (1-x^2) - 2a \sin^{-1} x + \log c_1 \\
= \log \left( \frac{c_1}{\sqrt{1-x^2}} \right) - 2a \sin^{-1} x
\]
\[
\therefore \frac{dv}{dx} = w \left[ \log \left( \frac{c_1}{\sqrt{1-x^2}} \right) - 2a \sin^{-1} x \right] = \frac{c_1}{\sqrt{1-x^2}} \cdot e^{-2a \sin^{-1} x}
\]
\[
V = \int \frac{c_1}{\sqrt{1-x^2}} e^{-2a \sin^{-1} x} \, dx + c_2 = \frac{c_1}{-2a} e^{-2a \sin^{-1} x} + c_2
\]
\[
\therefore \text{ general solution is } y = uv = e^{a \sin^{-1} x} \cdot v \\
y = \frac{-c_1}{2a} e^{-a \sin^{-1} x} + c_2 e^{a \sin^{-1} x}
\]

### 7.7 NORMAL FORM

General solution of \( \frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R \) by changing the dependent variable \( y \) and removing the first derivative.

Consider the equation

\[
\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R \quad \text{------- (1)}
\]

Let \( y = uv \quad \text{------- (2)} \) be the general solution of (1), where \( u, v \) are functions of \( x \) only.

\[
\frac{dy}{dx} = uv' + vu'
\]

and \( \frac{d^2y}{dx^2} = uv'' + 2u'v' + vu'' \quad \text{------- (3)} \)

Substituting the values of (2) and (3) in (1), we get

\[
uv'' + 2u'v' + vu'' + P(uv' + vu') + Quv = R
\]

\[
uv'' + [Pu + 2u']v' + [u'' + Pu' + Qu]v = R \quad \text{------- (4)}
\]

To remove the first derivative \( \frac{dy}{dx} \) in (4), choose \( u \) such that \( Pu + 2u' = 0 \quad \text{------- (5)} \)
Linear Equations of the Second Order with...

i.e. \( Pu + 2 \frac{du}{dx} = 0 \)

\[ \Rightarrow \frac{du}{dx} = -\frac{P}{2} u \Rightarrow \frac{du}{u} = -\frac{P}{2} dx \]

\[ \Rightarrow \text{Integrating, } \log u = -\frac{1}{2} \int P dx \Rightarrow u = e^{-\frac{1}{2} \int P dx} \]

from (5) \( \frac{du}{dx} = -\frac{1}{2} P du \)

\[ u^r = \frac{d^2 u}{dx^2} - \frac{1}{2} P \frac{du}{dx} - \frac{1}{2} u \frac{dP}{dx} = \frac{-1}{2} P - \frac{1}{2} u \frac{dP}{dx} \]

\[ u^r = \frac{P^2 u}{4} - \frac{1}{2} u \frac{dP}{dx} \quad (6) \]

Substituting the values of (5) and (6) in (4), we get

\[ uv^r + \left[ \frac{P^2 u}{4} - \frac{1}{2} u \frac{dP}{dx} - \frac{1}{2} P^2 u + Qu \right] v = R \]

\[ \Rightarrow v^r + \left[ Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} \right] v = R \]

\[ \Rightarrow \frac{d^2 v}{dx^2} + Iv = S \quad (7) \]

where \( I = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} \) and \( S = \frac{R}{u} \)

The equation (7) is called the normal forms of (1)

If \( I = K \) (constant) or \( I = \frac{K}{x^2} \), thus the solution of (7) can be obtained by the known methods.

The general solution of (1) is \( y = uv \).

7.7.1 SAQ : Find the normal form of the equation

\[ y^{''} - \frac{2}{x} y^{'} + \left( 1 + \frac{2}{x^2} \right) y = xe^x \]

7.7.2 Working Rule : Given equation is
\[ \frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R \]

**Step (1)**: Let \( y = uv \) be the general solution.

**Step (2)**: Find \( u = -e^{-\frac{1}{2} \int P \, dx} \)

**Step (3)**: Find \( I = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} \) and \( S = \frac{R}{u} \)

**Step (4)**: Then the equation reduces to normal form

\[ \frac{d^2v}{dx^2} + Iv = S \]

Solve for \( v \) depending on \( I = K \) (constant) or \( I = \frac{K}{x^2} \).

**Step (5)**: General solution is \( y = uv \).

**7.7.3 Example**: Solve \( y'' - 4xy' + 4x^2y = e^{x^2} \)

**Solution**: Given equation is

\[ \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 4x^2y = e^{x^2} \]  

Here \( P = -4x, \ Q = 4x^2, \ R = e^{x^2} \)

Let \( y = uv \) be the general solution.

\[ u = e^{-\frac{1}{2} \int P \, dx} = e^{-\frac{1}{2} \int -4x \, dx} = e^{x^2} \]

\[ u = e^{x^2} \]

\[ I = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} = 4x^2 - \frac{1}{4} (-4x)^2 - \frac{1}{2} \frac{d}{dx} (-4x) \]

\[ = 4x^2 - 4x^2 + 2 = 2 \]

\[ S = \frac{R}{u} = \frac{e^{x^2}}{e^{x^2}} = 1 \]

Then the equation (1) reduces to the normal form
\[
\frac{d^2v}{dx^2} + Iv = S
\]

i.e. \(\frac{d^2v}{dx^2} + 2v = 1\) \quad i.e. \((D^2 + 2)v = 1\) \quad \text{(2)}

A.E. is \(m^2 + 2 = 0 \Rightarrow m^2 = -2 \Rightarrow m = \pm \sqrt{2}i\)

C.F. is \(v_c = c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x\)

\[P.I. = v_p = \frac{1}{D^2 + 2} \cdot 1 = \frac{1}{D^2 + 2} e^{0x} = \frac{1}{0 + 2} = \frac{1}{2}\]

\[\therefore \text{ The general solution of (2) is } v = v_c + v_p\]

\[v = c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x + \frac{1}{2}\]

Hence the general solution of equations (1) is \(y = uv\).

\[y = e^{x^2} \left[ c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x + \frac{1}{2} \right]\]

**7.7.4 Example:** \(\frac{d^2y}{dx^2} - 2 \tan x \frac{dy}{dx} + 5y = 0\) \quad \text{(1)}

**Solution:** Here \(P = -2 \tan x, Q = 5, R = 0\)

Let \(y = uv\), be the general solution of (1)

\[u = e^{-\frac{1}{2} \int \frac{dP}{dx}} = e^{-\frac{1}{2} \int -2 \tan x dx} = e^{\int \frac{\sin x}{\cos x} dx} = e^{\log \sec x} = \sec x\]

Now \(u = \sec x\)

\[I = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} = 5 - \frac{1}{4} (-2 \tan x)^2 - \frac{1}{2} \frac{d}{dx} (-2 \tan x)\]

\[= 5 - \tan^2 x + \sec^2 x = 5 + 1 = 6\]

\[S = \frac{R}{u} = 0 = 0\]

Then the given equation reduces to the normal form

\[\frac{d^2v}{dx^2} + Iv = S\]
\[
\text{i.e. } \frac{d^2v}{dx^2} + 6v = 0 \Rightarrow (D^2 + 6)v = 0
\]

\[
\therefore \text{A.E. is } m^2 + 6 = 0 \Rightarrow m = \pm \sqrt{6}i
\]

C.F. of \( v = v_c = c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x \)

P.I. of \( v = v_p = 0 \) (\( \therefore S = 0 \))

\[
\therefore v = v_c + v_p = c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x
\]

\[
\therefore \text{The general solution of (1) is } y = uv
\]

\[
y = \sec x \left[ c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x \right]
\]

**7.7.5 Example**: Solve \( y'' - \frac{2}{x} y' + \left(1 + \frac{2}{x^2}\right) y = xe^x, \ x > 0 \) \( \text{-------- (1)} \)

**Solution**: Given equation is

\[
y'' - \left(\frac{2}{x}\right)y' + \left(1 + \frac{2}{x^2}\right)y = xe^x
\]

Here \( P = -\frac{2}{x} \), \( Q = 1 + \frac{2}{x^2} \), \( R = xe^x \)

Let \( y = uv \) be the general solution of (1)

\[
u = e^{-\frac{1}{2} \int P \, dx} = e^{\frac{1}{2} \int \frac{2}{x} \, dx} = e^{\int \log x} = x
\]

\[
I = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} = 1 + \frac{2}{x^2} - \frac{1}{4} \left(\frac{2}{x}\right)^2 - \frac{1}{2} \frac{d}{dx} \left(\frac{2}{x}\right)
\]

\[
= 1 + \frac{2}{x^2} - \frac{2}{x^2} = 1
\]

Then the equation reduces to the normal form

\[
\frac{d^2v}{dx^2} + \frac{dv}{dx} = S
\]

\[
\frac{d^2v}{dx^2} + v = e^x \ \text{i.e. } (D^2 + 1)v = e^x \ \text{-------- (2)}
\]

A.E. is \( m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i \)
C.F. of \( v = v_c = c_1 \cos x + c_2 \sin x \)

P.I. of \( v = v_p = \frac{1}{D^2 + 1} e^x = \frac{e^x}{2} \)

\( \therefore \) The general solution of (2) is \( v = v_c + v_p \)

\[ v = c_1 \cos x + c_2 \sin x + \frac{e^x}{2} \]

Hence, the general solution of (1) is \( y = uv \)

\[ y = x \left[ c_1 \cos x + c_2 \sin x + \frac{e^x}{2} \right] \]

7.7.6 Example: Solve \( \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 3)y = e^{x^2} \)

Solution: Given equation is

\[ \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 3)y = e^{x^2} \quad \text{-------- (1)} \]

Here \( P = -4x, \ Q = 4x^2 - 3, \ R = e^{x^2} \)

Let \( y = uv \) be the general solution of (1)

\[ u = e^{-\frac{1}{2} \int P \, dx} = e^{-\frac{1}{2} \int -4x \, dx} = e^{x^2} \]

\[ u = e^{x^2} \]

\[ I = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} = 4x^2 - 3 - \frac{1}{4} (-4x)^2 - \frac{1}{2} \frac{d}{dx} (-4x) \]

\[ = 4x^2 - 3 - 4x^2 + 2 = -1 \]

\[ I = -1 \]

\[ S = \frac{R}{u} - \frac{e^{x^2}}{e^{x^2}} = 1 \]

Then the equation (1) reduces to the normal form

\[ \frac{d^2 v}{dx^2} + Iv = S \]
\[ \text{i.e. } \frac{d^2v}{dx^2} - v = 1 \quad \text{i.e. } (D^2 - 1)v = 1 \quad \text{-------- (2)} \]

A.E. is \( m^2 - 1 = 0 \Rightarrow m = \pm 1 \)

C.F. is \( v_c = c_1e^x + c_2e^{-x} \)

P.I is \( v_p = \frac{1}{D^2 - 1} e^{0x} = \frac{1}{0 - 1} = -1 \)

\[ \therefore \text{The general solution of (2) is } v = v_c + v_p = c_1e^x + c_2e^{-x} - 1 \]

Hence the general solution of (1) is \( y = uv = e^x \left[ c_1e^x + c_2e^{-x} - 1 \right] \)

### 7.7.7 General solution of \( \frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R \) by changing the independent variable (x):

Let \( \frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R \quad \text{-------- (1)} \)

Let the independent variable \( x \) be changed to the new independent variable \( z \) which is a function of \( x \), i.e. \( z = f(x) \).

Now, \( \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \)

and \( \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dz} \cdot \frac{dz}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{dz} \right) \frac{dz}{dx} + \frac{d}{dx} \left( \frac{dz}{dx} \right) \frac{dy}{dz} \cdot \frac{dz}{dx} \)

\[ = \frac{d}{dz} \left( \frac{dy}{dz} \right) \cdot \frac{dz}{dx} \cdot \frac{dz}{dx} + \frac{d^2z}{dx^2} \frac{dy}{dz} \cdot \frac{dz}{dx} \]

\[ \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} \left( \frac{dz}{dx} \right)^2 + \frac{d^2z}{dx^2} \frac{dy}{dz} \cdot \frac{dz}{dx} \]

Substituting these values in (1), we get

\[ \frac{d^2y}{dz^2} \left( \frac{dz}{dx} \right)^2 + \frac{d^2z}{dx^2} \frac{dy}{dz} \cdot \frac{dz}{dx} + P \frac{dy}{dz} \cdot \frac{dz}{dx} + Q \cdot y = R \]

\[ \Rightarrow \frac{d^2y}{dz^2} \left( \frac{dz}{dx} \right)^2 + \left( \frac{d^2z}{dx^2} + P \frac{dz}{dx} \right) \frac{dy}{dz} \cdot \frac{dz}{dx} + Qy = R \]
Dividing by \( \left( \frac{dz}{dx} \right)^2 \) we get

\[
\frac{d^2y}{dz^2} + \frac{1}{\left( \frac{dz}{dx} \right)^2} \left[ \frac{d^2z}{dx^2} + \frac{P}{dz} \frac{dz}{dx} \right] \frac{dy}{dz} + \frac{Qy}{\left( \frac{dz}{dx} \right)^2} = \frac{R}{\left( \frac{dz}{dx} \right)^2}
\]

or

\[
\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} Q_1 y = R_1 \quad \text{-------- (2)}
\]

where

\[
P_1 = \left( \frac{d^2z}{dx^2} + P \frac{dz}{dx} \right) \left/ \left( \frac{dz}{dx} \right)^2 \right.
\]

\[
Q_1 = Q \left/ \left( \frac{dz}{dx} \right)^2 \right.
\]

\[
R_1 = R \left/ \left( \frac{dz}{dx} \right)^2 \right.
\]

Here \( P_1, Q_1, R_1 \) are functions of \( x \). But these can be changed into a function of \( z \) by using the relation \( z = f(x) \).

**Case (i)**: Choose \( z \) such that \( P_1 = 0 \)

\[
P_1 = 0 \Rightarrow \frac{d^2z}{dx^2} + P \frac{dz}{dx} = 0 \Rightarrow \frac{\frac{d^2z}{dx^2}}{\frac{dz}{dx}} = -P
\]

Integrating, we get

\[
\int \frac{dz}{dx} = \int -P \, dx
\]

(or)

\[
\frac{dz}{dx} = e^{-\int P \, dx} \Rightarrow z = \int e^{-\int P \, dx} \, dx
\]

Then the equation (2) becomes, \( \frac{d^2y}{dz^2} + Q_1 y = R_1 \) can be solved.

**Case (ii)**: Choose \( z \) such that \( Q_1 = a^2 \) (constant)

\[
\Rightarrow \frac{Q}{\left( \frac{dz}{dx} \right)^2} = \frac{Q}{a^2} \Rightarrow \frac{dz}{dx} = \pm \sqrt{\frac{Q}{a^2}}
\]
\[ z = \pm \int \frac{Q}{a^2} \, dx \]

In this we take \( \pm \) to make \( \left( \frac{dz}{dx} \right) \) real. Thus we get the value of \( z \). Secondly in this case \( P_1 \) should be either zero or constant. Then only the equation (2) can be solved by the methods discussed in the previous lesson.

**7.7.8 Example:** Solve \( \frac{d^2 y}{dx^2} - \frac{dy}{dx} + 4x^3y = x^5 \quad (x > 0) \)

**Solution:** The given equation can be written as

\[
\frac{d^2 y}{dx^2} - \frac{1}{x} \frac{dy}{dx} + 4x^2y = x^4 \quad \text{-------- (1)}
\]

Here \( P = -\frac{1}{x}, \quad Q = 4x^2, \quad R = x^4 \)

Choose \( z \) such that \( \left( \frac{dz}{dx} \right)^2 = Q \Rightarrow \frac{dz}{dx} = \sqrt{4x^2} = 2x \)

\[ z = \int 2x \, dx = x^2 \quad \text{------- (2)} \quad \frac{dz}{dx} = 2x, \quad \frac{d^2 z}{dx^2} = 2 \]

With this value of \( z \), (1) reduces to

\[
\frac{d^2 y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad \text{-------- (3)}
\]

where

\[
P_1 = \left( \frac{d^2 z}{dx^2} + P \frac{dz}{dx} \right) \left( \frac{dz}{dx} \right)^2 = 2 + \left( \frac{-1}{x} \right)(2x) = 2
\]

\[ Q_1 = \frac{Q}{\left( \frac{dz}{dx} \right)^2} = \frac{4x^2}{4x^2} = 1 \]
substituting these values in (3) we get

\[ \frac{d^2y}{dz^2} + y = \frac{1}{4} x^2 \Rightarrow (\theta^2 + 1) y = \frac{x^2}{4} = \frac{z}{4} \]

\[ \therefore \theta = \frac{dz}{dx}, \quad z = x^2 \]

The A.E. is \( m^2 + 1 = 0 \Rightarrow m = \pm i \)

\[ \therefore \text{C.F. of } y = y_c = c_1 \cos z + c_2 \sin z = c_1 \cos x^2 + c_2 \sin x^2 \]

\[ \text{P.I. of } y = y_p = \left( \frac{1}{\theta^2 + 1} \right) \frac{z}{4} = \frac{1}{4} \left( 1 + \theta^2 \right)^{-1} z = \frac{1}{4} \left[ 1 - \theta^2 + \ldots \right] z = \frac{1}{4} \frac{z}{4} = \frac{x^2}{4} \]

\[ \Rightarrow \text{The general solution of (1) is } y = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4} \]

**7.7.9 Example:** Solve \( \frac{d^2y}{dx^2} \frac{dy}{dx} - 4x^3 y = 8x^3 \sin x^2 \)

**Solution:** The given equation can be written as

\[ \frac{d^2y}{dx^2} - \frac{1}{x} \frac{dy}{dx} - 4x^2 y = 8x^2 \sin x^2 \]

Here \( P = -\frac{1}{x}, \quad Q = -4x^2, \quad R = 8x^2 \sin x^2 \)

Choose \( z \) such that

\[ \left( \frac{dz}{dx} \right)^2 = -1 \]

\[ \Rightarrow \left( \frac{dz}{dx} \right)^2 = 4x^2 \Rightarrow \frac{dz}{dx} = 2x \]

\[ \Rightarrow z = \int 2x dx = x^2 \]

\[ z = x^2 \quad \text{------- (2)} \]

\[ \frac{dz}{dx} = 2x, \quad \frac{d^2z}{dx^2} = 2 \]

with this value of \( z \), (1) reduces to
\[
\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad \text{-------- (3)}
\]

where \( P_1 = \left( \frac{d^2z}{dx^2} + P \frac{dz}{dx} \right) \left/ \left( \frac{dz}{dx} \right)^2 \right. = 2 + \left( \frac{-1}{x} \right) \frac{2x}{(2x)^2} = 0 \)

\[ Q_1 = \frac{Q}{\left( \frac{dz}{dx} \right)^2} = \frac{-4x^2}{(2x)^2} = -1 \]

\[ R_1 = \frac{R}{\left( \frac{dz}{dx} \right)^2} = \frac{8x^2 \sin x^2}{4x^2} = 2 \sin x^2 = 2 \sin z \]

Equation (3) reduces to

\[
\frac{d^2y}{dz^2} - y = 2 \sin z \quad \text{or} \quad \left( \theta^2 - 1 \right) y = 2 \sin z \quad \text{-------- (4)}
\]

A.E. is \( m^2 - 1 = 0 \Rightarrow m = \pm 1 \)

\( \therefore \) C.F. of \( y = y_c = c_1 e^z + c_2 e^{-z} \)

P.I. of \( y = y_p = \frac{1}{\theta^2 - 1} 2 \sin z = 2 \cdot \frac{1}{-1^2 - 1} \sin z = -\sin z \)

\( \therefore \) The general solution of (4) is \( y = y_c + y_p \)

\[ y = c_1 e^z + c_2 e^{-z} - \sin z \]

Hence the general solution of (1) is

\[ y = c_1 e^{z^2} + c_2 e^{-z^2} - \sin x^2 \quad \text{(Put } z = x^2) \]

**7.7.10 Example**: Solve \( y'' - \cot xy' - (\sin^2 x) y = 0 \)

**Solution**: The given equation is

\[
\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - \sin^2 x \ y = 0 \quad \text{-------- (1)}
\]

Here \( P = -\cot x, \ Q = -\sin^2 x, \ R = 0 \)
Choose $z$ such that \[
\left( \frac{dz}{dx} \right)^2 = -1
\]

\[
\left( \frac{dz}{dx} \right)^2 = \sin^2 x \Rightarrow \frac{dz}{dx} = \sin x
\]

\[
\Rightarrow z = \int \sin x \, dx = -\cos x
\]

\[
z = -\cos x \quad -------- (2)
\]

\[
\frac{dz}{dx} = \sin x, \quad \frac{d^2z}{dx^2} = \cos x
\]

With this value of $z$, (1) reduces to

\[
\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad -------- (3)
\]

where

\[
P_1 = \left( \frac{d^2z}{dx^2} + P \frac{dz}{dx} \right) \left( \frac{dz}{dx} \right)^2 = \frac{\cos x + (\cot x) \cdot \sin x}{\sin^2 x} = 0
\]

\[
Q_1 = \left( \frac{dz}{dx} \right)^2 \frac{-\sin^2 x}{\sin^2 x} = -1
\]

\[
R_1 = \frac{R}{\left( \frac{dz}{dx} \right)^2} = 0
\]

Substituting these values in (3) we get

\[
\frac{d^2y}{dz^2} - 1y = 0 \Rightarrow (\theta^2 - 1)y = 0 \quad -------- (4)
\]

where $\theta = \frac{d}{dz}$

The A.E. is $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

C.F. of $y$ is $y_c = c_1 e^z + c_2 e^{-z}$

P.I. of $y$ is $y_p = \frac{1}{\theta^2 - 1} \cdot 0 = 0$

$\therefore$ General solution of (4) is $y = y_c + y_p$
\[ y = c_1e^z + c_2e^{-z} \]

\[ \Rightarrow \text{The general solution of (1) is} \]

\[ y = c_1e^{-\cos x} + c_2e^{\cos x} \]

### 7.8 METHOD OF VARIATION OF PARAMETERS

This method is quite general and applies to equation of the form. If the previous method fails, then we may try for this method.

\[ y'' + Py' + Qy = R \quad \text{--------- (1)} \]

where \( P, Q \) and \( R \) are functions of \( x \). It gives

**Particular solution**

\[ = -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}dx \quad \text{--------- (2)} \]

where \( y_1 \) and \( y_2 \) are linearly independent solutions of \( y'' + Py_1 + Qy = 0 \quad \text{--------- (3)} \)

and \( w = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \) is called the Wronskian of \( y_1, y_2 \).

**Proof:** Let the C.F. of (1) be \( y = c_1y_1 + c_2y_2 \)

Replacing \( c_1, c_2 \) by unknown functions \( u(x) \) and \( v(x) \).

Let the P.I be

\[ y = uy_1 + vy_2 \quad \text{--------- (4)} \]

Differentiating (4) with respect to \( x \), we get

\[ y' = uy'_1 + vy'_2 + u'y_1 + v'y_2 \]

\[ = uy'_1 + vu'_2 \quad \text{--------- (5)} \]

On assuming that \( u'y_1 + v'y_2 = 0 \quad \text{--------- (6)} \)

Differentiating (4) twice and assuming (6) we get

\[ y = uy_1 + vy_2 \]

\[ y' = uy'_1 + vy'_2 \]

\[ y'' = uy''_1 + vy''_2 + u'y'_1 + v'y'_2 \]

Substituting these values in (1) we get

\[ uy''_1 + vy''_2 + u'y'_1 + v'y'_2 + P[uy'_1 + vy'_2] + Q[uy_1 + vy_2] = R \]

\[ \Rightarrow u[y''_1 + Py'_1 + Qy_1] + v[y''_2 + Py'_2 + Qy_2] + u'y'_1 + v'y'_2 = R \]

\[ \Rightarrow u'y'_1 + v'y'_2 = R \quad \text{--------- (7)} \quad \text{(since \( y_1 \) and \( y_2 \) satisfy (3))} \]

Solving for \( u', v' \) using (6) and (7) we get
7.33 Linear Equations of the Second Order with...  

7.8.1 Example: Solve \((D^2 - 3D + 2)y = \sin e^{-x}\)

Given equation in standard form is

\[
\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin e^{-x}
\]

Here \(P = -3, Q = 2, R = \sin e^{-x}\)

A.E. is \(m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \quad m = 1, 2\)

C.F. is \(y_2 = c_1e^x + c_2e^{2x}\)

Let the P.I. be \(y_p = uy_1 + vy_2\) where \(y_1 = e^x, y_2 = e^{2x}\), \(u, v\) are to be determined.

Now by the method of variation of parameters

\[
w = y_1y_2' - y_1' y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x \cdot 2e^{2x} - e^x \cdot e^{2x} = e^x e^{2x}
\]

\[
u = -\int \frac{y_2R}{w} dx = -\int \frac{e^{2x} \sin e^{-x}}{e^x \cdot e^{2x}} dx = \int \sin e^{-x}(-e^{-x}) dx
\]

\[
= -\cos e^{-x}
\]
\[ v = \int \frac{y_1 R}{w} \, dx = \int \frac{e^x \sin e^{-x}}{e^x e^{2x}} \, dx = \int e^{-2x} \sin(e^{-x}) \, dx \]

\[ v = \int \frac{y_1 R}{w} \, dx = \int \frac{e^x \sin e^{-x}}{e^x e^{2x}} \, dx = \int e^{-2x} \sin(e^{-x}) \, dx \]

\[ = -\int e^{-x} \sin e^{-x}\left(-e^{-x}\right) \, dx \quad \text{Put } e^{-x} = t, \quad e^{-x} \, dx = dt \]

\[ = -\int t \sin t \, dt \]

\[ = [t \cos t - \int \cos t \, dt] \]

\[ = [-t \cos t + \sin t] \]

\[ = e^{-x} \cos e^{-x} - \sin e^{-x} \]

\[ \therefore y_p = uy_1 + vy_2 \]

\[ = \left(-\cos e^{-x}\right)(e^x) + (e^{-x} \cos e^{-x} - \sin e^{-x})(e^{2x}) \]

\[ = -e^{2x} \sin e^{-x} \]

\[ \therefore \text{The required solution is } y = y_c + y_p \]

\[ y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin e^{-x} \]

**7.8.2 Example:** Solve \((D^2 + 1)y = \csc x\) by the method of variation of parameters.

**Solution:** Given equation in standard form is

\[ \frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = \csc x \]

Here \(P = 0, \quad Q = 1, \quad R = \csc x\)

The A.E. is \(f(m) = 0\)

\[ m^2 + 1 = 0 \Rightarrow m = \pm i \]
C.F. is \( y_c = c_1 \cos x + c_2 \sin x \)

Let the P.I. of (1) is \( y_P = u y_1 + v y_2 \)

where \( y_1 = \cos x = y_2 = \sin x \) and \( u, v \) are to be determined.

Now by the method of variation of parameters

\[
w = y_1 y_2 - y_1' y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1
\]

\[
u = -\int \frac{y_2 R}{w} \, dx = -\int \frac{\sin x \cdot \csc x}{1} \, dx = -\int dx = -x
\]

\[
v = \int \frac{y_1 R}{w} \, dx = \int \frac{\cos x \cdot \csc x}{1} \, dx = \int \frac{\cos x}{\sin x} \, dx = \log |\sin x|
\]

\[
\therefore y_P = u y_1 + v y_2 = -x \cos x + \log |\sin x| \cdot \sin x
\]

\[
\therefore \text{The general solution of (1) is}
\]

\[
y = y_c + y_P
\]

\[
= c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log |\sin x|
\]

**7.8.3 Example:** Solve \( [(x - 1)D^2 - xD + 1]y = (x - 1)^2 \) by the method of variation of parameters.

**Solution:** Given equation in standard form is

\[
\frac{d^2 y}{dx^2} - \frac{x}{x-1} \frac{dy}{dx} + \frac{1}{x-1} y = x - 1 \quad \text{-------- (1)}
\]

\[
P = -\frac{x}{x-1} \frac{dy}{dx} + \frac{1}{x-1} y = x - 1 \quad \text{-------- (1)}
\]

Here \( P = -\frac{x}{x-1}, \quad Q = \frac{1}{x-1}, \quad R = (x - 1) \)

Homogeneous part of equation (1) is

\[
\frac{d^2 y}{dx^2} - \frac{x}{x-1} \frac{dy}{dx} + \frac{1}{x-1} y = 0 \quad \text{-------- (2)}
\]

\[
1 + P + Q = 1 - \frac{x}{x-1} + \frac{x}{x-1} = 0 \Rightarrow y = e^x \quad \text{is a solution of (2)}
\]
Also \( P + Qx = -\frac{x}{x-1} + \frac{x}{x-1} = 0 \Rightarrow y = x \) is a solution of (2).

\[ y_c \text{ of } (1) = c_1 e^x + c_2 x \]

Let the P.I. of (1) be \( y_p = uy_1 + vy_2 \) where \( y_1 = e^x, \ y_2 = x \) and \( u, v \) are to be determined.

Now by the method of variation of parameters

\[
w = y_1y_2' - y_1'y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x - xe^x = e^x (1 - x)
\]

\[
u = -\int \frac{y_2 R}{w} \, dx = -\int \frac{x(x-1)}{e^x(1-x)} \, dx = \int e^{-x} \, dx = e^{-x}
\]

\[= -(x + 1)e^{-x}
\]

\[v = \int \frac{y_1 R}{w} \, dx = \int \frac{e^x(x-1)}{e^x(1-x)} \, dx = -\int 1 \, dx = -x
\]

\[y_p = uy_1 + vy_2 = -(x+1)e^{-x} \cdot e^x + (-x)x = -x^2 - x(1 + x^2) = -(1 + x + x^2)
\]

\[y = c_1 e^x + c_2 x - (1 + x + x^2)
\]

### 7.9 Answers to SAQ

#### 7.5.1 SAQ:

\[x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{x}
\]

or \(x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 1
\]

\[x^2D^2 + xD = 1, \] this is a Cauchy Euler equation.

Put \(x = e^t\), then \(t = \log x\)

Then the equation becomes
\[ \left[ \theta (\theta - 1) + \theta \right] y = 1 \quad \text{where} \quad \theta = \frac{\text{d}}{\text{d}t} \]

\[ \theta^2 y = 1 \]

\[ \theta y = \int \text{d}t = t + c_1 \]

\[ y = \int \left( t + c_1 \right) \text{d}t = \frac{t^2}{2} + c_1 t + c_2 \]

which is the general solution.

7.6.6 SAQ: Dividing, the given equation in standard form is

\[ y'' - 2 \left( \frac{1}{x} + 1 \right) y' + 2 \left( \frac{1}{x^2} + \frac{1}{x} \right) y = x \]

Here \( P = -\frac{2}{x} - 2, Q = \frac{2}{x^2} + \frac{2}{x}, R = x \)

Since \( P + Qx = -\frac{2}{x} - 2 + \frac{2}{x} + 2 = 0 \)

\[ \therefore y = x \] is a part of the C.F. of the given equation.

\[ \therefore u = x \]

7.7.1 SAQ: Here, \( P = -\frac{2}{x}, Q = 1 + \frac{2}{x^2}, R = xe^x \)

To remove the first derivation, choose

\[ u = e^{-\frac{1}{2} \int P \text{d}x} = e^{-\frac{1}{2} \int \frac{-2}{x} \text{d}x} = e^{\log x} = x \]

Let \( y = uv \) be the general solution. Then the normal form is

\[ \frac{d^2v}{dx^2} + Iv = S \quad \text{---------- (1)} \]

where \( I = Q - \frac{1}{4} p^2 - \frac{1}{2} \frac{dP}{dx} = 1 + \frac{2}{x^2} - \frac{1}{4} \left( \frac{4}{x^2} \right) = 1 \)

\[ S = \frac{R}{u} = \frac{xe^x}{x} = e^x \]

substituting these values in (1), we get the normal form
\[
\frac{d^2 v}{dx^2} + v = e^x
\]

7.10 SUMMARY

7.11 TECHNICAL TERMS
Cauchy-Euler Equation
Legendre’s Linear Equation
Variation of Parameters

7.12 EXERCISE
Solve the following differential equations

1. \[\frac{x^2 d^2 y}{dx^2} - x \frac{dy}{dx} + y = 2 \log x\]
2. \[\frac{x^4 d^3 y}{dx^3} + 2x^3 \frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1\]
3. \[\frac{x^2 d^2 y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x\]
4. \[\frac{x^2 d^2 y}{dx^2} + 2x \frac{dy}{dx} - 20y = (x + 1)^2\]
5. \[\frac{x^2 d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x\]
6. \[(2x + 3)^3 \frac{d^2 y}{dx^2} - 2(2x + 3) \frac{dy}{dx} - 12y = 6x\]
7. \[(x + 3)^2 \frac{d^2 y}{dx^2} - 4(x + 3) \frac{dy}{dx} + 6y = \log(x + 3)\]
8. \[x \frac{d^2 y}{dx^2} + (1 - x) \frac{dy}{dx} - y = e^x\]
9. \[xy'' - (2x - 1)y'' + (x - 1)y' + (x - 1)y = e^x\]
10. \[y'' - x^2 y' + xy = x\]
11. \[y'' - 2 \tan xy' + 3y = 2 \sec x . \text{ Given that } y = \sin x \text{ is a solution.}\]
12. \[(x - 1) y'' - xy' + y = (x - 1)^2\]
13. \( \frac{d^2y}{dx^2} - 2 \tan x \frac{dy}{dx} + 5y = 0 \) by reducing it into normal form.

14. \( \frac{d^2y}{dx^2} - 2xy' + (x^2 + 5)y = xe^{-x^2/2} \)

15. \( \frac{d^2y}{dx^2} - 2xy' + (x^2 + 2)y = e^{(x^2+2x)/2} \)

16. \( \frac{d^2y}{dx^2} + (\tan x)y' + (\cos^2 x)y = 0 \)

17. \( \frac{d^2y}{dx^2} + \frac{2}{x}y' + \frac{a^2}{x^4}y = 0 \)

18. \( (1 + x^2)y'' + xy' + 2y = 0 \)

19. \( \cos xy'' + \sin x y' - 2 \cos^3 xy = 2 \cos^5 x \)

20. \( \frac{d^2y}{dx^2} + (3\sin x - \cot x)y' + 2y \sin^2 x = \sin^2 x e^{-\cos x} \)

Solve the Following Differential Equations by the Method of Variation of Parameters:

21. \( \frac{d^2y}{dx^2} + 4y = 2 \tan x \)

22. \( (D^2 + 1)y = \csc x \cot x \)

23. \( \frac{d^2y}{dx^2} + 4y = 4 \sec^2 2x \)

24. \( (D^2 + 1)y = \sec x \)

25. \( (D^2 + a^2)y = \sec ax \)

26. \( (D^2 - 2D)y = e^x \sin x \)

7.13 ANSWERS TO EXERCISES

1. \( y = (c_1 + c_2 \log x) x + 2 \log x + 4 \)

2. \( y = c_1 x^{-1} + c_2 x + c_3 x \log x + \frac{1}{4} x^{-1} \log x \)

3. \( y = x \left[ c_1 \cos (\log x) + c_2 \sin (\log x) \right] + x \log x \)

4. \( y = c_1 x^4 + c_2 x^{-5} - \frac{1}{14} x^2 - \frac{1}{9} x - \frac{1}{20} \)
5. \[ y = c_1 x^{-1} + c_2 x^{-2} + x^{-2}e^{-x} \]

6. \[ y = c_1 (2x + 3)^{-1} + c_2 (2x + 3)^3 - \frac{3}{4} (2x + 3) + 3 \]

7. \[ y = c_1 (x + 3)^2 + c_2 (x + 3)^3 + \frac{1}{6} \log(x + 3) + \frac{5}{36} \]

8. \[ y = e^x \log x + c_1 e^x \int x^{-1}e^{-x}dx + c_2 e^x \]

9. \[ y = e^x \left[ c_1 + c_2 \log x + \frac{1}{x} \right] \]

10. \[ y = 1 + c_1 x \int x^{-2} e^{x^3/3} dx + c_2 x \]

11. \[ y = -\frac{1}{2} \sec x - c_1 \left( \cos x - \frac{1}{2} \sec x \right) + c_2 \sin x \]

12. \[ y = c_1 e^x + c_2 x - x^2 - 1 \]

13. \[ y = \left[ c_1 \cos(x\sqrt{6}) + c_2 \sin(x\sqrt{6}) \right] \sec x \]

14. \[ y = \left[ c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4} \right] e^{\frac{x^2}{2}} \]

15. \[ y = e^{x^2/2} \left[ c_1 \cos(x\sqrt{3}) + c_2 \sin(x\sqrt{3}) + \frac{e^x}{4} \right] \]

16. \[ y = c_1 \cos (\sin x) + c_2 \sin (\sin x) \]

17. \[ y = c_1 \cos \left( \frac{a}{x} \right) + c_2 \sin \left( \frac{a}{x} \right) \]

18. \[ y = c_1 \cos (x + 1)e^{-x} - c_2 \sin (x + 1)e^{-x} + (x + 1)e^{-x} \]

19. \[ y = c_1 e^{\sqrt{2} \sin x} + c_2 e^{-\sqrt{2} \sin x} + \sin^2 x \]

20. \[ y = c_1 e^{\cos x} + c_2 e^{2\cos x} + \frac{1}{6} e^{-\cos x} \]

21. \[ y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log |\sec 2x + \tan 2x| \]

22. \[ y = c_1 \cos x + c_2 \sin x - x \sin x - \cos x \log \sin x \]
23. \( y = c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \cdot \log |\sec 2x + \tan 2x| \)

24. \( y = c_1 \cos x + c_2 \sin x + \cos x \log (\cos x) + x \sin x \)

25. \( y = c_1 \cos ax + c_2 \sin ax + \frac{\cos ax}{a^2} \log (\cos x) + \frac{x}{a} \sin ax \)

26. \( y = c_1 + c_2 e^{2x} - \frac{e^x \sin x}{2} \)

### 7.14 MODEL QUESTIONS

1. Solve \( \frac{d^2 y}{dx^2} + y = \cosec x \) by the method of variation of parameters.

2. Solve \( \frac{d^2 y}{dx^2} - \cot x \frac{dy}{dx} \left( 1 - \cot x \right) y = e^x \sin x \)

3. If \( y = x \) and \( y = xe^{2x} \) are solutions of the homogeneous equation corresponding to

\[ x^2 \frac{d^2 y}{dx^2} - 2x (1 + x) \frac{dy}{dx} + 2(x + 1)y = x^3 \]

Solve it by the method of variation of parameters.

4. Solve \( x^6 \frac{d^2 y}{dx^2} + 3x^5 \frac{dy}{dx} + a^2 y = \frac{1}{x^2} \) by changing the independent variable.

5. Solve \( y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2} \sin 2x \)

6. Solve \( y'' + (1 - \cot x)y' - y \cot x = \sin^2 x \), given that \( y = e^{-x} \) is a part of C.F.
Lesson - 8

METHOD OF UNDETERMINED COEFFICIENTS AND SIMULTANEOUS LINEAR EQUATIONS

8.1 OBJECTIVE OF THE LESSON

In this lesson, we learn how to solve the differential equation \( f(r)y = Q(x) \) by the method of undetermined coefficients and also to solve simultaneous linear differential equations. In this lesson, we deal with systems of linear differential equations with constant coefficients only. They have a wide variety of applications in physics, engineering, medicine and ecology.

8.2 STRUCTURE OF THE LESSON

This lesson has the following components.

- 8.3 Introduction
- 8.4 Method of Undetermined Coefficients
- 8.5 Simultaneous linear equations
- 8.6 An Equivalent Triangular System
- 8.7 Summary
- 8.8 Technical Terms
- 8.9 Exercises
- 8.10 Answers to Exercises
- 8.11 Model Examination Questions
- 8.12 Reference Books

8.3 INTRODUCTION

In this lesson will learn the following method for solving linear differential equation \( f(r)y = Q(x) \) by method of undetermined coefficients and system of equations is solved by eliminating all but one of the dependent variables and then solving the resulting equations as before. Each of the dependent variables is obtained in a similar manner.

8.4 METHOD OF UNDETERMINED COEFFICIENTS

A general solution of a non-homogeneous linear equation
\[
a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = Q(x) \quad \text{(1)}
\]

where \( a_0 \neq 0 \), \( Q(x) \neq 0 \) and \( a_i \) are constants \( i = 0, 1, 2, \ldots, n \) is of the form

\[
y = y_c + y_p
\]

where \( y_c \) is the complementary function and \( y_p \) is any particular integral of the non-homogeneous equations. There is a much simpler special method of particular interest, which we discuss now. This method is called the method of undetermined coefficients.

**RULES FOR THE METHOD OF UNDETERMINED COEFFICIENTS**

8.4.1 : No term of \( Q(x) \) in equation (1) is the same as a term of \( y_c \). In this case if \( Q(x) \) in (1) is one of the function in the first column in Table 1. Choose the corresponding function \( y_p \) in the second column and determine its undetermined coefficients by substituting \( y_p \) and its derivatives into (1).

**Table 1**

<table>
<thead>
<tr>
<th>Term in ( Q(x) )</th>
<th>Choice for ( y_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Ke^{\alpha x} )</td>
<td>( ce^{\alpha x} )</td>
</tr>
<tr>
<td>( Kx^n ) (( n = 0, 1, 2, \ldots ))</td>
<td>( c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 )</td>
</tr>
<tr>
<td>( K \cos wx )</td>
<td>( c_1 \cos wx + c_2 \sin wx )</td>
</tr>
<tr>
<td>( K \sin wx )</td>
<td>( e^{ax} (c_1 \cos wx + c_2 \sin wx) )</td>
</tr>
</tbody>
</table>

8.4.2 Example : Solve \((D^2 + 4D + 4)y = 4x^2 + 6e^x\).

Here A.E. is \( m^2 + 4m + 4 = 0 \Rightarrow (m + 2)^2 = 0 \Rightarrow m = -2, -2 \)

\( \therefore \) C.F. is \( y_c = (c_1 + c_2 x)e^{-2x} \)

Since \( Q(x) = 4x^2 + 6e^x \) has no term common with \( y_c \),

\( y_p = Ax^2 + Bx + C + De^x \)
where $A$, $B$, $C$, $D$ are to be determined.

\[ y_p' = 2Ax + B + De^x \]

\[ y_p'' = 2A + De^x \]

Substituting these values in given equation we get

\[ 2A + De^x + 4\left(2Ax + B + De^x\right) + 4\left(Ax^2 + Bx + C + De^x\right) = 4x^2 + 6e^x \]

Simplifying and equating the coefficients of like terms we get

\[ 4A = 4, \quad 8A + 4B = 0 \]

\[ 2A + 4B + 4C = 0, \quad 9D = 6 \]

\[ \Rightarrow A = 1, \quad B = -2, \quad C = \frac{3}{2}, \quad D = \frac{2}{3} \]

\[ \therefore y_p = x^2 - 2x + \frac{2}{3}e^x + \frac{3}{2} \]

Hence the general solution is $y = y_c + y_p$

\[ y = \left(c_1 + c_2x\right)e^{-2x} + x^2 - 2x + \frac{2}{3}e^x + \frac{3}{2} \]

### 8.4.3 Example:

Solve $\left(D^2 + 2D + 5\right)y = 12e^x - 34 \sin 2x$.

**Solution:** Here A.E. is $m^2 + 2m + 5 = 0 \Rightarrow m = -1 \pm 2i$

C.F. is $y_c = e^{-x}\left[C_1 \cos 2x + C_2 \sin 2x\right]$

Also, $y_p = Ae^x + B \sin 2x + C \cos 2x$

\[ y_p' = Ae^x + 2B \cos 2x - 2C \sin 2x \]

\[ y_p'' = Ae^x + 4B \sin 2x - 4C \cos 2x \]

Substituting these values in equation, we get

\[ \left(Ae^x - 4B \sin 2x - 4C \cos 2x\right) + 2\left(Ae^x + 2B \cos 2x + -2C \sin 2x\right) + 5\left(Ae^x + B \sin 2x + C \cos 2x\right) \]

\[ = 12e^x - 34 \sin 2x \]
Equating the coefficients of like terms, we get

\[ 8A = 12, \ B - 4c = -34, \ 4B + C = 0 \]

Thus, \( A = \frac{3}{2}, \ B = -2, \ C = 8 \), and

\[ y_p = \frac{3}{2}e^x - 2\sin 2x + 8\cos 2x \]

\[ \therefore \text{The general solution of the given equation is } y = y_c + y_p \]

\[ y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{3}{2}e^x - 2\sin 2x + 8\cos 2x \]

8.4.4 Example: Solve \( y'' + 4y = 8x^2 \)

Solution: The given equation may be written using the operator \( D \) as \( (D^2 + 4)y = 8x^2 \).

A.E. is \( m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \pm 2i \)

C.F. is \( y_c = c_1 \cos 2x + c_2 \sin 2x \)

Also \( y_p = Ax^2 + Bx + C \) Then

\[ y'_p = 2Ax + B \]

\[ y''_p = 2A \]

substituting these values is given equation, we get

\[ 2A + 4(Ax^2 + Bx + C) = 8x^2 \]

Equating the coefficients of like terms, we get

\[ 4A = 8, \quad 4B = 0, \quad 2A + 4C = 0 \]

Thus \( A = 2, \ B = 0, \ C = -1 \)

Hence \( y_p = 2x^2 - 1 \)

\[ \therefore \text{The general solution of the given problem is} \]

\[ y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + 2x^2 - 1 \]
**Rule 2**: When \( Q(x) \) is (1) contains a term which is \( x^K \) times a term \( f(x) \) of \( y_c \); Where \( K \) is zero or a positive integer, the particular integral \( y_p \) of equation (1) will be a linear combination of \( x^{K+1} f(x) \) and all its linearly independent derivatives (2) (ignoring the constant coefficients). If in addition \( Q(x) \) contains terms which correspond to Rule 1, then the proper terms required by this case must be included in \( y_p \).

**8.4.6 Example**: Solve \( (D^2 - 3D + 2)y = 2x^2 + 3e^{2x} \). A.E. is \( m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2 \)

\[
\therefore \text{C.F is } y_c = c_1e^x + c_2e^{2x}
\]

Comparing right hand side of given equation with \( y_c \), we observe that \( Q(x) \) contains \( e^{2x} \) which is \( x^0 \) times the same terms in \( y_c \). Hence for this term, \( y_p \) must contain a linear combination of \( x^{0+1} e^{2x} \) and all its linearly independent derivatives. Also \( Q(x) \) has the term \( x^2 \) which belongs to Rule 1. For this term \( y_p \) must include a linear combination of it and all its linearly independent derivatives, we can neglect the function \( e^{2x} \) as it already appeared in \( y_c \). Therefore

\[
y_p = Ax^2 + Bx + C + Dxe^{2x}
\]

Differentiate it twice to get

\[
y'_p = 2Ax + B + 2Dxe^{2x} + De^{2x}
\]

\[
y''_p = 2A + 4Dxe^{2x} + 4De^{2x}
\]

Substituting the values of \( y_p, y'_p \) and \( y''_p \) in given differential equation, we get

\[
2A + 4Dxe^{2x} + 4De^{2x} - 3(2Ax + B + 2Dxe^{2x} + De^{2x}) + 2(Ax^2 + Bx + C + Dxe^{2x})
\]

\[
= 2x^2 + 3e^{2x}
\]

\[
\Rightarrow 2Ax^2 + (2B - 6A)x + (2A - 3B + 2C) + De^{2x} = 2x^2 + 3e^x
\]

Equating the coefficients of like terms on the two sides of this equation, we get

\[
2A = 2, \ 2B - 6A = 0; \ 2A - 3B + 2C = 0, \ D = 3
\]

\[
\Rightarrow A = 1, \ B = 3, \ C = \frac{7}{2}, \ D = 3
\]
The general solution is
\[ y = y_c + y_p = c_1e^x + c_2e^{2x} + x^2 + 3x + \frac{7}{2} + 3xe^{2x} \]

**8.4.7 Example:** Solve \( \left(D^2 - 3D + 2\right)y = xe^{2x} + \sin x \)

**Solution:**
A.E. is \( m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2 \)

C.F. is \( y_c = c_1e^x + c_2e^{2x} \)
\( y_p = Ax^2e^{2x} + Bxe^{2x} + C\sin x + D\cos x \)

Differentiate \( y_p \) twice to get
\[ y_p' = 2Axe^{2x} + 2Ax^2e^{2x} + Be^{2x} + 2Bxe^{2x} + C\cos x - D\sin x \]
\[ y_p'' = \left(4Ax^2 + (8A + 4B)x + 2A + 4B\right)e^{2x} - C\sin x - D\cos x \]

Substituting \( y_p, y_p', y_p'' \) in the given differential equation to get
\[ \left(4Ax^2 + (8A + 4B)x + 2A + 4B\right)e^{2x} - C\sin x - D\cos x \]
\[ -3\left(2Ax^2 + (2A + 2B)x + B\right)e^{2x} + C\cos x - D\sin x \]
\[ + 2\left(Ax^2 + Bx\right)e^{2x} + C\sin x + D\cos x \]
\[ = xe^{2x} + \sin x \]

Equating the coefficients of like terms on the two sides of this equation, to get
\[ 4A - 6A + 2A = 0 \]
\[ 8A + 4B - 6A - 6B + 2B = 1 \]
\[ 2A + 4B - 3B = 0 \]
\[ -C + 3D + 2C = 1 \]
\[ -D - 3C + 2D = 0 \]
Then we obtain \( y_p = \frac{1}{2}x^2e^{2x} - xe^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x \)

\[ \therefore \text{The general solution is} \quad y = y_c + y_p \]

\[ y = C_1e^x + C_2e^{2x} + \frac{1}{2}x^2e^{2x} - xe^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x \]

**Rule 3**: If (i) the A.E. of (1) has an \( r \) multiple root and (ii) \( Q(x) \) contains a term which is \( x^k f(x) \), \( f(x) \) is a term in \( y_c \) and is obtained from the \( r \) multiple root. Then \( y_p \) will be a linear combination of \( x^{k+r} f(x) \) and all its linearly independent derivatives. If in addition, \( Q(x) \) contains terms that arise as in Rule 1 and Rule 2, then the proper terms, which these rules demand, must also be included in \( y_p \).

**8.4.9 Example**: Solve \( (D^2 + 4D + 4)y = 3xe^{-2x} \)

**Solution**: A.E. is \( m^2 + 4m + 4 = 0 \Rightarrow m = -2, -2 \)

\[ \therefore \text{C.F. is} \quad y_c = (C_1 + C_2x)e^{-2x} \].

Note that the A.E. has a multiple root \( m = -2 \), and \( Q(x) \) contains the term \( xe^{-2x} \) which is \( x \) times the term \( e^{-2x} \) in \( y_c \), and that this term in \( y_c \) came from a multiple root. Here \( r = 2 \) & \( k = 1 \). \( \therefore y_p \) must be a linear combination of \( x^3 e^{-2x} \) and all its linearly independent derivatives. Thus

\[ y_p = Ax^3e^{-2x} + Bxe^{2x} \quad \text{(neglect} \ e^{-2x} \text{and} \ xe^{-2x} \text{they are already in} \ y_c \)\]

Differentiate it twice to get

\[ y_p' = (3Ax^2 + 2Bx)e^{-2x} - 2(Ax^3 + Bx^2)e^{-2x} \]
\[ y'_p = (-2Ax^3 + (3A - 2B)x^2 + 2Bx)e^{-2x} \]
\[ y''_p = (-6Ax^2 + 2(3A - 2B)x + 2B)e^{-2x} - 2\left[-2Ax^3 + (3A - 2B)x^2 + 2Bx\right]e^{-2x} \]
\[ = \left[4Ax^3 + (-12A + 4B)x^2 + (6A - 8B)x + 2B\right]e^{-2x} \]

Substituting \( y_p, y'_p, y''_p \) in the given differential equations to get
\[
\left\{4Ax^3 + (-12A + 4B)x^2 + (6A - 8B)x + 2B + 4\left[-2Ax^3 + (3A - 2B)x^2 + 2Bx\right] + 4\left[Ax^3 + Bx^2\right]\right\}e^{-2x} = 3xe^{-2x}
\]
\[ \Rightarrow (6Ax + 2B)e^{-2x} = 3xe^{-2x} \]

Equating like terms on both sides to get
\[ 6A = 3 \Rightarrow A = \frac{1}{2} \]
\[ 2B = 0 \Rightarrow B = 0 \]

Hence, \( y_p = \frac{1}{2}x^3 e^{-2x} \)

\[ \therefore \text{The general solution is } y = y_c + y_p \]
\[ y = (c_1 + c_2x)e^{-2x} + \frac{1}{2}x^3 e^{-2x} \]

### 8.5 SIMULTANEOUS LINEAR EQUATION

**Definition:** The system of equations

\[
\begin{align*}
P_{11}(D)y_1 + P_{12}(D)y_2 + \cdots + P_{1n}(D)y_n &= Q_1(t) \\
P_{21}(D)y_1 + P_{22}(D)y_2 + \cdots + P_{2n}(D)y_n &= Q_2(t) \\
\vdots & \quad \vdots \\
P_{n1}(D)y_1 + P_{n2}(D)y_2 + \cdots + P_{nn}(D)y_n &= Q_n(t)
\end{align*}
\]

where \( D = \frac{d}{dt}, y_1, y_2, \cdots, y_n \) are functions and \( P_{ij}(D) \) polynomial operators, is called a system of \( n \) linear differential equations.
8.5.1 Solution of system of linear equations: A solution of the system of linear equations (1) is a set of functions \( y_1(t), y_2(t), \ldots, y_n(t) \) each defined on a common interval \( I \), satisfying all equations of (1) for all \( t \) in the interval \( I \). The solution is general if the set of functions \( y_1(t), y_2(t), \ldots, y_n(t) \) contains the correct number of constants (see theorem 8.5.2).

We shall solve the system (1) by means of operators. Since the polynomial operator obeys all the rules of algebra, we shall use the method for solving system of equations (1) similar to that used in solving an algebraic system of simultaneous equations. However, there are two important differences between the two systems:

1. The operator symbol \( D \) is not a numerical quantity. It is a differential operator, operating on a function. Hence, the order in which the operators are written is important.
2. Solutions of algebraic systems do not contain arbitrary constants, while the general solution of (1) does have constants.

To decide about the number of arbitrary constants appearing in the general solution of (1), we accept the following theorem without proof.

8.5.2 Theorem: Consider the pair of equations

\[
\begin{align*}
P_1(D)x + P_2(D)y &= q_1(t) \\
P_3(D)x + P_4(D)y &= q_2(t)
\end{align*}
\]

The number of arbitrary constants in the general solution \( x(t), y(t) \) of (2) is equal to the order of

\[
P_1(D)P_4(D) - P_2(D)P_3(D)
\]

provided that \( P_1(D)P_4(D) - P_2(D)P_3(D) \neq 0 \).

If \( P_1(D)P_4(D) - P_2(D)P_3(D) = 0 \), then the system (2) is called degenerate, and if it is non-zero, then it is non-degenerate.

Since, equation (3) has the same form as a determinant write

\[
\begin{vmatrix}
P_1(D) & P_2(D) \\
P_3(D) & P_4(D)
\end{vmatrix} = P_1(D)P_4(D) - P_2(D)P_3(D)
\]

8.5.3 Example: Solve

\[
\begin{align*}
\frac{dx}{dt} - x + \frac{dy}{dt} + 4y &= 1 \\
\frac{dx}{dt} - \frac{dy}{dt} &= t - 1
\end{align*}
\]

---

Differential Equation,
Abstract Algebra...
Solution : Letting \( D = \frac{d}{dt} \), we have

\[
(2D - 1)x + (D + 4)y = 1 \\
Dx - Dy = t - 1
\]

-------- (2)

Multiplying the first equation by \( D \), the second by \((D + 4)\), and adding then, we get

\[
D(2D - 1)x + D(D + 4)y = D(1) = 0 \\
(D + 4)Dx - D(D + 4)y = (D + 4)(t - 1) = 4t - 3
\]

adding \( (3D^2 + 3D)x = 4t - 3 \) -------- (3)

A.E. of (3) is \( 3m^2 + 3m = 0 \Rightarrow m(m + 1) = 0 \Rightarrow m = 0, -1 \)

C.F. of (3) is \( x_c = c_1 + c_2 e^{-t} \)

P.I. of (3) is \( x_p = \frac{1}{3D^2 + 3D} (4t - 3) = \frac{1}{3D} \frac{1}{D + 1} (4t - 3) \)

\[
= \frac{1}{3D} (1 + D)^{-1} (4t - 3) \\
= \frac{1}{3D} (1 - D + D^2 - \cdots) (4t - 3) \\
= \frac{1}{3D} (4t - 3 - 4) = \frac{1}{3D} (4t - 7) \\
= \frac{1}{3} \int (4t - 7) dt = \frac{1}{3} \left( \frac{4t^2}{2} - 7t \right) \\
= \frac{2}{3} t^2 - \frac{7}{3} t
\]

The general solution of (3) is \( x = x_c + x_p \)

\[
x(t) = c_1 + c_2 e^{-t} + \frac{2}{3} t^2 - \frac{7}{3} t \quad \text{-------- (4)}
\]

substituting (4) in to the second equation of (2)
Differential Equation, Abstract Algebra...

8.11 Method of Undetermined...

\[ Dx - Dy = t - 1 \]
\[ Dy = Dx - t + 1 \]

\[ = D \left[ c_1 + c_2 e^{-t} + \frac{2}{3} t^2 - \frac{7}{3} t \right] - t + 1 \]
\[ = -c_2 e^{-t} + \frac{4}{3} t - \frac{7}{3} - t + 1 \]
\[ Dy = -c_2 e^{-t} + \frac{t - 4}{3} \]

Integrating on both sides we get

\[ y(t) = \left( -c_2 e^{-t} + \frac{t - 4}{3} \right) dt + c_3 \]
\[ y(t) = c_2 e^{-t} + \frac{t^2}{6} - \frac{4}{3} t + c_3 \quad \text{(5)} \]

Here, the determinant of (2) is

\[
\begin{vmatrix}
2D - 1 & D + 4 \\
D & -D
\end{vmatrix} = -3D^2 - 3D
\]

which is of order two. Hence by theorem 8.5.2, the number of constants appearing in the general solution must be two, but equating (4) and (5), we have three constants. To find the relation between these three constants, we use the fact that the solution of a system of equations is a set of functions, which satisfies each equation of the system identically. Now, substituting, \( x(t) \), \( y(t) \) from (4) & (5) in (2), we get

\[
(2D - 1) \left[ c_1 + c_2 e^{-t} + \frac{2}{3} t^2 - \frac{7}{3} t \right] + (D + 4) \left[ c_2 e^{-t} + \frac{t^2}{6} - \frac{4}{3} t + c_3 \right] = 1 \quad \text{(6)}
\]
\[ \Rightarrow c_3 = \frac{c_1 + 7}{4} \]

with this values of \( c_3 \), (6) will be true for all \( t \). Putting this value in (5), we have

\[ y(t) = c_2 e^{-t} + \frac{t^2}{6} - \frac{4}{3} t + \frac{c_1 + 7}{4} \quad \text{(7)} \]

The pair of functions \( x(t) \), \( y(t) \) defined by equations (4) and (7) contain only two arbitrary constants, and is the general solution of (2).
8.5.4 Example: Solve \( \frac{dx}{dt} = 3x + 2y \)
\( \frac{dy}{dt} + 5x + 3y = 0 \)

Solution: The given equations can be written as
\[
(D - 3)x - 2y = 0 \\
5x + (D + 3)y = 0
\]
where \( D = \frac{d}{dt} \) \quad \quad (1)

To eliminate \( y \) we operate first equation of (1) by \( (D + 3) \) and multiplying second by (2), we have
\[
(D + 3)(D - 3)x - 2(D + 3)y = 0 \\
10x + 2(D + 3)y = 0
\]

Adding \( (d^2 + 1)x = 0 \) \quad \quad (2)

A.E. of (2) is \( m^2 + 1 = 0 \Rightarrow m = \pm i \)

solution of (2) is \( x(t) = c_1 \cos t + c_2 \sin t \) \quad \quad (3)

substituting the value of \( x \) in first equation of (1)
\[
(D - 3)[c_1 \cos t + c_2 \sin t] - 2y = 0
\]
\( \Rightarrow 2y = -c_1 \sin t + c_2 \cos t - 3c_1 \cos t - 3c_2 \sin t \)
\( \Rightarrow y = \frac{1}{2}[(c_2 - 3c_1) \cos t - (c_1 + 3c_2) \sin t] \) \quad \quad (4)

since the determinant of (1) is
\[
\begin{vmatrix} D - 3 & -2 \\ 5 & D + 3 \end{vmatrix} = D^2 - 9 + 10 = D^2 + 1
\]

which is of order two. Hence, the number of constants in \( x(t), y(t) \) is equal to the order.

\( \therefore \) The pair of functions \( x(t), y(t) \) from equations (3) & (4) contains only two constants and is the general solution of (1).
8.5.7 Example: Solve \[
\frac{dy}{dx} + \frac{dz}{dx} + 2y + z = 0
\]
\[
\frac{dz}{dx} + 5y + 3z = 0
\]

**Solution:** Given equation in operator form \( D \equiv \frac{d}{dx} \)

\[
(D + 2)y + (D + 1)z = 0 \quad \text{-------- (1)}
\]
\[
(D + 3)z + 5y = 0 \quad \text{-------- (2)}
\]

Eliminating \( y \) from (1) & (2)

multiplying (1) by 5 and operating (2) by \((D+2)\)

\[
5(D + 2)y + 5(D + 1)z = 0
\]
\[
5(D + 2)y + (D + 2)(D + 3)z = 0
\]
\[
-(5(D + 1) - (D + 2)(D + 3))z = 0
\]
\[
-(D^2 + 1)z = 0
\]
\[
\Rightarrow (D^2 + 1)z = 0 \quad \text{-------- (3)}
\]

A.E. is \( m^2 + 1 = 0 \Rightarrow m = \pm i \)

solution of (3) is
\[
z = c_1 \cos x + c_2 \sin x
\]

substituting \( z \) in (2) we get
\[
(D + 3)[c_1 \cos x + c_2 \sin x] + 5y = 0
\]
\[
(-c_1 \sin x + c_2 \cos x + 3c_1 \cos x + 3c_2 \sin x) + 5y = 0
\]
\[
5y = (c_1 - 3c_2) \sin x - (c_2 + 3c_1) \cos x
\]
\[
y = \frac{1}{5}(c_1 - 3c_2) \sin x - \left(\frac{c_2 + 3c_1}{5}\right) \cos x
\]

Now
\[
\begin{vmatrix}
D + 2 & D + 1 \\
D + 3 & 5
\end{vmatrix}
= 5D + 10 - D^2 - 4D - 3
\]
= \(- (D^2 - D - 7)\) which is of order 2.

So, the general solution must contain two arbitrary constants.

So, the general solution of (1) & (2) is

\[ z = c_1 \cos x + c_2 \sin x \]

\[ y = \frac{(c_1 - 3c_2)}{5} \sin x - \frac{(c_2 + 3c_1)}{5} \cos x \]

8.5.8 Example : Solve

\[ \frac{dx}{dt} + 4x + 3y = t \]

\[ \frac{dy}{dx} + 2x + 5y = e^t \]

Solution : Given equations in operator form

\[ (D + 4)x + 3y = t \quad \text{------- (1)} \]

\[ 2x + (D + 5)y = e^t \quad \text{------- (2)} \]

Let us eliminate \( y \) from (1) & (2)

operating on (1) by \((D + 5)\) and multiplying (2) by 3.

\[ (D + 5)(D + 4)x + 3(D + 5)y = (D + 5)t \quad \text{------- (3)} \]

\[ 6x + 3(D + 5)y = 3e^t \quad \text{------- (4)} \]

Subtracting \((D^2 + 9D + 20 - 6)x = 1 + 5t - 3e^t \quad \text{------- (5)} \)

A.E. is \( m^2 + 9m + 14 = 0 \Rightarrow m = -2, -7 \)

\[ \therefore x_c = c_1 e^{-2t} + c_2 e^{-7t} \]

\[ x_p = \frac{1}{D^2 + 9D + 14}(1 + 5t - 3e^t) \]

\[ = \frac{1}{D^2 + 9D + 14}(1 + 5t - 3 \cdot \frac{1}{D^2 + 9D + 14} \cdot e^t) \]
\[
\frac{1}{14} \left[ 1 + \frac{D^2 + 9D}{14} \right] (1 + 5t) - \frac{3e^t}{1^2 + 9 \cdot 1 + 14} = \frac{1}{14} \left[ 1 + \frac{D^2 + 9D}{14} \right] (1 + 5t) - \frac{3e^t}{24} \\
= \frac{1}{14} \left[ 1 - \frac{D^2 + 9}{14} + \ldots \right] (1 + 5t) - \frac{e^t}{8} \\
= \frac{1}{14} \left[ 1 + 5t - \frac{9.5}{14} \right] - \frac{e^t}{8} = \frac{5t}{14} - \frac{3t}{196} - \frac{1}{8} e^t
\]

\[ \therefore \text{Solution of (5) is} \]

\[ x = c_p + x_p \]

\[ x = c_1 e^{-2t} + c_2 e^{-7t} + \frac{5t}{14} - \frac{31}{196} - \frac{1}{8} e^t \]

\[ \frac{dx}{dt} = -2c_1 e^{-2t} - 7c_2 e^{-7t} + \frac{5t}{14} - \frac{31}{196} - \frac{1}{8} e^t \]

Substituting \( x, \frac{dx}{dt} \) in (1) we get

\[ (D + 4) x + 3y = t \]

\[ -2c_1 e^{-2t} - 7c_2 e^{-7t} + \frac{5t}{14} - \frac{1}{8} e^t - 4 \left( c_1 e^{-2t} + c_2 e^{-7t} + \frac{5t}{14} - \frac{37}{196} - \frac{1}{8} e^t \right) + 3y = t \]

\[ \Rightarrow y = \frac{1}{3} \left[ 3e_1 e^{-7t} \cdot 2c_2 e^{-2t} - \frac{3}{7} t + \frac{27}{98} + \frac{5e^t}{8} \right] \]

\[ \text{Now} \quad \begin{vmatrix} D + 4 & 3 \\ 2 & D + 5 \end{vmatrix} = D^2 + 9D + 14, \text{ which is of order 2.} \]

\[ \therefore \text{General solution must contain two arbitrary constants.} \]

Hence the general solution of (1) & (2) is

\[ x = c + e^{-2t} - 7c_2 e^{-7t} + \frac{5}{14} - \frac{1}{8} e^t \]
8.6 AN EQUIVALENT TRIANGULAR SYSTEM

Consider the pair of equation

\[
\begin{align*}
    P_1(D)x + P_2(D)y &= q_1(t) \\
    P_3(D)x + P_4(D)y &= q_2(t)
\end{align*}
\]

---------- (1)

We obtain a new system as follows: We retain one of the equations in (1), let us say the first, and change the second. This is achieved by multiplying the retained equation by any arbitrary operator \( K(D) \) and adding it to the second. The new system thus, takes the form

\[
\begin{align*}
    P_1(D)x + P_2(D)y &= q_1(t) \\
    \left[ P_1(D)K(D) + P_3(D) \right]x + \left[ P_2(D)K(D) + P_4(D) \right]y &= K(D)q_1(t) + q_2(t)
\end{align*}
\]

-------- (2)

The two systems (1) and (2) are equivalent in the sense that a pair of functions \( x(t), y(t) \) which satisfy the first system will also satisfy the second and vice versa. The determinents of the both systems are same.

Thus, by retaining always one equation in the system and changing the other by the above procedure, we obtain a new system, equivalent to (1), in the form

\[
\begin{align*}
    P_1(D)x &= q_1(t) \\
    P_3(D)x + P_4(D)y &= q_2(t)
\end{align*}
\]

---------- (3)

Such system, i.e. one in which a coefficient of \( x \) or of \( y \) is zero, is called an equivalent triangular systems.

If the original system is already in the form

\[
\begin{align*}
    P_1(D)x &= q_1(t) \\
    P_4(D)y &= q_2(t)
\end{align*}
\]

-------- (4.a)

or, in the triangular form

\[
\begin{align*}
    P_1(D)x &= q_1(t) \\
    P_3(D)x + P_4(D)y &= q_2(t)
\end{align*}
\]

-------- (4.b)

Then the pair of functions \( x(t), y(t) \) obtained by solving the system will contain the correct number of constants.
8.6.1 Example: Solve \[
\begin{cases}
(3D^2 + 3D)x = 4t - 3 \\
(D - 1)x - D^2y = t^2
\end{cases}
\] \hspace{1cm} \text{------- (1)}

Solution: The first equation of (1) is \((3D^2 + 3D)x = 4t - 3\)

The A.E. is \(3m^2 + 3m = 0 \Rightarrow 3m(m + 1) = 0 \Rightarrow m = 0, -1\)

C.F. is \(x_c = c_1 + c_2e^{-t}\)

P.I. = \(x_p = \frac{1}{3D(D+1)}(4t - 3)\)

\[
= \frac{1}{3D}(1 + D)^{-1}(4t - 3)
= \frac{1}{3D}(1 - D + D^2 - \cdots)(4t - 3)
= \frac{1}{3D}(4t - 3 - 4) = \frac{1}{3D}(4t - 7)
= \frac{1}{3D}\int (4t - 7) dt
= \frac{1}{3}\left(\frac{4t^2}{2} - 7t\right) = \frac{2}{3}t^2 - \frac{7}{3}t
\]

The general solution of the first equations of (1) is

\[x(t) = x_c + x_p = c_1 + c_2e^{-t} + \frac{2}{3}t^2 - \frac{7}{3}t \hspace{1cm} \text{------- (2)}\]

Substituting it is the second equation of (1), we get

\[
(D - 1)\left[ -c_1 + c_2e^{-t} + \frac{2}{3}t^2 + \frac{7}{3}t \right] - D^2y = t^2
\]

\[D^2y = \frac{7}{3} - c_1 - c_2e^{-t} + \frac{11}{3}t - \frac{5}{3}t^2\]

\[\Rightarrow Dy = \int \left(\frac{7}{3} - c_1 - 2c_2e^{-t} + \frac{11}{3}t - \frac{5}{3}t^2\right) dt + c_3\]
The pair of functions in equations (2) and (3) is the general solution of (1) and from theorem 8.5.2 it contains the correct number of four arbitrary constants.

8.6.2 Example: Solve \[
\begin{align*}
(3D - 1)x + 4y &= t \\
Dx - Dy &= t - 1
\end{align*}
\] \[\text{---------- (1)}\]

**Solution:** We retain the first equation and obtain a second equation by multiplying the first by \(\frac{D}{4}\) and adding it to the second. We get

\[
\begin{align*}
\frac{D}{4}(3D - 1)x + Dy &= \frac{D}{4}(t) \\
Dx - Dy &= t - 1
\end{align*}
\]

\[
\text{adding} \quad \frac{1}{4}(3D^2 + 3D)x = \frac{1}{4}(1 + 4t - 4)
\]

i.e. \(3D^2 + 3D)x = 4t - 3\)

Thus, the equivalent triangular system is

\[
\begin{align*}
(3D - 1)x + 4y &= t \\
(3D^2 + 3D)x &= 4t - 3
\end{align*}
\] \[\text{---------- (2)}\]

Consider second equation of (2) is

\(3D^2 + 3D)x = 4t - 3\)

A.E. is \(3m^2 + 3m = 0 \Rightarrow m(m + 1) = 0\), \(m = 0, -1\)

C.F. is \(x_c = c_1 + c_2 e^{-t}\)

P.I. is \(x_p = \frac{1}{3D^2 + 3D}(4t - 3) = \frac{1}{3D(D + 1)}(4t - 3)\)
The general solution of the second equation of (2) is
\[ x(t) = c_1 + c_2 e^{-t} + \frac{2}{3} t^2 - \frac{7}{3} t \]
substituting it in the first equation of (2), we get
\[
(3D - 1) \left[ c_1 + c_2 e^{-t} + \frac{2}{3} t^2 - \frac{7}{3} t \right] + 4y = t
\]
\[
3 \left( -c_2 e^{-t} + \frac{4}{3} t - \frac{7}{3} - c_1 - c_2 e^{-t} - \frac{2}{3} t^2 + \frac{7}{3} t \right) - t = -4y
\]
\[
\Rightarrow y(t) = \frac{c_1 + 7}{4} + c_2 e^{-t} + \frac{1}{6} t^2 - \frac{4}{3} t
\]

The determinant of (1) is \((3D - 1)(-D) - 4D = -3D^2 - 3D\), whose order is two. Thus the pair of functions \(x(t), y(t)\) contains the correct number of constants and is the general solution of equation (1).

**8.6.3 Example** : Solve the systems

\[
\begin{cases}
(D + 4)x + Dy = 1 \\
(D - 2)x + y = t^2
\end{cases}
\]

**Solution** : Here, we retain the second equation of (1) and obtain a new first equation by multiplying the second by -D and adding it to the first
\[(D + 4)x + Dy = 1\]
\[-D(D - 2)x - Dy = -D(t^2)\]

Adding \((-D^2 + 3D + 4)x = -2t + 1\)

Thus, the equivalent triangular system is

\[
\begin{align*}
(-D^2 + 3D + 4)x &= -2t + 1 \\
(D - 2)x + y &= t^2
\end{align*}
\]

---------- (2)

Consider the first equation of (2) is

\[(-D^2 + 3D + 4)x = -2t + 1\]

A.E. is \(-m^2 + 3m + 4 = 0 \Rightarrow m = -1, 4\)

C.F. is \(x_c = c_1e^{4t} + c_2e^{-t}\)

P.I. is \(x_p = \frac{1}{-D^2 + 3D + 4}(-2t + 1) = \frac{1}{-D^2 + 3D + 4}(-2t + 1)\)

\[= \frac{1}{4(1 + D)}\left(\frac{1 + D}{4} + \frac{D^2}{16} + \cdots\right)(-2t + 1)\]

\[= \frac{1}{4(1 + D)}\left(-2t + 1 - \frac{1}{2}\right)\]

\[= \frac{1}{4}\left(1 - D + D^2 - \cdots\right)(-2t + 1)\]

\[= \frac{1}{4}\left(-2t + 1 + 2\right) = -\frac{t}{2} + \frac{5}{8}\]

The general solution of the first equation of (2) is

\[x(t) = x_c + x_p = c_1e^{4t} + c_2e^{-t} - \frac{t}{2} + \frac{5}{8} \quad \text{---------- (3)}\]
Substituting it in the second equation of (1), we get

\[(D - 2)\left[c_1e^{4t} + c_2e^{-t} - t/2 + 5/8\right] + y = t^2\]

\[\Rightarrow y(t) = -2c_1e^{4t} + 3c_2e^{-t} + t^2 - t + 7/4 \quad \text{(4)}\]

The determinant of (1) is of order two and the pair of the functions (3) and (4) contains two constants and it is a general solution of (1).

8.6.4 Degenerate Case: The system of linear differential equations

\[
\begin{align*}
P_1(D)x + P_2(D)y &= q_1(t) \\
P_3(D)x + P_4(D)y &= q_2(t)
\end{align*}
\]

is said to be degenerate if its determinant

\[
\begin{vmatrix}
P_1(D) & P_3(D) \\
P_2(D) & P_4(D)
\end{vmatrix} = 0
\]

If by eliminating \(x\) or \(y\), the right side of the system (1) is not zero, then there will be no solution; these will be infinitely many solutions if the right side is zero.

8.6.5 Example: Show that the system

\[
\begin{align*}
Dx - Dy &= t \\
Dx - Dy &= t^2
\end{align*}
\]

is degenerate and find the number of solutions it has?

Solution: Here the determinant

\[
\begin{vmatrix}
D & -D \\
D & -D
\end{vmatrix} = 0
\]

Thus, the system is degenerate. When we eliminate \(x\) or \(y\), the right side of the given system does not reduce to zero and thus it has no solution.

8.6.6 Example: Solve the system

\[
\begin{align*}
Dx - Dy &= t \\
4Dx - 4Dy &= 4t
\end{align*}
\]

Solution: Here, the determinant, \[
\begin{vmatrix}
D & -D \\
4D & -4D
\end{vmatrix} = 0
\]. Thus the system is degenerate. However, its right side reduces to zero, when \(x\) or \(y\) is eliminated. In this case, there are infinitely many solutions of the system.
8.7 SUMMARY

In this lesson we discussed the procedure for finding the Particular Integral (P.I) of the differential equation \( f(D)y = Q(x) \) by the method of undetermined coefficients. We also discussed simultaneous linear differential equations in various cases. We also discussed some problems.

8.9 TECHNICAL TERMS

Undetermined Coefficients
Multiple root
Simultaneous linear equations
Determinant
Triangular system
Degenerate

8.9 EXERCISE

1. Solve \( \frac{d^2x}{dt^2} + 4x + y = te^{3t} \)

2. Solve \( \frac{d^2y}{dt^2} + y - 2x = \cos^2 t \)

2. Solve \( (5D + 4)y = (2D + 1)z = e^{-x} \)

\( (D + 8)y - 3z = 5e^{-x} \)

3. \( \frac{dx}{dt} + y - 1 = \sin t \)

\( \frac{dy}{dt} + x = \cos t \)

4. Solve \( \frac{dx}{dt} + y = \sin t \)

\( \frac{dy}{dx} + x = \cos t \) given that \( x = 2 \) and \( y = 0 \), when \( t = 0 \)

5. Solve \( \frac{dx}{dt} + 2y = e^t \)
\[ \frac{dy}{dt} - 2x = e^{-t} \]

6. Solve \[ \frac{dx}{dt} + \frac{dy}{dt} - 2x = \sin 2t \]

\[ \frac{dx}{dt} - \frac{dy}{dt} + 2y = \cos 2t \]

Solve the following equations by the method of undetermined coefficients.

7) \[ \frac{d^2y}{dx^2} + \frac{3}{2} \frac{dy}{dx} + 2y = 12e^x \]

8) \[ \frac{d^2y}{dx^2} + \frac{3}{2} \frac{dy}{dx} + 2y = \sin x \]

9) \[ \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2 \]

10) \[ (D^2 - 2D - 8)y = 9xe^x + 10e^{-x} \]

8.10 ANSWERS TO EXERCISES

1) \[ x = \left( c_1 \cos \sqrt{3}t + c_2 \sin \sqrt{3}t \right) + \left( c_3 \cos \sqrt{2}t + c_4 \sin \sqrt{2}t \right) + \frac{5}{1452} e^{3t} - \frac{49}{1452} e^{3t} - \frac{1}{12} - \frac{1}{4} \cos 2t \]

\[ y = -\left( c_1 \cos \sqrt{3}t + c_2 \sin \sqrt{3}t \right) - 2\left( c_3 \cos \sqrt{2}t + c_4 \sin \sqrt{2}t \right) + \frac{1}{1452} e^{3t} - \frac{23}{1452} e^{3t} + \frac{1}{3} \]

2) \[ y = c_1 e^{-2x} + c_2 e^x + 2e^{-x} \]

\[ z = 2c_1 e^{-2x} + 3c_2 e^x + 3e^{-x} \]

3) \[ x = c_1 e^t + c_2 e^{-t} \]

\[ y = 1 + \sin t - c_1 e^t + c_2 e^{-t} \]

4) \[ x = e^t + e^{-t}, \quad y = -e^t + e^{-t} + \sin t \]

5) \[ x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{5} e^t - \frac{2}{5} e^{-t} \]

\[ y = \frac{1}{2}\left[ e^t + 2c_1 \sin 2t + 2c_2 \cos 2t - \frac{1}{5} e^t - \frac{2}{5} e^{-t} \right] \]
6) \( x = e^t (c_1 \cos t + c_2 \sin t) - \frac{1}{2} \cos 2t \)

\( y = e^t (c_1 \sin t - c_2 \cos 2t) - \frac{1}{2} \sin 2t \)

7) \( y = c_1 e^{-2x} + c_2 e^{-x} + 2e^x \)

8) \( y = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{10} (\sin x - 3 \cos x) \)

9) \( y = e^{-x/2} \left[ c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right] + x^2 - 2x \)

10) \( y = c_1 e^{4x} + c_2 e^{-2x} - xe^x - 2e^{-x} \)

8.11 MODEL QUESTIONS

1) Solve \( \frac{dx}{dt} = x - 2y \)

\( \frac{dy}{dt} = 5x + 3y \)

2) Solve \( \frac{dx}{dt} = -ay \)

\( \frac{dy}{dt} = ax \)

3) Solve \( \frac{dy}{dx} + \frac{dz}{dx} + 2y + z = 0 \)

\( \frac{dz}{dx} + 5y + 3z = 0 \)

4) Solve \( (D^2 - 3D)y = 2e^{2x} \sin x \) by the method of undetermined coefficients.

5) Solve \( (D^2 + D)y = x^2 + 2x \) by the method of undetermined coefficients.
Lesson - 9

NUMBER THEORY; GROUPS; SUBGROUPS

9.1 OBJECTIVE OF THE LESSON

The objectives of this lesson are to introduce to the students some basic results in number theory and an important algebraic structure called "GROUP" and its properties.

9.2 STRUCTURE OF THE LESSON

This lesson has the following components.

9.3 Introduction
9.4 Divisors
9.5 Prime Numbers
9.6 Congruences
9.7 Solutions of Linear Congruences
9.8 Euler's and Fermat's Theorems
9.9 Euler's function \( \phi(n) \)
9.10 Binary Operation
9.11 Groups
9.12 Subgroups
9.13 Answers to Self Assessment Questions (SAQ's)
9.14 Exercises
9.15 Answers to Exercises in Number theory
9.16 Model Examination Questions
9.17 Reference Books

9.3 INTRODUCTION

Sections 9.4 through 9.9 deal with number theory. The concepts of greatest common divisor (g.c.d) and least common multiple (l.c.m.) are defined and the Euclidean Algorithm to find the g.c.d. is proved. The notion of prime numbers is introduced and the unique factorization theorem, for integers (greater than 1) as a product of prime numbers is presented. It is also shown that the number of prime numbers is infinite. Solutions to linear congruences, the famous theorems such as Chinese remainder theorem, Euler's theorem, Fermat's theorem, and Wilson's theorem have been proved.
Sections 9.10 through 9.12 deal with group theory. The concepts of Binary relation groups, subgroups, cyclic groups are presented and some important properties of groups have been presented.

A good number of examples have been provided for a better understanding of the concepts involved in number theory and group theory.

**NUMBER THEORY**

**9.4 DIVISORS**

We denote the set of all integers by Z. All lower case letters represent integers unless stated otherwise.

**9.4.1 Definition**: An integer \( b \) is said to be divisible by an integer \( a \), \( a \neq 0 \), if there is an integer \( x \) such that \( b = ax \), and we write \( a | b \). If \( b \) is not divisible by \( a \), we write \( a \nmid b \).

If \( a | b \) then we say that \( a \) is a factor or divisor of \( b \) and \( b \) is multiple of \( a \).

**9.4.2 Examples**: \( 5 | 25 \), \( 8 | 56 \), \( 17 | 51 \), 12 \( \nmid 17 \), 2 \( \nmid 13 \), ....

**9.4.3 Definition**: If \( a | a \) and \( 0 < a < b \) then \( a \) is called a proper divisor of \( b \).

eg: 3 is a proper divisor of 36.

Note that, if \( a \neq 0 \) then \( a | a \) and \( a | 0 \), and that when \( a | 1 \), then \( a = \pm 1 \).

**9.4.4 Theorem**:

1. \( a | b \Rightarrow a | bc \) for any integer \( c \).
2. \( a | b \) and \( b | c \Rightarrow a | c \)
3. \( a | b \) and \( a | c \Rightarrow a | (bx + cy) \) for any \( x, y \in \mathbb{Z} \).
4. \( a | b \) and \( b | a \Rightarrow a = \pm b \)
5. \( a | b \), \( a > 0 \), \( b > 0 \) \( \Rightarrow a \leq b \)
6. If \( m \neq 0 \), \( a | b \Leftrightarrow ma | mb \)

**Proof**:

1. \( a | b \Rightarrow b = ax \) for some \( x \in \mathbb{Z} \Rightarrow bc = axc \Rightarrow a | (bc) \)
2. \( a | b \) and \( b | c \Rightarrow b = ax \) and \( c = by \) for some \( x, y \in \mathbb{Z} \)
   \[ \Rightarrow c = by = axy \Rightarrow a | c \]
(3) \( a \mid b \) and \( a \mid c \) \( \Rightarrow \) \( b = aq_1 \) and \( c = aq_2 \) for some \( q_1, q_2 \in \mathbb{Z} \)

\[
bx + cy = aq_1x + aq_2y = a(q_1x + q_2y) \Rightarrow a \mid (bx + cy).
\]

(4) \( a \mid b, b \mid a \) \( \Rightarrow b = aq_1, a = bq_2 \) for some \( q_1, q_2 \in \mathbb{Z} \) \( \Rightarrow a = aq_1q_2 \Rightarrow a - aq_1q_2 = 0 \)

\[
\Rightarrow a(1-q_1q_2) = 0 \Rightarrow 1-q_1q_2 = 0 (\therefore a \neq 0) \Rightarrow q_1q_2 = 1 \Rightarrow q_1 = q_2 = 1 \text{ or } q_1 = q_2 = -1 \Rightarrow a = \pm b
\]

(5) \( a \mid b \) \( \Rightarrow b = aq \) for some \( q \in \mathbb{Z} \).

\[
a > 0, \ b > 0, \ b = aq \Rightarrow q > 0 \Rightarrow q \geq 1 \Rightarrow aq \geq a \Rightarrow b \geq a \ (\text{or } a \leq b).
\]

(6) \( a \mid b \) \( \Rightarrow b = aq \) for some \( q \in \mathbb{Z} \) \( \Rightarrow mb = maq \Rightarrow ma \mid mb \).

By induction we obtain the following corollary.

9.4.5 **Corollary**: If \( a \neq 0, b_1, b_2, \ldots, b_n, x_1, x_2, \ldots, x_n \in \mathbb{Z} \), then \( a \mid b_1, a \mid b_2, \ldots, a \mid b_n \) \( \Rightarrow a \mid (b_1x_1 + \ldots + b_nx_n) \).

9.4.6 **Theorem (Division Algorithm)**: Given any integers \( a \) and \( b \) with \( a > 0 \), there exist unique integers \( q \) and \( r \) such that \( b = aq + r, 0 \leq r < a \).

If \( a \nmid b \), then \( r \) satisfies the stronger inequality \( 0 < r < a \).

**Proof**: Let \( S = \{b - ma/b - ma > 0; m \in \mathbb{Z}\} \). If \( m \leq \frac{b}{a} \) then \( b - ma \geq 0 \Rightarrow S \) is non-empty. By well-ordering principle, \( S \) has a least element, say \( r \).

Thus \( r \in S \) and is of the form \( r = b - qa \) for some \( q \in \mathbb{Z} \).

\[
\Rightarrow b = qa + r
\]

Clearly \( r \geq 0 \). If \( r \geq a \) then \( 0 < r - a = b - qa - a = b - (q+1)a \in S \)

\[
\therefore r - a \in S, \ r - a < r \text{ and } r \text{ is the least element in } S.
\]

This is a contradiction

So, \( 0 \leq r < a \).

To prove uniqueness. Suppose that \( b = aq_1 + r_1, \ 0 \leq r_1 < a, q_1, r_1 \in \mathbb{Z} \).

\[
qa + r = q_1a + r_1 \Rightarrow (q - q_1)a = r_1 - r
\]
9.4.7 Corollary: If \( a, b \in \mathbb{Z}, a \neq 0 \) then \( \exists \) unique \( r, q \in \mathbb{Z} \) such that \( b = qa + r, 0 \leq r < |a| \).

Proof: It suffices that if we consider the case when \( a < 0 \).

Then \( |a| = -a > 0 \)

By the theorem 9.4.6 there exist unique integers \( q', r' \) such that \( b = q'|a| + r', 0 \leq r' < |a| \).

Then \( b = q'(-a) + r' = (-q')a + r', 0 \leq r' < |a| \).

:. \( b = qa + r \) where \( q = -q' \) and \( r = r' \) and \( 0 \leq r < |a| \)

9.4.8 Definition: If \( b = aq + r, 0 \leq r < |a| \) then \( r \) is called remainder and \( q \) is called quotient.

9.4.9 Definition: If \( a | b, a | c \) then \( a \) is called common divisor (factor) of \( b \) and \( c \).

9.4.10 Definition: Let \( b \) and \( c \) be integers not both zero. A positive integer \( g \) is said to be the greatest common divisor (g.c.d) of \( b \) and \( c \) if

(i) \( g \) is a common divisor of \( b \) and (i.e.) \( g | b, g | c \)

(ii) any common divisor of \( b \) and \( c \) is a divisor of \( g. \) (i.e.) \( a | b, a | c \Rightarrow a | g \).

By part 3 of theorem 9.4.4, g.c.d. of \( b \) and \( c \) is unique.

The g.c.d. of \( b \) and \( c \) is denoted by \( (b, c) \). Similarly the g.c.d. of integers \( b_1, \ldots, b_n \), not all zero, can be defined. It is denoted by \( (b_1, \ldots, b_n) \).

9.4.11 Theorem: If \( k \in \mathbb{Z}, a = bk + c \), then \( (a, b) = (b, c) \).

Proof: Let \( (a, b) = g, (b, c) = g_1 \).

Then \( g, g_1 \) are positive integers. Now

\[
(a, b) = g \Rightarrow g | a, g | b
\]

By part 3 of theorem 9.4.4 \( g | (a - bk) \).
⇒ \( g \mid c \) ⇒ \( g \) is a common divisor of \( b \) and \( c \).

⇒ \( g \mid g_1 \) \hspace{1cm} (1)

Also \((b, c) = g_1 \Rightarrow g_1 \mid b, g_1 \mid c\)

By part 3 of theorem 9.4.4, \( g_1 \mid bk + c \)

⇒ \( g_1 \mid a \Rightarrow g_1 \) is a common divisor of \( a \) and \( b \).

⇒ \( g_1 \mid g \) \hspace{1cm} (2)

From (1) and (2), \( g = g_1 \).

9.4.12 Theorem: If \( b \) and \( c \) are integers, not both zero, then \((b, c)\) exists, more over, we can find integers \( x_0 \) and \( y_0 \) such that \((b, c) = bx_0 + cy_0\).

Proof: Let \( S = \{bx + cy/bx + cy > 0 \text{ and } x, y \in Z\} \). Since, one of \( b \) or \( c \) is not zero, \( b^2 + c^2 > 0 \), \( b^2 + c^2 = b \cdot b + c \cdot c \in S \).

Thus \( S \neq \emptyset \) and \( S \) is a subset of natural numbers. By well ordering principle, \( S \) has aleast element, say, \( s \).

Since \( s \in S \), \( s \) has the form \( s = bx_0 + cy_0 \), for some \( x_0, y_0 \in Z \).

We claim that \( s = (b, c) \).

\( d \mid b, d \mid c \Rightarrow d \mid (bx_0 + cy_0) \Rightarrow d \mid s \) \hspace{1cm} (1)

We now show that \( s \mid b \) and \( s \mid c \). By division algorithm, \( \exists q, r \in Z \) such that \( b = sq + r, 0 \leq r < s \).

If \( r \neq 0 \), then \( r = b - sq = b - (bx_0 + cy_0)q \)

\( = b(1 - x_0q) + c(-y_0q) \in S \)

This is a contradiction to the fact that \( s \) is the least element of \( S \).

\( \therefore r = 0 \) and hence \( b = sq \Rightarrow s \mid b \)

Similarly we can show that \( s \mid c \).

\( s \mid b, s \mid c \Rightarrow s \) is a common divisor of \( b \) and \( c \) \hspace{1cm} (2)

From (1) and (2), we get \( s = (b, c) \).

Hence \((b, c) = bx_0 + cy_0\) for some \( x_0, y_0 \in Z \).
9.4.13 Corollary: The g.c.d. of $b$ and $c$, not both zero, can be characterized in the following two ways.

1. It is the least positive value of $bx + cy$ where $x$ and $y$ range over all integers.
2. It is the greatest among all the common divisors of $b$ and $c$.

Proof: (1) Proof follows from the proof of theorem 9.4.12.

(2) If $d$ is any common divisor of $b$ and $c$, then $d \mid g$, by the definition of $g$.

If $d \leq 0$, then clearly $d \leq g$ since $g > 0$.

If $d > 0$, then by part 5 of theorem 9.4.4, $d \leq g$. Thus $g$ is the greatest among all common divisors of $b$ and $c$.

9.4.14 SAQ: Let $a, b,$ and $c$ be non-zero integers. Then $(a, (b,c)) = ((a,b), c)$.

9.4.15 Corollary: If $b_1, b_2, \ldots, b_n \in \mathbb{Z}$, not all zero, with g.c.d. $g$ then there exist integers $x_1, \ldots, x_n$ such that

$$g = (b_1, \ldots, b_n) = \sum_{i=1}^{n} b_i x_i$$

Proof: Let $T = \left\{ \sum_{i=1}^{n} b_i x_i \middle| \sum_{i=1}^{n} b_i x_i > 0 \text{ and } x_i \in \mathbb{Z} \text{ for } i = 1, \ldots, n \right\}$

Then $T$ is a non-empty subset of $\mathbb{N}$. Then the least element of $T$ is the g.c.d of $b_1, \ldots, b_n$.

9.4.16 SAQ: Let $a$ and $b$ be integers, not both zero. If $g = (a, b)$ then an integer $m$ is a multiple of $g$, iff, $m = ax + by$ for some integers $x$ and $y$.

9.4.17: Let $a$ and $b$ be integers, not both zero. Then for any positive integer $m$, $(ma, mb) = m(a, b)$.

Proof: By corollary 9.4.13,

$$ma, mb \text{ least positive value of } max + mby$$

$$= m\text{ (least positive value of } ax + by)$$

$$= m(a, b)$$

9.4.18 Theorem: (i) If $d \mid a$ and $d \mid b$ and $d > 0$, then $\left( \frac{a}{d}, \frac{b}{d} \right) = \left( \frac{a,b}{d} \right)$. 

(ii) If \((a, b) = g\) then \(\left(\frac{a}{g}, \frac{b}{g}\right) = 1\).

**Proof:**

(i) \((a, b) = d(\frac{a}{d}, \frac{b}{d}) = d\left(\frac{a}{d}, \frac{b}{d}\right)\)

\[\Rightarrow \left(\frac{a}{d}, \frac{b}{d}\right) = \frac{1}{d}(a, b)\]

(ii) \(\left(\frac{a}{g}, \frac{b}{g}\right) = \frac{1}{g}(a, b) = \frac{1}{g} \cdot g = 1\)

**9.4.19 Theorem:** If \((a, m) = (b, m) = 1\) then \((ab, m) = 1\)

**Proof:** By theorem 9.4.12, \(\exists\) integers \(x_0, y_0, x_1, y_1\) such that \(1 = ax_0 + my_0\) and \(1 = bx_1 + my_1\).

Now \((ax_0)(by_1) = (1 - my_0)(1 - my_1)\)

\[= 1 - my_1 - my_0 + m^2y_0y_1\]

\[= 1 - m(y_0 + y_1 - my_0y_1)\]

\[= 1 - my_2\text{ where } y_2 = y_0 + y_1 - my_0y_1\]

\[\therefore abx_0y_1 + my_2 = 1\]

\[\therefore \text{if } g = (ab, m) \text{ then } g | ab, g | m \Rightarrow g | (abx_0y_1 + my_2)\]

\[\Rightarrow g | 1 \Rightarrow g = 1\]

\[\therefore (ab, m) = 1\]

**9.4.20 Definition:** Two integers \(a\) and \(b\) are said to be relatively prime or coprime if \((a, b) = 1\). In this case we also say that \(a\) is prime to \(b\).

**9.4.21 Definition:** \(a_1, a_2, \ldots, a_n\) are said to be relatively prime if \((a_i, a_j) = 1\) for all \(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n, i \neq j\).

**9.4.22 Theorem:** For any \(x, (a, b) = (b, a) = (a, -b) = (a, b + ax)\).

**Proof:** Let \((a, b) = d\) and \((a, b + ax) = g\). It is clear that \((b, a) = (a, -b) = d\).

By application of parts 3 and 4 of theorem 9.4.4, we obtain \(d | g, g | d\) and hence \(g = d\).
9.4.23 Theorem: Let $a$ and $b$ be integers, not both zero. A necessary and sufficient condition for $a$ and $b$ to be relatively prime is that $ax + by = 1$ for some integers $x$ and $y$.

Proof: If $a$, $b$ are relatively prime, then $(a, b) = 1$. By theorem 9.4.12, exist integers $x$ and $y$ such that $ax + by = 1$.

Conversely, suppose that $ax + by = 1$ for some integers $x$ and $y$. Let $d = (a, b)$. Then $d | a$ and $d | b$ and hence $d | ax + by$.

$\therefore d = 1$. Thus $(a, b) = 1$

$\therefore a$ and $b$ are relatively prime.

9.4.24 Theorem: If $c | ab$ and $(b, c) = 1$ then $c | a$.

Proof: By theorem 9.4.17, $(ab, ac) = a(b, c) = a$. But $c | ab$ and $c | ac \Rightarrow c | a$ by corollary 9.4.13.

9.4.25 The Euclidean Algorithm: Given integers $b$ and $c > 0$, we make a repeated application of the division algorithm, to obtain a series of equations.

\[
\begin{align*}
  b &= cq_1 + r_1, \quad 0 \leq r_1 < c \\
  c &= r_2q_2 + r_2, \quad 0 \leq r_2 < r_1 \\
  r_1 &= r_3q_3 + r_3, \quad 0 \leq r_3 < r_2 \\
  \vdots & \vdots \\
  r_{j-2} &= r_{j-1}q_j + r_j, \quad 0 \leq r_j < r_{j-1} \\
  r_{j-1} &= r_jq_{j+1}
\end{align*}
\]

The g.c.d. of $b$ and $c$ is $r_j$, the least non-zero remainder in the division process. (We consider, here, $r_0 = c$). The values of $x_0$ and $y_0$ in $(b, c) = bx_0 + cy_0$ can be obtained by eliminating $r_{j-1}$, $r_{j-2}$, $\ldots$, $r_2$, $r_1$ from the above set of equations.

Proof: Since the remainders $r_1 > r_2 > \cdots$ are decreasing and all non-negative, there is a first integer $j$ such that $r_{j+1} = 0$.

We claim that $r_j = (b, c)$. We start from the bottom equation.

Since $r_j | r_{j-1}$, it follows that $r_j | r_{j-1}q_j + r_j = r_{j-2}$. Now, $r_j | r_{j-2}$, $r_j | r_{j-1}$ and hence, $r_j | r_{j-2} q_{j-1} + r_{j-1} = r_{j-3}$. In this way, by using the equations in the Euclidean Algorithm,
we obtain that \( r_j \mid r_{j-1}, r_j \mid r_{j-2}, \ldots, r_j \mid r_1, r_j \mid c, r_j \mid b \). Thus \( r_j \) is a common divisor of \( b \) and \( c \).

The \( d \) is a common divisor of \( b \) and \( c \), then \( d \mid b, d \mid c \Rightarrow d \mid (b - cq_1) \Rightarrow d \mid r_1 \nabla \). Thus \( r_j = (b, c) \) as claimed.

**9.4.26 SAQ :** In the Euclidean Algorithm (Theorem 9.4.25), considering the top two equations, prove that \((b, c) = (r_1, r_2)\). 

**9.4.27 Example :** If \( d = (1769, 2378) \) using Euclid Algorithm, compute \( d \) and then express it as a linear combination of 1769, 2378.

**Solution :** Here \( b = 2378, c = 1769 \). By division algorithm we can do as follows.

\[
\begin{align*}
\text{c = 1769) 2378 = b(1 = q_1) } \\
1769 \\
1769 \div 1218 = 2 \text{ with remainder } 551 \Rightarrow r_1 = 609 \\
1218 \\
1218 \div 551 = 2 \text{ with remainder } 58 \Rightarrow r_2 = 551 \\
551 \div 522 = 1 \text{ with remainder } 58 \Rightarrow r_3 = 551 \\
522 \div 58 = 9 \text{ with remainder } 0 \Rightarrow r_4 = 58 \\
58 \div 29 = 2 \text{ with remainder } 0 \Rightarrow r_5 = 29
\end{align*}
\]

\[
\begin{align*}
2378 &= 1 \times 1769 + 609(b = cq_1 + r_1) \\
1769 &= 2 \times 609 + 551(c = r_1q_2 + r_2) \\
609 &= 1 \times 551 + 58(r_1 = r_2q_3 + r_3)
\end{align*}
\]
Since 29 is the least non-zero remainder, we get that \((1769, 2378) = 29\).

Again 
\[ d = 29 = r_4 = r_2 - r_3 q_4 = r_2 - (q_1 - r_2 q_3) q_4 \]
\[ = r_2 - q_1 q_4 + r_2 q_3 q_4 = r_2 (1 + q_3 q_4) - q_1 q_4 = (c - q_1 q_2) (1 + q_3 q_4) - q_1 q_4 \]
\[ = c (1 + q_3 q_4) - q_1 (q_2 + q_2 q_3 q_4 + q_4) \]
\[ = c (1 + q_3 q_4) - (b - c q_1) (q_2 + q_2 q_3 q_4 + q_4) \]
\[ = c (1 + q_3 q_4 + q_1 q_2 + q_1 q_2 q_3 q_4 + q_1 q_4) - b (q_2 + q_2 q_3 q_4 + q_4) \]

Substituting \(q_1, q_2, q_3\) and \(q_4\) we get
\[ 29 = c (1 + 1\times 9 + 1\times 2 + 1\times 9 + 1\times 2\times 1\times 9) - b (2 + 9 + 2\times 1\times 9) \]
\[ = 39 c - 29 b = (39)(1769) + (-29)(2378) \]

**9.4.28 Definition** : Let \(a, b\) be two non-zero integers. The least common multiple (L.C.M.) of \(a, b\) is the unique positive integer \(m\) such that

(i) \(a | m, b | m\) and

(ii) \(a | k, b | k \Rightarrow m | k\)

we denote L.C.M. of \(a, b\) by \([a, b]\).

**9.4.29 Example** : \([5, 10] = 10,\ \ [16, 20] = 80\).

**9.4.30 SAQ** : Prove that the L.C.M. of two consecutive natural numbers is their product.

**Note** : (i) \([a, b] = [-a, b] = [a, -b] = [-a, -b]\)

(ii) If \(a \neq 0, b \neq 0 \in \mathbb{Z}\) then \(a | c, b | c \Rightarrow [a, b] | c\).

(iii) \([a, b] ab\) for any non-zero integers \(a\) and \(b\).

**9.4.31 Theorem** : If \(a, b\) are positive integers then \([a, b] (a, b) = ab\).
**Proof:** Let \((a, b) = d\)

Then \(\exists r, s \in \mathbb{Z}\) such that \(a = dr, b = ds\).

Put \(m = \frac{ab}{d}\)

we prove that \([a, b] = m\)

Now, \(m = \frac{ab}{d} = \frac{ads}{d} = as; m = \frac{ab}{d} = \frac{drb}{d} = rb\)

\(\Rightarrow a | m, b | m\)

If \(c\) is any common multiple of \(a\) and \(b\) then \(\exists A, B \in \mathbb{Z}\) such that \(c = aA, c = bB\). Again \((a, b) = d \Rightarrow \exists x, y \in \mathbb{Z}\) such that \(d = ax + by\).

For these \(\frac{c}{m} = \frac{c}{ab} = \frac{cd}{ab} = \frac{c(ax + by)}{ab} = \frac{c}{b}x + \frac{c}{a}y = Bx + Ay\)

\(\therefore \frac{c}{m}\) is an integer. \(\Rightarrow m | c\)

\(\therefore m = [a, b] = \frac{ab}{d} = \frac{ab}{(a, b)}\)

or \([a, b] = (a, b)\)

**9.4.32 Example:** If \(a = 2210, b = 493\) find \((a, b)\) and also \([a, b]\). By division algorithm

\[
\begin{array}{c|cc}
493 & 2210 & 4 \\
1972 & 4 & 2 \\
238 & 493 & 2 \\
476 & 238 & 2 \\
17 & 238 & 14 \\
17 & 14 & 0 \\
\end{array}
\]

\(\therefore (a, b) = 17\)

\([a, b] = \frac{|ab|}{(a, b)} = \frac{2210 \times 493}{17} = 64090\)
9.4.33 **Corollary**: If \( a \) and \( b \) are non-zero integers then \( \lfloor a, b \rfloor (a, b) = |a b| \)

**Proof**: Clearly \(|a|\) and \(|b|\) are positive integers.

By the above theorem, \( \lfloor |a|, |b| \rfloor (|a|, |b|) = |a||b| = |a b| \)

But \( [a, b] = [a, b] \) and \( (|a|, |b|) = (a, b) \).

Hence \( [a, b] (a, b) = |a, b| \)

### 9.5 PRIME NUMBERS

#### 9.5.1 Definition

An integer \( p > 1 \) is called a prime number or a prime in case there is no divisor \( d \) of \( p \) satisfying \( 1 < d < p \).

If an integer \( a > 1 \) is not a prime, it is called a composite numbers.

Thus for an example 2,3,5,7, ...... are primes, whereas 4, 6, 7, 8, 9 are composite.

Note that the integer 1 is neither a prime nor a composite number.

#### 9.5.2 Theorem

If \( p \) is prime and \( a, b \in \mathbb{Z} \) then \( p|ab \Rightarrow p|a \) or \( p|a \).

**Proof**: Suppose \( p \nmid a \)

Then \( (p, a) = 1 \).

By theorem 9.4.24 \( p|b \)

#### 9.5.3 Corollary

If \( p \) is a prime, and \( a_1, a_2, \ldots, a_n \in \mathbb{Z} \) then \( p|a_1 a_2 \ldots a_n \Rightarrow p|a_i \) for some \( i \in \{1, 2, \ldots, n\} \).

**Proof**: Proof is by induction on \( n \). By theorem 9.5.2, the theorem holds for \( n = 2 \).

Assume that the theorem holds whenever \( p \) divides a product with fewer than \( n \) factors.

If \( p|a_1 a_2 \ldots a_n \) then \( p|ac \) where \( c = a_2 \ldots a_n \). By the theorem 9.5.2 \( p|a \) or \( p|c \).

If \( p|c \), then by induction hypothesis, \( p|a_i \) for some \( i \).

#### 9.5.4 Corollary

If \( p, q_1, \ldots, q_k \) are prime numbers then \( p|q_1 q_2 \ldots q_k \Rightarrow p = q_i \) for some \( i \).
Proof: By the corollary 9.5.3, if \( p | q_1 q_2 \cdots q_k \) then \( p | q_j \) for some \( j \). Since \( q_j \) is prime it follow that \( p = q_j \).

9.5.5 Theorem: (The fundamental theorem of arithmetic / The unique factorization theorem).

Every integer \( a > 1 \), can be expressed as a product of primes, and this factorization is unique apart from the order of the prime factors.

Proof: Existence: We prove that 'a' can be expressed as product of primes.

Let \( S = \{ a > 1/ a \text{ can not expressed as a product of primes} \} \).

If \( S = \emptyset \) then the theorem is proved. Suppose \( S \neq \emptyset \).

By well ordering principle, \( S \) has least member, say \( m \).

If \( m \) is prime, then \( m \in S \), since \( m \) itself stands as a product with a single factor. \( m \) can be factored as \( m = m_1 m_2 \), therefore \( 1 < m_1 < m, 1 < m_2 < m \).

\( m \) is the least member of \( S \) and \( m_1, m_2 < m \). \( \Rightarrow m_1, m_2 \) can be expressed as a product of primes.

\[ \Rightarrow m = m_1 m_2 \text{ can be expressed as a product of primes.} \]

This is a contradiction.

\[ \therefore S = \emptyset \]

Hence, every integer \( a > 1 \) can be expressed as a product of primes.

Uniqueness: If \( a = p_1 p_2 \cdots p_s = q_1 q_2 \cdots q_t \) are two prime factorizations for \( a \), we prove that \( s = t \) and every \( p_i = q_j \) for some \( j \). We use induction on the positive integers. Let \( s = 1 \).

There \( p_1 = q_1 q_2 \cdots q_t \)

Since \( p_1 \) is prime, \( t_1 = 1 \) and \( p_1 = q_1 \).

\[ \therefore \text{The statement is true for } s = 1. \]

Suppose that the statement is true for \( s = r \). (i.e.) \( p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_t \).

Then \( t = r \) and every \( p_i = q_j \) for some \( j \). Suppose that \( p_1 p_2 \cdots p_{r+1} = q_1 q_2 \cdots q_t \). Because \( p_{r+1} | q_1 q_2 \cdots q_t \), by theorem 9.5.2 \( p_{r+1} | q_j \) for some \( j \).

That is \( p_{r+1} = q_j \), since \( q_j \) is prime.

\[ \therefore p_1 p_2 \cdots p_r = q_1 \cdots q_{j-1} q_{j+1} \cdots q_t. \]

By induction hypothesis, \( r = t - 1 \) and each \( p_j = q_k \) for some \( k \neq j \).
9.5.6 Theorem (Euclid): The number of primes is infinite. That is there is no end to the sequence of primes 2, 3, 5, 7, 11, 13, .......

Proof: Suppose there were only a finite number of primes $p_1, p_2, \ldots, p_r$. Then form the number $n = 1 + p_1 p_2 \cdots p_r$.

The number $n$ is not divisible by $p_1$ or $p_2$ or $\ldots$ or $p_r$ (because, when $n$ is divided by $p_i$, we have the remainder 1)

Hence, any prime divisor $p$ of $n$ is a prime distinct from $p_1, p_2, \ldots, p_r$. Since $n$ is either a prime or has a prime factor $p$, this implies that there is a prime distinct from $p_1, p_2, \ldots, p_r$. Hence the number of primes is infinite.

9.5.7 Note: Every integer $n > 1$ can be uniquely written in the form $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $p_1, p_2, \ldots, p_k$ are primes such that $1 < p_1 < p_2 < \cdots < p_k$ and $\alpha_1, \ldots, \alpha_k$ are positive integers. This representation of $n$ is called prime factorization of $n$ in the canonical form or prime power factorization of $n$.

9.5.8 Note: If $a = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, $b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$ then $(a, b) = \prod_{j=1}^{n} p_j^{\gamma_j}$ where $\gamma_j = \min\{\alpha_j, \beta_j\}$.

9.6 CONGRUENCES:

9.6.1 Definition: If an integer $m$, not zero, divides the difference $a - b$, we say that $a$ is congruent to $b$ modulo $m$ and write $a \equiv b \pmod{m}$. If $a - b$ is not divisible by $m$, we say that $a$ is not congruent to $b \pmod{m}$, and in this case we write $a \not\equiv b \pmod{m}$.

Note: $m \mid (a - b) \iff -m \mid (a - b)$

Through out the lesson congruences modulo positive integers $m$ are considered.

For example, $73 \equiv 4 \pmod{23}$; $21 \equiv -9 \pmod{10}$; $-26 \equiv 1 \pmod{3}$

9.6.2 Theorem: Two integers $a$ and $b$ are congruent modulo $m$, iff they leave the same remainder when divided by $m$.

Proof: Suppose $a \equiv b \pmod{m}$.

We prove that $a$, $b$ have the same remainder upon division by $m$.

$a \equiv b \pmod{m} \Rightarrow m \mid a - b \Rightarrow \exists k \in \mathbb{Z} \ a - b = km \text{ or } a = b + km$ \quad \text{--------- (1)}

By division algorithm for $b, m \exists q, r \in \mathbb{Z}$ such that $b = mq + r$, $0 \leq r < m$.

$\Rightarrow r$ is the remainder when $b$ is divided by $m$ \quad \text{------ (2)}
From (1), \( a = b + km = mq + r + km = m(q + k) + r \)

\[ \Rightarrow a = m(q + k) + r, \quad 0 \leq r < m. \]

\[ \Rightarrow r \] is the remainder when \( a \) is divided by \( m \). ----------- (3)

From (2) and (3) \( a, b \) have the same remainder when divided by \( m \).
Conversely suppose that \( a, b \) have the same remainder \( r \) when divided by \( m \).

Then \( \exists q_1, q_2 \in \mathbb{Z} \) such that \( a = q_1m + r, b = q_2m + r \).

So \( a - b = (q_1 - q_2)m \Rightarrow m|(a - b) \Rightarrow a \equiv b \mod m \)

9.6.3 Theorem: Let \( m > 0 \) and let \( a, b, c, d, x, y \) be integers. Then

(1) \( a \equiv a \mod m \) (reflexive property)

(2) \( a \equiv b \mod m \Rightarrow b \equiv a \mod m \) (symmetric property)

(3) \( a \equiv b \mod m \Rightarrow b \equiv c \mod m \Rightarrow a \equiv c \mod m \) (transitive property)

(4) \( a \equiv b \mod m, \ c \equiv d \mod m \Rightarrow a + c \equiv b + d \mod m, \ ac \equiv bd \mod m \)

(5) \( a \equiv b \mod m \Rightarrow a + c \equiv b + c \mod m, \ ac \equiv bc \mod m \)

(6) \( a \equiv b \mod m \Rightarrow a^k = b^k \mod m, \ k \geq 1, k \in \mathbb{Z} \)

(7) \( ac = bc \mod m, \ (c, m) = 1 \Rightarrow a \equiv b \mod m \)

(8) \( ac = bc \mod m, \ c | m \Rightarrow a \equiv b \left( \mod \frac{m}{c} \right) \)

Proof: (1) \( a - a = 0 \cdot m \Rightarrow a \equiv a \mod m \)

(2) \( a \equiv b \mod m \Rightarrow m|(a - b) \Rightarrow m|(-1)(a - b) \Rightarrow m|(b - a) \Rightarrow b \equiv a \mod m \)

(3) \( a \equiv b \mod m, \ b \equiv c \mod m \Rightarrow m|(a - b), \ m|(b - c) \)

\[ \Rightarrow m|(a - b) + (b - c) \Rightarrow m|(a - c) \Rightarrow a \equiv c \mod m \)

(4) \( a \equiv b \mod m, \ c \equiv d \mod m \)

\[ \Rightarrow m|(a - b), \ m|(c - d) \Rightarrow m|(a - b) + (c - d) \]

\[ \Rightarrow m|(a + c) - (b + d) \Rightarrow (a + c) \equiv (b + d) \mod m \)
\[ m \mid (a - b), m \mid (c - d) \Rightarrow \exists k_1, k_2 \in Z \text{ such that} \]

\[ a - b = mk_1, c - d = mk_2 \Rightarrow a = b + mk_1 c = d + mk_2 \]

\[ \Rightarrow ac = (b + mk_1)(d + mk_2) = bd + (bk_2 + dk_1)m + k_1k_2m^2 \]

\[ \Rightarrow ac - bd = m(bk_2 + dk_1 + k_1k_2m) \]

\[ \Rightarrow m \mid (ac - bd) \Rightarrow ac \equiv bd \pmod{m} \]

**(5)** \[ a \equiv b \pmod{m} \Rightarrow m \mid (a - b) \Rightarrow m \mid (a + c) - (b + c) \]

\[ \Rightarrow (a + c) \equiv (b + c) \pmod{m} \]

and \[ m \mid a - b \Rightarrow m \mid c(a - b) \Rightarrow m \mid (ca - cb) \]

\[ \Rightarrow ac \equiv bc \pmod{m} \]

**(6)** It is proved by induction for \( k = 1 \), \( a \equiv b \pmod{m} \). Assume that \( a^k \equiv b^k \pmod{m} \) is true for \( k = r \)

i.e. \( a^r \equiv b^r \pmod{m} \)

Now, \( a \equiv b \pmod{m} \), \( a^r \equiv b^r \pmod{m} \)

By using (4) we get \( aa^r \equiv bb^r \pmod{m} \)

\[ \Rightarrow a^{r+1} \equiv b^{r+1} \pmod{m} \]

Hence \( a^k \equiv b^k \pmod{m} \) is true for \( k = r + 1 \).

\[ \therefore \text{By induction} \ a^k \equiv b^k \pmod{m} \text{ is true } \forall k. \]

**(7)** \[ ac \equiv bc \pmod{m}, (c, m) = 1 \Rightarrow m \mid (ac - bc) \Rightarrow m \mid c(a - b) \Rightarrow m \mid (a - b) \text{ since } (c, m) = 1 \text{ by theorem 9.4.24}. \]

\[ m \mid (a - b) \Rightarrow a \equiv b \pmod{m} \]

**(8)** \[ ac \equiv bc \pmod{m}, c \mid m \]

\[ \Rightarrow m \mid c(a - b) \text{ and } c \mid m \Rightarrow \left( \frac{m}{c} \right)(a - b) \]

\[ \Rightarrow a \equiv b \left( \frac{m}{c} \right) \]
9.6.4 Theorem: Let \( f \) be a polynomial with integer coefficients. If \( a \equiv b \pmod{m} \) then \( f(a) \equiv f(b) \pmod{m} \).

Proof: Let \( f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n \) where \( \alpha_0, \alpha_1, \cdots, \alpha_n \in \mathbb{Z} \).

\[
\alpha_i \equiv b \pmod{m} \Rightarrow \alpha_i \equiv b^i \pmod{m} \quad \text{for} \quad i = 1, 2, \cdots, n \quad \text{and hence,} \quad \alpha_i a^i \equiv \alpha_i b^i \pmod{m} \quad \text{for} \quad i = 1, 2, \cdots, n. \]

Finally, \( (\alpha_0 + \alpha_1 a + \alpha_2 a^2 + \cdots + \alpha_n a^n) \equiv (\alpha_0 + \alpha_1 b + \cdots + \alpha_n b^n) \pmod{m} \)

\[
\Rightarrow f(a) \equiv f(b) \pmod{m}
\]

9.6.5 Theorem: \( a x \equiv a y \pmod{m} \Leftrightarrow x \equiv y \left( \mod \frac{m}{(a, m)} \right) \)

Proof: Let \( (a, m) = d \). Then \( \left( \frac{a}{d}, \frac{m}{d} \right) = 1 \)

\[
a x \equiv a y \pmod{m} \Leftrightarrow m \mid (a x - a y) \Leftrightarrow a x - a y = m q \quad \text{for some} \quad q \in \mathbb{Z}.
\]

\[
\Leftrightarrow \frac{a}{d} (x - y) = \frac{m}{d} q
\]

\[
\Leftrightarrow \frac{m}{d} \left| \frac{a}{d} (x - y) \right| \left( x - y \right) \Leftrightarrow \frac{m}{d} \left( x - y \right) \quad \Rightarrow \frac{a}{d} = 1
\]

\[
\Rightarrow x \equiv y \left( \mod \frac{m}{(a, m)} \right)
\]

9.6.6 SAQ: If \( a x \equiv a y \pmod{m} \) and \( (a, m) = 1 \) then \( x \equiv y \pmod{m} \).

9.6.7 Theorem: If \( a \equiv b \pmod{m_1} \), \( a \equiv b \pmod{m_2} \) and \( m = [m_1, m_2] \) (i.e.) \( m \) is the L.C.M. of \( m_1 \) and \( m_2 \) then \( a \equiv b \pmod{m} \).

Proof: \( a \equiv b \pmod{m_1} \Rightarrow m_1 \mid (a - b) \Rightarrow a - b = q_1 m_1 \) for some \( q_1 \in \mathbb{Z} \)

\[
a \equiv b \pmod{m_2} \Rightarrow m_2 \mid (a - b) \Rightarrow a - b = q_2 m_2 \quad \text{for some} \quad q_2 \in \mathbb{Z}
\]

\[
\therefore \ a - b \quad \text{is a multiple of both} \quad m_1 \quad \text{and} \quad m_2 \quad \text{and hence a multiple of} \quad m = [m_1, m_2] \quad \therefore \ a \equiv b \pmod{m}
\]
9.6.8 SAQ: $x \equiv y \mod m_i$ for $i = 1, 2, \ldots, r \Leftrightarrow x \equiv y \mod [m_1, m_2, \ldots, m_r]$

9.6.9 Definition: If $m$ is a positive integer then "congruence modulo $m$" is an equivalence relation on $\mathbb{Z}$. The equivalence classes with respect to this equivalence relation are called residue classes modulo $m$.

9.6.10 Notation: (1) The residue class containing an integer $a$ is denoted by $[a]$ or $\bar{a}$.

(2) The set of all residue classes modulo a positive integer $m$ is denoted by $\mathbb{Z}_m$.

9.6.11 Theorem: If $m$ is a positive integer then $\{0, 1, 2, \ldots, m-1\}$ and $O(\mathbb{Z}_m) = m$.

Proof: Let $a \in \mathbb{Z}$. By division algorithm, $\exists q, r \in \mathbb{Z}$ such that $a = mq + r$, $0 \leq r < m$

Now, $a - r = mq \Rightarrow m | (a - r) \Rightarrow a \equiv r \mod m$

$\Rightarrow \bar{a} = \bar{r}$ where $\bar{r} = 0, 1, 2, \ldots, m-1$

Suppose $\bar{r} = \bar{s}$ where $0 \leq r < m$, $0 \leq s < m$.

Assume that $r > s$. Then $r - s > 0$.

Now $r < m$, $s < m \Rightarrow r - s < m \Rightarrow 0 < r - s < m$. $\bar{r} = \bar{s} \Rightarrow r \equiv s \mod m \Rightarrow m | (r - s) \Rightarrow m$ divides any integer between 0 and $m$. It is a contradiction. Therefore $r \geq s$. Similarly $s \geq r$.

$\therefore r = s$

$\therefore Z_m = \{0, 1, 2, \ldots, m-1\}$.

9.7 SOLUTIONS OF LINEAR CONGRUENCES:

9.7.1 Definition: Let $f(x)$ denote a polynomial with integer coefficients. We write $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$. If $u$ is an integer such that $f(u) \equiv 0 \mod m$, then we say that $u$ is a solution of the congruence $f(x) \equiv 0 \mod m$.

If $f(u) \equiv 0 \mod m$ and $v \equiv u \mod m$ then $f(v) \equiv f(u) \mod m$.

$\Rightarrow f(v) \equiv 0 \mod m$ (\because $u$ is a solution)

$\Rightarrow v$ is a solution of the congruence $f(x) \equiv 0 \mod m$.

Therefore every integer congruent to $u \mod m$ is a solution of $f(x) \equiv 0 \mod m$.

9.7.2 Definition: Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ be a polynomial with integer coefficients. If $a_0 \not\equiv 0 \mod m$, then the degree of the congruence $f(x) \equiv 0 \mod m$ is defined as $n$. 
If \(a_0 \equiv 0 \pmod{m}\), let \(j\) be the smallest positive integer such that \(a_j \not\equiv 0 \pmod{m}\). Then the degree of the congruence is defined as \(n - j\). If there is no such integer \(j\), that is, if all the coefficients of \(f(x)\) are multiples of \(m\), no degree is assigned to the congruence.

**Note:**

(i) The degree of the congruence \(f(x) \equiv 0 \pmod{m}\) is not the same thing as the degree of the polynomial \(f(x)\).

(ii) The degree of congruence depends on modulus.

If \(g(x) = 6x^3 + 3x^2 + 1\), then degree of the congruence \(g(x) \equiv 0 \pmod{5}\) is 3 and degree of \(g(x) \equiv 0 \pmod{2}\) is 2, where as \(g(x)\) is of degree 3.

**9.7.3 Definition:** A polynomial congruence of first degree is called a linear congruence.

Any linear congruence of degree 1 can be put in the form \(ax \equiv b \pmod{m}\), \(a \not\equiv 0 \pmod{m}\).

**9.7.4 Theorem:** If \(x_0\) is a solution of \(ax \equiv b \pmod{m}\), and \(x_1 \equiv x_0 \pmod{m}\) then \(x_1\) is also a solution of \(ax \equiv b \pmod{m}\).

**Proof:** This follows directly from 9.7.1.

**9.7.5 Definition:** Suppose \(x_1\) and \(x_2\) are two solutions of the congruence \(ax \equiv b \pmod{m}\). If \(x_1 \not\equiv x_2 \pmod{m}\), then \(x_1, x_2\) are called incongruent solutions of \(ax \equiv b \pmod{m}\).

**Existence of solution for** \(ax \equiv b \pmod{m}\):

**9.7.6 Theorem:** The linear congruence \(ax \equiv b \pmod{m}\) has a solution iff the g.c.d. of \(a\) and \(m\) (i.e.) \(d = (a, m)\) divides \(b\).

In this case the congruence \(ax \equiv b \pmod{m}\) has exactly \(d\) incongruent solutions.

**Proof:** Suppose \(x_0\) is a solution of \(ax \equiv b \pmod{m}\) then \(ax_0 \equiv b \pmod{m}\).

\[\Rightarrow \exists y_0 \in \mathbb{Z} \text{ such that } ax_0 - b = y_0m \text{ or } ax_0 - y_0m = b \text{ or } ax_0 + m(-y_0) = b \Rightarrow d \mid b\]

Conversely suppose \(d \mid b\)

Since \((a, m) = d\)

\[\exists k, \ell \in \mathbb{Z} \text{ such that } ak + m\ell = d\]

\[d \mid b \Rightarrow b = dd' \text{ for some } d' \in \mathbb{Z}.\]
Now, \( b = dd' = (ak + m\ell)d' = a(kd') + m(\ell d') \)

\[ \Rightarrow a(kd') - b = m(-\ell d') \Rightarrow m \mid a(kd') - b \]

\[ \Rightarrow a(kd') \equiv b(\text{mod} \ m) \Rightarrow x_0 = kd' \text{ is a solution of the congruence } ax \equiv b(\text{mod} \ m). \]

**Now we prove that if \( x_0 \) is a solution of the congruence \( ax \equiv b(\text{mod} \ m) \) then,**

\[
 x_0, x_0 + \frac{m}{d}, \ldots, x_0 + (d-1)\frac{m}{d} \quad \text{-------- (1)}
\]

**are \( d \) incongruent solutions of** \( ax \equiv b(\text{mod} \ m) \):

\[ ax_0 = b(\text{mod} \ m) \Rightarrow m \mid (ax_0 - b) \Rightarrow ax_0 - b = mq \text{ for some } q \in \mathbb{Z} \Rightarrow ax_0 = b + mq \text{ ----- (2)} \]

For any integer \( x_0 + k\frac{m}{d}, 0 \leq k \leq d-1 \) we have

\[
 a\left(x_0 + \frac{m}{d}\right) = ax_0 + \frac{a}{d}km = b + mq + \frac{a}{d}km \text{ from (2)}
\]

\[ = b + m\left(q + \frac{a}{d}k\right) \Rightarrow a\left(x_0 + \frac{m}{d}\right) - b = m\left(q + \frac{a}{d}k\right) \]

\[ \Rightarrow m\left| a\left(x_0 + \frac{m}{d}\right) - b \Rightarrow a\left(x_0 + \frac{m}{d}\right) \equiv b(\text{mod} \ m) \right. \]

\[ : \text{ The integers in (1) are all solutions of the congruence } ax \equiv b(\text{mod} \ m) \]

Now, we show that the integers in (1) are in congruent modulo \( m \).

Suppose \( x_0 + \frac{m}{d}k \equiv x_0 + \frac{m}{d}\ell(\text{mod} \ m), 0 \leq k < \ell \leq d \).

Then \( \frac{m}{d}k \equiv \frac{m}{d}\ell(\text{mod} \ m) \Rightarrow k \equiv \ell(\text{mod} \ m) \)

which is a contradiction because \( 0 < k - \ell < d \)

\[ : \text{ The integers in (1) are solutions such that no two are congruent modulo } m. \]

Finally we prove that every solution of \( ax \equiv b(\text{mod} \ m) \) is congruent to one of the solutions in (1).
Let $y_0$ be a solution of $ax \equiv b \pmod{m}$.

Then $ay_0 \equiv b \pmod{m}$

Then $ay_0 \equiv ax_0 \equiv b \pmod{m} \Rightarrow ay_0 \equiv ax_0 \pmod{m}$

$$\Rightarrow m \mid a(y_0 - x_0) \Rightarrow \frac{m}{d} \bigg| a(y_0 - x_0) \Rightarrow \frac{m}{d} \bigg| (y_0 - x_0)$$

$$\left( \therefore (m, a) = d \Rightarrow \left( \frac{m}{d}, \frac{a}{d} \right) = 1 \right)$$

$$\Rightarrow \exists t \in \mathbb{Z} \text{ such that } y_0 - x_0 = \frac{m}{d} \Rightarrow y_0 = x_0 + \frac{m}{d}.$$ 

By division algorithm $\exists q, r \in \mathbb{Z}$ such that $t = qd + r, 0 \leq r < d$

$$\therefore y_0 = x_0 + t \frac{m}{d} = x_0 + (qd + r) \frac{m}{d} = x_0 + qm + r \frac{m}{d}$$

$$y_0 = x_0 + qm + r \frac{m}{d} \Rightarrow y_0 \equiv x_0 + r \frac{m}{d} \pmod{m}. 0 \leq r < d$$

Hence, $y_0$ is congruent to one of the solutions in (1).

Hence the theorem.

9.7.8 Corollary: If $(a, m) = 1$, the congruence $ax \equiv b \pmod{m}$ has a unique solution.

9.7.9 Note: If $(a, m) = d$ and $d \nmid b$, then the congruence $ax \equiv b \pmod{m}$ has no solution.

9.7.10 Definition (Complete residue system modulo $m$): Let $m$ be a positive integer. A set $x_1, \cdots, x_m$ of integers is called a complete residue system modulo $m$ if every integer $y$ is congruent to one and only one $x_j$ modulo $m$.

9.7.11 Example: Do the following congruences possess a solution?

(i) $135x \equiv 6 \pmod{10}$  (ii) $84x \equiv 16 \pmod{35}$  (iii) $66x \equiv 8 \pmod{78}$  (iv) $12x \equiv 7 \pmod{21}$

Solution: We know that the congruence $ax \equiv b \pmod{m}$ has a solution if $(a, m) = d$ divides $b$.

Hence, the congruence is $135x \equiv 6 \pmod{10}$, $a = 135, b = 6, m = 10$

d = (a, m) = (135, 10) = 5$ does not divide $b = 6$.

The congruence $135x \equiv 6 \pmod{10}$ does not possess solutions.
(ii) \( a = 84, \ b = 16, \ m = 35 \)

\( d = (a, m) = (84, 35) = 7 \) does not divide \( b = 16 \).

\( \therefore \) The congruence \( 84x \equiv 16 \pmod{35} \) does not have a solution.

(iii) \( a = 66, \ b = 8, \ m = 78 \)

\( d = (a, m) = (66, 78) = 6 \) does not divide \( b = 8 \).

\( \therefore \) The congruence \( 12x \equiv 7 \pmod{21} \) does not have a solution.

9.7.12 Example : Solve the following congruences.

(i) \( 3x + 2 \equiv 0 \pmod{7} \)

(ii) \( 2x + 1 \equiv 0 \pmod{7} \)

(iii) \( 3x \equiv 1 \pmod{125} \)

Solution : (i) Here, the congruence is \( 3x \equiv -2 \pmod{7} \)

\( a = 3, \ b = -2, \ m = 7 \)

\( d = (a, m) = (3, 7) = 1 \) and hence the congruence \( 3x \equiv -2 \pmod{7} \) has a unique solution.

Now \( 3x \equiv -2 \pmod{7} \) \( \quad \text{----------- (1)} \)

Also \( 0 \equiv 14 \pmod{7} \) \( \quad \text{----------- (2)} \)

Adding (1) and (2), \( 3x \equiv 12 \pmod{7} \)

\( \Rightarrow x \equiv 4 \pmod{7} \) \( \quad (\because (3, 7) = 1) \)

\( \therefore x = 4 \) is the solution of the congruence \( 3x + 2 \equiv 0 \pmod{7} \).

(ii) The congruence is \( 2x \equiv -1 \pmod{7} \)

\( a = 2, \ b = -1, \ m = 7 \)

\( d = (a, m) = (2, 7) = 1 \)

Clearly \( d | b \)

\( \therefore \) The congruence \( 2x \equiv -1 \pmod{7} \) has only one solution.
Now \(2x \equiv -1 \pmod{7} \quad \text{(1)}\)

Also \(0 \equiv 7 \pmod{7} \quad \text{(2)}\)

Adding (1) and (2), \(2x \equiv b \pmod{7} \implies x = 3 \pmod{7} \quad (\because (2,7) = 1)\)

\(\therefore x = 3\) is the solution of the congruence \(2x + 1 \equiv 0 \pmod{7}\).

(iii) The congruence is \(3x \equiv 1 \pmod{125}\)

\(a = 3, b = 1, m = 125\)

\(\therefore d = (a,m) = (3,125) = 1\)

\(\therefore\) The congruence \(3x \equiv 1 \pmod{125}\) has only one solution.

Now \(3x \equiv 1 \pmod{125} \quad \text{(1)}\)

\(0 \equiv 125 \pmod{125} \quad \text{(2)}\)

Adding (1) and (2), \(3x \equiv 126 \pmod{125}\)

\(\implies x \equiv 42 \pmod{125} \quad (\because (3,125) = 1)\)

\(\therefore x = 42\) is the solution of the congruence \(3x \equiv 1 \pmod{125}\).

9.7.13 SAQ: Solve

(i) \(4x \equiv 5 \pmod{6}\)

(ii) \(3x \equiv 5 \pmod{7}\)

9.7.14 Example: Solve the following congruences.

(i) \(259x \equiv 5 \pmod{11}\)

(ii) \(342x \equiv 5 \pmod{13}\)

Solutions: (i) Given congruence is \(259x \equiv 5 \pmod{11}\) \quad \text{(1)}

Now \(259 \equiv 6 \pmod{11}\)

\(\therefore 259x \equiv 6x \pmod{11}\)

By transitive property, \(6x \equiv 5 \pmod{11}\) \quad \text{(2)}

Here \(a = 6, b = 5, m = 11, d = (a,m) = (6,11) = 1\) divides \(b = 5\)

\(\therefore\) The congruences (1) and (2) have only one solution.
Now \( 6x \equiv 5 \pmod{11} \) \quad \text{-------- (3)}

Also \( 0 \equiv 55 \pmod{11} \) \quad \text{-------- (4)}

Adding (3) and (4), \( 6x \equiv 60 \pmod{11} \)

\[ \Rightarrow x \equiv 10 \pmod{11} \quad (\because (6,11) = 1) \]

\( \therefore x = 10 \) is a solution of (3) and hence of (1).

(ii) Given congruence is \( 342x \equiv 5 \pmod{13} \) \quad \text{-------- (1)}

Now \( 342 \equiv 4 \pmod{13} \)

\[ \Rightarrow 342x \equiv 4x \pmod{13} \quad \text{-------- (2)} \]

By transitive property \( 4x \equiv 5 \pmod{13} \)

Here \( a = 4, b = 5, m = 13 \)

\[ \therefore d = (a, m) = (4, 13) = 1 \text{ divides } b = 5. \]

\( \therefore \) The congruence (3) and hence (1) has only one solution.

Now \( 4x \equiv 5 \pmod{13} \) \quad \text{-------- (4)}

Also \( 0 \equiv 39 \pmod{13} \) \quad \text{-------- (5)}

Adding (4) and (5) we get \( 4x \equiv 44 \pmod{13} \)

\[ \Rightarrow x \equiv 11 \pmod{13} (\because (4,13) = 1) \]

\( \therefore x \equiv 11 \) is an incongruent solution of (1).

9.7.15 Example : Solve the following congruences :

(i) \( 13x \equiv 10 \pmod{28} \)

(ii) \( 16x \equiv 25 \pmod{19} \)

Solution : (i) The given congruence is \( 13x \equiv 10 \pmod{28} \).

Here \( a = 13, b = 10, m = 28 \)

\[ d = (a, m) = (13, 28) = 1 \text{ divides } b = 10. \]

\( \therefore \) The congruence \( 13x \equiv 10 \pmod{28} \) has only one solution.

Now \( 13x \equiv 10 \pmod{28} \) \quad \text{-------- (1)}

Also \( 0 \equiv 224 \pmod{28} \) \quad \text{-------- (2)}
Adding (1) and (2), \(13x \equiv 224 \pmod{28}\)

\[\Rightarrow x = 18 \pmod{28}\]

\(\therefore x = 18\) is a solution of the congruence \(13x \equiv 10 \pmod{28}\)

(ii) Try yourself: (Ans: \(x = 17\))

9.7.16 Example: Solve (i) \(15x \equiv 6 \pmod{21}\) (ii) \(222x \equiv 12 \pmod{18}\) (iii) \(15x \equiv 25 \pmod{35}\)

Solutions: (i) The congruence is \(15x \equiv 6 \pmod{21}\) \(---------- (1)\)

Here \(a = 15, b = 6, m = 21\)

\[d = (a, m) = (15, 21) = 3\] which divides \(b = 6\).

\(\therefore\) The given congruence (1) has \(3\) (= \(d\)) incongruent solutions \(\pmod{21}\).

Given congruence is \(3 \times 5x \equiv 3 \times 2 \pmod{3 \times 7}\)

\[\Rightarrow 5x \equiv 2 \pmod{7}\]

Also \(0 \equiv 28 \pmod{7}\)

Adding, \(5x \equiv 30 \pmod{7}\)

\[\Rightarrow x \equiv 6 \pmod{7} (\because (5, 7) = 1)\]

\(\therefore x_0 = 6\) is a solution of \(5x \equiv 2 \pmod{7}\) and hence is a solution of (1).

\(\therefore\) All the three incongruent solutions \(\pmod{21}\) of \(15x \equiv 6 \pmod{21}\) are given by

\[x_0 + K \frac{m}{d}, K = 0, 1, 2, \ldots, d - 1\]

Here \(x_0 = 6, d = 3, \frac{m}{d} = \frac{21}{3} = 7\)

\(\therefore\) The three solutions are \(x = x_0 + K \frac{m}{d} = 6 + 7K; K = 0, 1, 2\)

\(\therefore x \equiv 6, 13, 20 \pmod{21}\)
(ii) The given congruence is \(222x \equiv 12 \pmod{18}\) \hspace{1cm} (1)

Here \(a = 222, b = 12, m = 18\)

\[d = (a, m) = (222, 18) = 6\] which divides \(b = 12\).

\[\therefore\] The given congruence has 6 incongruent solutions mod 18.

\[222x \equiv 12 \pmod{18}\]

Also \(222 \equiv 6 \pmod{18}\)

\[\therefore 222x \equiv 6x \pmod{18}\]

By transitive property, \(6x \equiv 12 \pmod{18}\) \hspace{1cm} (2)

\[\Rightarrow x \equiv 2 \pmod{3}\]

\[\therefore x_0 = 2\] is a solution of the congruence (2) and hence of the congruence (1).

\[\therefore\] The six incongruent solutions of (1) mod 18 are given by

\[x \equiv x_0 + K \frac{m}{d} = 2 + K \frac{18}{6} = 2 + 3K\]

where \(K = 0, 1, 2, 3, 4, 5\).

\[\therefore x = 2, 5, 8, 11, 14, 17\] are the six incongruent solutions of (1).

(iii) The given congruence is \(15x \equiv 25 \pmod{35}\) \hspace{1cm} (1)

Here \(a = 15, b = 25, m = 35\)

\[d = (a, m) = (15, 35) = 5\] which divides \(b = 25\).

\[\therefore\] The given congruence (1) has 5 incongruent solutions mod 35.

\[15x \equiv 25 \pmod{35} \Rightarrow 3x \equiv 5 \pmod{7}\] \hspace{1cm} (2)

Also \(0 \equiv 7 \pmod{7}\) \hspace{1cm} (3)

Adding (2) and (3), \(3x \equiv 12 \pmod{7}\)

\[\Rightarrow x \equiv 4 \pmod{7}\] \hspace{1cm} (\because (3, 7) = 1)

\[\therefore x_0 = 4\] is a solution of the congruence (2) and hence of the congruence (1).

\[\therefore\] The five incongruent solutions of (1) mod 35 are given by

\[x = x_0 + K \frac{m}{d} = 4 + K \frac{35}{5} = x + 7K\] for \(K = 0, 1, 2, 3, 4\).

\[\therefore x = 4, 11, 18, 25, 32\] are the five solutions of (1) which are incongruent mod 35.
9.7.17 Theorem (Chinese Remainder Theorem): Let \( m_1, m_2, \ldots, m_r \) denote \( r \) positive integers that are relatively prime in pairs, and let \( a_1, a_2, \ldots, a_r \) denote any \( r \) integers. Then the congruences \( x \equiv a_i \pmod{m_i}, i = 1, 2, \ldots, r \) have common solutions. Any two solutions are congruent modulo \( m_1 m_2 \cdots m_r \).

**Proof:** The simultaneous linear congruences are

\[
\begin{align*}
x & \equiv a_1 \pmod{m_1} \\
x & \equiv a_2 \pmod{m_2} \\
& \quad \vdots \\
x & \equiv a_r \pmod{m_r}
\end{align*}
\]

--------- (1)

Let \( m = m_1 m_2 \cdots m_r \) and \( M_j = \frac{m}{m_j} = m_1 \cdots m_{j-1} m_{j+1} \cdots m_r \)

Then \((M_j, m_j) = 1\)

\[\therefore \text{By theorem (9.7.6), the congruence } M_j x \equiv 1 \pmod{m_j} \text{ has a unique solution, say } x_j.\]

Then \( M_j x_j \equiv 1 \pmod{m_j} \)--------- (2)

Consider \( X = a_1 x_1 M_1 + a_2 x_2 M_2 + \cdots + a_r x_r M_r \)

Now, we show that \( X \) is a common solution of (1).

For \( i \neq j, m_j \mid M_i \Rightarrow M_i \equiv 0 \pmod{m_j} \)--------- (3)

\[
X = a_1 M_1 x_1 + a_2 M_2 x_2 + \cdots + a_j M_j x_j + \cdots + a_r M_r x_r.
\]

\[= a_j M_j x_j \pmod{m_j} \quad \text{from (3).}\]

\[= a_j \pmod{m_j} \quad \text{from(2).}\]

So, \( X \) is a common solution of the given congruences (1)

**Uniqueness:** If \( X \) and \( Y \) are both common solutions of \( x \equiv a_i \pmod{m_i}, i = 1, 2, \ldots, r \)

then \( X \equiv a_i \pmod{m_i} \) and \( Y \equiv a_i \pmod{m_i} \) for \( i = 1, 2, \ldots, r \).
\( \Rightarrow X \equiv Y \pmod{m_i} \) for \( i = 1, 2, \ldots, r \)

\( \Rightarrow X \equiv Y \pmod{m} \) by Theorem 9.6.7

9.7.18 Example: Find the smallest positive integer (except \( x = 1 \)) that satisfies the following congruences simultaneously.

\[
\begin{align*}
  x &\equiv 1 \pmod{3} \\
  x &\equiv 1 \pmod{5} \\
  x &\equiv 1 \pmod{7}
\end{align*}
\]

Solution: Here \( a_1 = a_2 = a_3 = 1, \ m_1 = 3, \ m_2 = 5, \ m_3 = 7 \). \( m = m_1 m_2 m_3 = 3 \times 5 \times 7 = 105 \)

\[
\begin{align*}
M_1 &= \frac{105}{3} = 35, \quad M_2 = \frac{105}{5} = 21, \quad M_3 = \frac{105}{7} = 15
\end{align*}
\]

For \( j = 1, 2, 3 \), \( M_jx_j \equiv 1 \pmod{m_j} \) have unique solutions.

For \( j = 1, 35x_1 \equiv 1 \pmod{3} \Leftrightarrow x_1 = 2 \)

\( j = 2, 21x_2 \equiv 1 \pmod{5} \Leftrightarrow x_2 = 1 \)

\( j = 3, 15x_3 \equiv 1 \pmod{7} \Leftrightarrow x_3 = 1 \)

Solution of the given congruences is

\[
X = a_1x_1M_1 + a_2x_2M_2 + a_3x_3M_3 = 1 \times 2 	imes 35 + 1 \times 1 \times 21 + 1 \times 1 \times 15 = 106
\]

\( X = 106 \) is a common solution.

9.8 Euler's and Fermat's Theorems:

9.8.1 Definition: The operation \( \oplus \) defined on \( \mathbb{Z}_m \) as \( \overline{a} \oplus \overline{b} = \overline{a+b} \), \( \forall \overline{a}, \overline{b} \in \mathbb{Z}_m \) is called residue class addition or addition modulo \( m \). It is denoted by \( +_m \).

9.8.2 Definition: The operation \( \odot \) defined on \( \mathbb{Z}_m \) as \( \overline{a} \odot \overline{b} = \overline{ab} \), \( \forall \overline{a}, \overline{b} \in \mathbb{Z}_m \) is called residue class multiplication or multiplication modulo \( m \). It is denoted by \( \cdot_m \).

9.8.3 Definition: If \( x \equiv y \pmod{m} \) then \( y \) is called a residue of \( x \) modulo \( m \). A set \( \{x_1, x_2, \ldots, x_m\} \) is called a complete residue system modulo \( m \) if for every integer \( y \) there is one and only one \( x_j \) such that \( y \equiv x_j \pmod{m} \).
Note: If a set of \( m \) integers forms a complete residue system modulo \( m \) then no two integers in the set are congruent modulo \( m \).

9.8.4 Example: If \( m = 6 \), the sets \( \{0,1,2,3,4,5\} \), \( \{1,2,3,4,5,6\} \), \( \{-6, -5, -4, -3, -2, -1\} \) are complete residue systems modulo 6.

9.8.5 Theorem: If \( x \equiv y \pmod{m} \) then \( (x, m) = (y, m) \).

Proof: \( x \equiv y \pmod{m} \Rightarrow m \mid (x - y) \Rightarrow x - y = mq \) for some \( q \in \mathbb{Z} \).

Since \( (x, m) \mid x \) and \( (x, m) \mid m \), we have \( (x, m) \mid y \) and hence, \( (x, m) \mid (y, m) \)

In the same way we find \( (y, m) \mid (x, m) \)

\[ \therefore (x, m) = (y, m) \quad (\because (x, m) \) and \( (y, m) \) are positive) \]

9.8.6 Definition: A reduced residue system modulo \( m \) is a set of integers \( r_i \) such that \( (r_i, m) = 1, r_i \not\equiv r_j \pmod{m} \) if \( i \neq j \) and such that every \( x \) prime to \( m \), is congruent modulo \( m \) to some integer \( r_i \) of the set.

Note: In view of theorem 9.8.5, a reduced residue system modulo \( m \) can be obtained by deleting from a complete residue system modulo \( m \) those members that are not relatively prime to \( m \). All reduced residue system \( b \) modulo \( m \) will contain the same number of members, a number that is denoted by \( \phi(m) \).

9.8.7 Definition. Euler's \( \phi \)-function: Let \( n \geq 1 \). Then \( \{1, 2, \cdots, n\} \) is a complete residue system modulo \( n \). Thus the number \( \phi(n) \) is the number of positive integers less than or equal to \( n \) that are relatively prime to \( n \). This function \( \phi \) is called Euler's \( \phi \)-function.

9.8.8 Theorem: Let \( (a, m) = 1 \). Let \( r_1, r_2, \cdots, r_n \) be a complete (reduced), residue system modulo \( m \). Then \( a_{r_1}, a_{r_2}, \cdots, a_{r_n} \) is complete (reduced) residue system modulo \( m \).

Proof: \( (a, m) = 1 \) if \( (r_i, m) = 1 \), then \( (a_{r_i}, m) = 1 \) by theorem 9.4.19. We show that \( a_{r_1}, a_{r_2}, \cdots, a_{r_n} \) are all distinct.

Suppose \( a_{r_i} \equiv a_{r_j} \pmod{m} \)

Then \( r_i \equiv r_j \pmod{m} \)

\[ \Rightarrow i = j \]

\[ \therefore a_{r_i} \not\equiv a_{r_j} \pmod{m} \) when \( i \neq j \).

Hence \( a_{r_1}, a_{r_2}, \cdots, a_{r_n} \) is a complete residue (reduced) system modulo \( m \).
9.8.9 Theorem (Euler's Theorem) : If \((a, m) = 1\) then \(a^{\phi(m)} \equiv 1 \pmod{m}\).

Proof : Let \(r_1, r_2, \cdots, r_{\phi(m)}\) be a reduced residue system modulo \(m\).

Then by theorem 9.8.8 \(ar_1, ar_2, \cdots, ar_{\phi(m)}\) is also a reduced residue system modulo \(m\).

Hence, to each \(r_i\) \exists one and only one \(ar_j\) such that \(r_i \equiv ar_j \pmod{m}\).

Further more, different \(r_i\) will have different corresponding \(ar_j\).

(i.e.) the numbers \(ar_1, ar_2, \cdots, ar_{\phi(m)}\) are just the residues modulo \(m\) of \(r_1, r_2, \cdots, r_{\phi(m)}\) but not necessarily in the same order. Multiplying and using 5 of theorem 9.6.3 we obtain
\[
\prod_{j=1}^{\phi(m)} (ar_j) = \prod_{i=1}^{\phi(m)} r_i (\pmod{m})
\]

Hence, \(a^{\phi(m)} \prod_{j=1}^\phi(m) r_j \equiv \prod_{i=1}^\phi(m) r_i \pmod{m}\)

Now, \((r_j, m) = 1\) so we use theorem 9.4.19 and theorem 9.6.5 to cancel the \(r_j\) and we obtain \(a^{\phi(m)} \equiv 1 \pmod{m}\).

9.8.10 Theorem (Fermat's Theorem) (Corollary of Euler's theorem) : Let \(p\) denote a prime.

(i) If \(p \nmid a\) then \(a^{p-1} \equiv 1 \pmod{p}\)

(ii) For every integer \(a\), \(a^p \equiv a \pmod{p}\)

Proof : (i) If \(p \nmid a\) then \((a, p) = 1\) and \(a^{\phi(p)} \equiv 1 \pmod{p}\) (by theorem 9.8.9)

Since \(\phi(p) = p - 1\), \(a^{p-1} \equiv 1 \pmod{p}\)

(ii) By (i) \(a^{p-1} \equiv 1 \pmod{p}\)

By 5 of theorem 9.6.3, \(a^p = a^{p-1} \cdot a \equiv 1 \cdot a = a \pmod{p}\)
9.8.11 Theorem (Wilson's theorem): If \( p \) is a prime then \( p! \equiv -1 \pmod{p} \).

**Proof:** If \( p = 2 \) or 3 the congruence is easily verified.

Suppose \( p \geq 5 \)

Let \( S = \{1, 2, 3, \ldots, p-1\} \)

Then \((a, p) = 1 \forall a \in S\)

\[ \therefore \text{The linear congruence } ax \equiv 1 \pmod{p} \text{ has a unique solution, say } a' (1 \leq a' \leq p-1) \text{ by theorem 9.7.6.} \]

Now \( aa' \equiv 1 \pmod{p} \)

But if \( a' = a \) then \( aa \equiv 1 \pmod{p} \)

\[ \iff a^2 \equiv 1 \pmod{p} \iff p \mid (a^2 - 1) \]

\[ \iff p \mid (a+1)(a-1) \iff p \mid (a+1) \text{ or } p \mid (a-1) \]

\[ \iff a \equiv 0 \text{ or } p = a + 1 \]

\[ \iff a = 1 \text{ or } a = p - 1 \]

\[ \therefore \text{Thus for each } a \in S, a \neq 1 \text{ and } a \neq p - 1, \text{ there exists unique } a' \in S \text{ such that } aa' \equiv 1 \pmod{p} \text{ and } a' \neq a. \]

Multiplying all these congruences we get \( 2 \cdot 3 \cdot 4 \cdots (p-2) \equiv 1 \pmod{p} \)

\[ \Rightarrow 2 \cdot 3 \cdot 4 \cdots (p-2)(p-1) \equiv (p-1) \pmod{p} \]

\[ \Rightarrow p-1 \equiv (p-1) \pmod{p} \equiv -1 \pmod{p} \]

\[ \Rightarrow p-1 \equiv -1 \pmod{p} \]

Wilson's theorem and Fermat's theorem can be used to determine those primes for which \( x^2 \equiv -1 \pmod{p} \) has a solution.

9.8.12 Theorem: Let \( p \) denote a prime. Then \( x^2 \equiv -1 \pmod{p} \) has a solution if and only if \( p = 2 \) or \( p \equiv 1 \pmod{4} \).

**Proof:** If \( p = 2 \), then the congruence \( x^2 \equiv -1 \pmod{p} \) has a solution \( x = 1 \).
For any odd prime \( p \), we can write Wilson’s theorem in the form

\[
\left( 1 \cdot 2 \cdot 3 \cdots \frac{p-1}{2}\right) \left( \frac{p+1}{2} \cdots (p-j) \cdots (p-2)(p-1) \right) \equiv -1 \pmod{p}
\]

The product on the left hand side has been divided into two parts, each with the same number of factors.

Pairing off \( j \) in the first half with \( (p-j) \) in the second half, we can rewrite the above congruence in the form.

\[
1 \cdot (p-1) \cdot 2(p-2) \cdots 3(p-3) \cdots j(p-j) \cdots \left( \frac{p-1}{2} \right) \left( \frac{p+1}{2} \right) \equiv -1 \pmod{p}
\]

or

\[
\prod_{j=1}^{p-1} j(p-j) \equiv -1 \pmod{p} \quad \text{------- (1)}
\]

But \( j(p-j) = -j^2 \pmod{p} \)

If \( p \equiv 1 \pmod{4} \) then \( 4 \mid (p-1) \Rightarrow 2 \left\lfloor \frac{p-1}{2} \right\rfloor \Rightarrow \frac{p-1}{2} \) is an even integer.

From (1),

\[
\prod_{j=1}^{p-1} j(p-j) \equiv \prod_{j=1}^{p-1} (-j)^2
\]

\[
\equiv (-1)^{\prod_{j=1}^{p-1} j^2}
\]

\[
\equiv (-1)^\frac{p-1}{2} \left( \prod_{j=1}^{p-1} j \right)^2 \pmod{p} \quad \text{------- (2)}
\]

From (1) and (2)

\[
\left( \prod_{j=1}^{\frac{p-1}{2}} j \right)^2 = -1 \pmod{p}
\]
⇒ we have a solution \( x = \prod_{j=1}^{p-1} j \) of \( x^2 \equiv -1 \mod p \):

If \( p \neq 2 \) and \( p \not\equiv 1 \mod 4 \) then \( p \equiv 3 \mod 4 \) since \( \{0,1,2,3\} \) is a complete residue system modulo 4. \( p \neq 2 \Rightarrow p \not\equiv 2 \mod 4 \).

\[ \Rightarrow 4/(p-3) \Rightarrow p-3 = 4q \text{ for some } q \in \mathbb{Z}. \]

\[ \Rightarrow \frac{p-1}{2} = 2q + 1 \Rightarrow \frac{p-1}{2} \text{ is an odd integer.} \]

If for some integer \( x, x^2 \equiv -1 \mod p \) then

\[ x^{p-1} \equiv \left( x^2 \right)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} \equiv -1 \mod p \] \hspace{1cm} (3)

But, clearly \( p \nmid x \). So we have

\[ x^{p-1} \equiv 1 \mod p \] \hspace{1cm} (by Fermat's theorem).

From (3) and (4) we get \( 1 \equiv -1 \mod p \)

\[ \Rightarrow 2 \equiv 0 \mod p \Rightarrow p \mid 2 \Rightarrow p = 2 \]

This contradiction shows that \( x^2 \equiv -1 \mod p \) has no solution in this case.

**9.8.13 :** Show that \( 18 \equiv -1 \mod 437 \)

**Proof :** By Wilson's theorem, \( 23 \equiv -1 \mod 23 \)

\[ \Rightarrow 22 \equiv -1 \mod 23 \]
\[ \Rightarrow 22 \times 21 \times 20 \times 19 \times 18 \equiv -1 \mod 23 \]
\[ \Rightarrow (-1)(-2)(-3)(-4)18 \equiv -1 \mod 23 \]
\[ \Rightarrow 24 \times 18 \equiv -1 \mod 23 \]
\[ \Rightarrow 1 \times 18 \equiv -1 \mod 23 \]
\[ \Rightarrow 18 \equiv -1 \mod 23 \] \hspace{1cm} (1)

\[ 437 = 19 \times 23 \]
By Wilson’s theorem \(19 - 1 \equiv -1 \pmod{19}\)

i.e. \(18 \equiv -1 \pmod{19}\) \(\cdots \cdots (2)\)

From (1) and (2) \(18 \equiv -1 \pmod{[19, 23]}\)

\[\Rightarrow 18 \equiv -1 \pmod{437}\]

### 9.9 THE EULER FUNCTION \(\phi (n)\)

In this section we will use the Chinese remainder theorem to obtain an important property of the function \(\phi (n)\) of Definition 9.8.7.

**9.9.1 Theorem**: Let \(m\) and \(n\) denote any two positive, relatively prime integers. Then

\[\phi (mn) = \phi (m) \phi (n)\]

**Proof**: Let \(\phi (m) = j\) and let \(r_1, r_2, \ldots, r_j\) be a reduced residue system modulo \(m\).

Similarly let \(\phi (n) = k\) and let \(s_1, s_2, \ldots, s_k\) be a reduced residue system modulo \(n\).

**Claim**: \(x\) is in a reduced residue system modulo \(mn\).

\[\Leftrightarrow (x, m) = (x, n) = 1,\ \text{and hence} \ x \equiv r_h \pmod{m} \ \text{and} \ x \equiv s_i \pmod{n} \ \text{for some} \ h \ \text{and} \ i.\]

Suppose \(x\) is in a reduced residue system modulo \(mn\)

Then \((x, mn) = 1 \Rightarrow (x, m) = (x, n) = 1\)

(because if \((x, m) = d\) then \(d | x, \ d | m \Rightarrow d | x, \ d | mn\)

\[\Rightarrow d | 1 \Rightarrow d = 1. \ \text{So} \ (x, m) = 1. \ \text{Similarly} \ (x, n) = 1\)

\((x, m) = 1, (x, n) = 1 \Rightarrow x \equiv r_h \pmod{m} \ \text{and} \ x \equiv s_i \pmod{n} \ \text{for some} \ h \ \text{and} \ i.\]

Conversely, if \(x \equiv r_h \pmod{m}\) and \(x \equiv s_i \pmod{n}\) then \((x, m) = (x, n) = 1\) and

\[(x, n) = (x, s_i) = 1\]

\[\Rightarrow (x, m) = (x, n) = 1 \Rightarrow (x, mn) = 1\]

\[\Rightarrow x \ \text{is in a reduced residue system modulo} \ mn. \ \text{Thus our claim is proved.}\]

Thus reduced residue system modulo \(mn\) can be obtained by determining all \(x\) such that \(x \equiv r_h \pmod{m}\) and \(x \equiv s_i \pmod{n}\) for some \(h\) and \(i\). According to the Chinese remainder theorem, each pair \(h, i\) determines a single \(x\) modulo \(mn\). Clearly, different pairs \(h, i\) yield different \(x\) modulo \(mn\). There are \(jk\) of these pairs. Therefore a reduced residue system modulo \(mn\) contains \(jk = \phi (m) \phi (n)\) numbers, and we have \(\phi (mn) = \phi (m) \phi (n)\).
9.9.2 **Theorem**: If \( p \) is a prime, then
\[
\phi(p^n) = p^n - p^{n-1} - p^n \left(1 - \frac{1}{p}\right)
\]
for every positive integer \( n \).

**Proof**: \( \phi(p^n) \) is the number of integers \( x \) such that \( 1 \leq x \leq p^n, (x, p^n) = 1 \).

For every integer \( x \), \((x, p^n) = 1 \iff (x, p) = 1 \).

Hence \((x, p^n) \neq 1 \iff (x, p) \neq 1 \).

.: The number of integers that are not relatively prime to \( p^n \) are only those which are multiples of \( p \).

Now, the multiples of \( p \) are
\[p, 2p, \ldots, (p-1)p, pp, (p+1)p, \ldots, (p^{n-1}-1)p\]
which are \( p^{n-1} - 1 \) in number.

Hence \( \phi(n) = \) the number of integers \( x \) such that \( 1 \leq x \leq p^n, (x, p^n) = 1 \)
\[= (p^n - 1) - (p^{n-1} - 1) = p^n - p^{n-1}\]

9.9.3 **Theorem**: If \( n > 1 \), then \( \phi(n) = n \prod_{p | n} \left(1 - \frac{1}{p}\right) \). Also \( \phi(1) = 1 \).

**Proof**: If \( n = 1 \) then
\[n \prod_{p | n} \left(1 - \frac{1}{p}\right) = 1 \quad \therefore \phi(1) = 1\]

If \( n > 1 \), write \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) in the canonical form.

Now \( \left(\frac{p_j^{\alpha_j}}{p_j}, \frac{p_{j+1}^{\alpha_{j+1}}}{p_{j+1}}, \ldots, \frac{p_r^{\alpha_r}}{p_r}\right) = 1 \)

Applying theorem 9.9.1 repeatedly we obtain
\[
\phi(n) = \phi(p_1^{\alpha_1}) \phi(p_2^{\alpha_2}) \cdots \phi(p_r^{\alpha_r})
\]
\[= \prod_{j=1}^{r} \phi(p_j^{\alpha_j}) \quad \text{(1)}\]

Also by theorem 9.9.2, \( \phi(p^\alpha) = p^\alpha \left(1 - \frac{1}{p}\right) \)
From (1), $\phi(n) = p_1^{\alpha_1} \left(1 - \frac{1}{p_1}\right) p_2^{\alpha_2} \left(1 - \frac{1}{p_2}\right) \ldots p_r^{\alpha_r} \left(1 - \frac{1}{p_r}\right)$

$$= p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} \left(1 - \frac{1}{p_1}\right) \ldots \left(1 - \frac{1}{p_r}\right)$$

$$= n \prod_{j=1}^{r} \left(1 - \frac{1}{p_j}\right) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

9.9.4 Theorem: For $n \geq 1$, we have $\sum_{d|n} \phi(d) = n$.

Proof: If $n = 1$, then the theorem is obvious. If $n > 1$, let $S = \{1, 2, \ldots, n\}$.

Define a relation $\sim$ on $S$ as follows.

$$a \sim b \iff (a, n) = (b, n) \ \forall \ a, b \in S$$

$\sim$ is reflexive: $a \sim a \iff (a, n) = (a, n)$

Symmetry: If $a \sim b$, then $(a, n) = (b, n)$

$$\Rightarrow (b, n) = (a, n) \Rightarrow b \sim a$$

Transitive: If $a \sim b, b \sim c$ then

$$(a, n) = (b, n) \text{ and } (b, n) = (c, n)$$

$$\Rightarrow (a, n) = (c, n) \Rightarrow a \sim c$$

Hence $\sim$ is an equivalence relation on $S$. For every $a \in S$, $\exists$ unique divisor $d$ of $n$ such that $(a, n) = d$.

The equivalence class of $a$ is $[a] = \{b \in S \mid a \sim b\}$

$$= \{b \mid 1 \leq b \leq n, (b, n) = (a, n) = d\}$$

$$= \left\{\frac{b}{d} \mid 1 \leq \frac{b}{d} \leq \frac{n}{d}, \left(\frac{b}{d}, \frac{n}{d}\right) = 1\right\}$$

$$= \left\{\frac{b'}{d'} \mid 1 \leq b' \leq \frac{n}{d}, \left(b', \frac{n}{d}\right) = 1\right\}$$
where \( b' = \frac{b}{d} \)

The number of elements in \( \bar{\mathbb{A}} = \phi \left( \frac{n}{d} \right) \)

\( S \) in the union of disjoint equivalence classes.

Hence \( n = \sum \) number of elements in \( \bar{\mathbb{A}} \), where the summation is taken over all the distinct equivalence classes.

\[ = \sum_{d|n} \phi \left( \frac{n}{d} \right) \]

**GROUP THEORY**

**9.10 BINARY OPERATION**

**9.10.1 Notation**: We use the following notation throughout this material.

- \( Z \): The set of all integers.
- \( \mathbb{Z}^+ \): The set of all non-zero positive integers.
- \( \mathbb{Q} \): The set of all rational numbers.
- \( \mathbb{Q}^+ \): The set of all non-zero positive rational numbers.
- \( \mathbb{R} \): The set of all real numbers.
- \( \mathbb{R}^+ \): The set of all non-zero positive real numbers.
- \( \mathbb{C} \): The set of all complex numbers.

**9.10.2 Definition**: Let \( G \) be a non-empty set. Any mapping from \( G \times G \) into \( G \) is called a binary operation on \( G \).

**9.10.3 Notation**: If \( \circ \) is a binary operation on \( G \), then \( \circ \) is a mapping from \( G \times G \) into \( G \). We write \( \circ : G \times G \rightarrow G \) is a mapping. It is customary to denote the image \( \circ((a, b)) \) of \( (a, b) \in G \times G \) by \( a \circ b \) or simply \( ab \). We call \( ab \) the product of \( a \) and \( b \) under the operation. However if the binary operation symbol is other than \( \circ \), we retain the symbol between the elements.

**9.10.4 Definition**: Let \( \circ \) be a binary operation on \( G \). Then

1. \( \circ \) is said to be associative if \( (a \circ b) \circ c = a \circ (b \circ c) \) for all \( a, b, c \in G \).
2. \( \circ \) is said to be commutative if \( a \circ b = b \circ a \) for all \( a, b \in G \).
9.10.5 Example :  Let \( G = \mathbb{Z}^+ \). For all \( a, b \in G \) define \( a \, \Box \, b = \text{minimum of } a \text{ and } b \). Then \( \Box \) is a commutative and associative binary operation on \( G \).

9.10.6 Example :  Let \( G \) be any non-empty set containing at least two elements. Define \( a \, \Box \, b = a \, \forall \, a, b \in G \). Then \( \Box \) is an associative, non-commutative binary operation on \( G \).

9.10.7 SAQ :  Let \( G = \mathbb{Q} \). Define \( \Box \) on \( G \) by \( a \, \Box \, b = \sqrt{a/b} \). Show that \( \Box \) is not a binary operation on \( G \).

9.10.8 Example :  Let \( G = \mathbb{Q}^+ \). Define \( \Box \) on \( G \) by \( a \, \Box \, b = a/b \). Then \( \Box \) is a binary operation on \( G \) which is neither commutative nor associative.

\[
2, 4, 8 \in G.
\]
\[
\begin{align*}
2 \, \Box \, 4 &= \frac{2}{4} = \frac{1}{2}; \\
4 \, \Box \, 2 &= \frac{4}{2} = 2; \\
2 \, \Box \, 4 &= \frac{2}{4} \neq 4 \, \Box \, 2
\end{align*}
\]
\[
(2 \, \Box \, 4) \, \Box \, 8 = \left(\frac{2}{4}\right) \, \Box \, 8 = \left(\frac{1}{2}\right) \, \Box \, 8 = \frac{1}{16}
\]
\[
2 \, \Box \left(4 \, \Box \, 8\right) = 2 \, \Box \left(\frac{4}{8}\right) = 2 \, \Box \left(\frac{1}{2}\right) = 2 \times \frac{1}{4} = 4
\]
\[
(2 \, \Box \, 4) \, \Box \, 8 \neq 2 \, \Box \left(4 \, \Box \, 8\right)
\]

9.10.9 Example :  Let \( G = \mathbb{Z}^+ \). Define \( \Box \) on \( G \) by \( a \, \Box \, b = \sqrt{a/b} \). Then \( \Box \) is not a binary operation on \( G \).

\[
1, 2 \in G. \text{ But } 1 \, \Box \, 2 = \frac{1}{2} = 0.5 \notin G.
\]

9.10.10 SAQ :  Let \( G \) be the set of all real valued functions defined on \( \mathbb{R} \). For \( f, g \in G \) define \( + \) on \( G \) by \( f + g = h \) where \( h(x) = f(x) + g(x) \) for all \( x \in \mathbb{R} \).

Then prove that \( + \) is a commutative and associative binary operation on \( G \).

9.10.11 SAQ :  Let \( X \) be a set. Let \( G \) be the set of all subsets of \( X \). Define (i) \( + \) on \( G \) by \( A + B = (A - B) \cup (B - A) \) (ii) \( A \cdot B = A \cap B \). Prove that \( + \) and \( \cdot \) are commutative and associative binary operations on \( G \).

9.10.12 Tabular representation of a binary operation :  When \( G \) is a finite set and \( \Box \) is a binary operation on \( G \), this binary operation \( \Box \) can be described by a table.

Let \( G = \{a_1, a_2, \ldots, a_n\} \)

Then the tabular presentation of \( \Box \) is given below.
If \( x, y \in G \) then \( x = a_i \) and \( y = a_j \) for some \( i, j \in \{1, 2, \ldots, n\} \). Then \( x \cdot y = a_i \cdot a_j \) is the element that occurs in the row of \( a_i \) and the column \( a_j \).

\( \cdot \) is commutative iff \( a_i \cdot a_j = a_j \cdot a_i \) for all \( i \) and \( j \). Thus the binary operation is commutative iff the table is symmetric about the diagonal from the top left corner to the bottom right corner (called the leading or principal diagonal). We note that the principal diagonal is the diagonal consisting of \( a_1 \cdot a_1, a_2 \cdot a_2, \ldots, a_n \cdot a_n \).

**9.10.13 Example:** \( G = \{0, 1, 2, 3, 4\} \). The following table gives a binary operation \( + \) on \( G \) which is commutative and associative.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**9.10.14 Example:** \( G = \{a, b, c, d\} \). The following table describes a binary operation \( \cdot \) on \( G \) which is commutative and associative.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>d</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>d</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>c</td>
<td>b</td>
<td>a</td>
</tr>
</tbody>
</table>

**9.10.15 Example:** \( G = \{e, a, b\} \).
Here \( \circ \) is commutative but not associative.

\[
(a \circ b) \circ c = e \circ b = b, \quad a \circ (b \circ c) = a \circ e = a
\]

\[
\therefore (a \circ b) \circ c \neq a \circ (b \circ c)
\]

### 9.11 GROUPS

In this section we introduce the concept of a group which is a very important concept in Algebra.

#### 9.11.1. Definition:
A group is a pair \( (G, \circ) \) where \( G \) is a non-empty set and \( \circ \) is a binary operation on \( G \) with the following properties.

(i) \( \circ \) is associative

(ii) There exists an element \( e \in G \) such that \( e \circ a = a \circ e = a \) for all \( a \in G \).

(iii) For each \( a \in G \) there exists \( b \in G \) such that \( a \circ b = b \circ a = e \)

\( e \) is called the identity element of \( (G, \circ) \) and \( b \) is called the inverse of \( a \).

#### 9.11.2 Lemma:
Let \( (G, \circ) \) be a group

(i) Let \( e \in G \) such that \( a \circ e = e \circ a = a \ \forall \ a \in G \). Then this \( e \) is unique w.r.t. this property.

(ii) Let \( e \) be the identity element of the group. Let \( a \in G \). Then the element \( b \) such that \( a \circ b = b \circ a = e \) is unique with this property.

**Proof:**

(i) Suppose \( \exists e' \in G \) such that \( a \circ e' = e' \circ a = a \ \forall \ a \in G \).

Now \( e \circ e' = e' \circ e = e \)

\[
\therefore e = e'
\]

(ii) Suppose \( a \circ b = b \circ a = e \) and \( a \circ c = c \circ a = e \)

\[
b = b \circ c = b \circ (a \circ c) = (b \circ a) \circ c = e \circ c = c
\]

\[\therefore b = c\]
9.11.3 Notation: (i) When we say that $G$ is a group, we mean $G$ is a group w.r.t. a certain binary operation on $G$. We write $ab$ for the image of the ordered pair $(a, b)$ under the binary operation. $ab$ is called the product of $a$ and $b$ in the group $G$.

(ii) When the binary operation in the group is denoted by the symbol $+$, we say that the group is an additive group. The image of $(a, b)$ under $+$ is denoted by $a + b$ and is called the sum of $a$ and $b$. The group $\langle G, + \rangle$ is called an additive group.

9.11.4 Definition: Let $G$ be a group. By (i) of lemma 9.11.2, the element $e \in G$ such that $ac = ca = a \ \forall a \in G$ is unique. This $e$ is called the identity element of the group.

By (ii) of lemma 9.11.2, for $a \in G$ the element $b \in a$ such that $ab = ba = e$ is unique. $b$ is called the inverse of $a$. The inverse of $a$ is denoted by $a^{-1}$.

9.11.5 Notation: In an additive group $\langle G, + \rangle$,

(i) the identity element is denoted by 0 and

(ii) the inverse of an element $a \in G$ is denoted by $-a$.

9.11.6 Theorem: Let $G$ be a group. Let $a, b, c \in G$.

(i) If $ab = ac$ then $b = c$ (left cancellation law)

(ii) If $ba = ca$ then $b = c$ (right cancellation law)

Proof: Let $e$ be the identity element of $G$.

(i) $ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c \Rightarrow eb = ec \Rightarrow b = c$

(ii) $ba = ca \Rightarrow (ba)a^{-1} = (ca)a^{-1}$

$\Rightarrow b(aa^{-1}) = e(aa^{-1}) \Rightarrow be = ce \Rightarrow b = c$

9.11.7 Theorem: Let $G$ be a group. Let $a, b \in G$. Then (i) $(a^{-1})^{-1} = a$ (ii) $(ab)^{-1} = b^{-1}a^{-1}$.

Proof: (i) $(a^{-1})(a^{-1})^{-1} = e, \ a^{-1}a = e$

$\therefore a^{-1}a = (a^{-1})(a^{-1})^{-1}$
By left cancellation law we have $a = (a^{-1})^{-1}$

(ii) $(ab)(b^{-1}a^{-1}) = ((ab)b^{-1})a^{-1} = (a(bb^{-1}))a^{-1} = (ae)a^{-1} = aa^{-1} = e$

$(b^{-1}a^{-1})(ab) = ((b^{-1}a^{-1})a)b = (b^{-1}(a^{-1}a))b$

$= (b^{-1}e)b = b^{-1}b = e$

Since the inverse is unique, $(ab)^{-1} = b^{-1}a^{-1}$.

**9.11.8 Lemma:** Let $G$ be a non-empty set. Let $\circ$ be an associative binary operation on $G$, such that (i) $\exists e \in G$ such that $e \circ a = a \forall a \in G$ (left identity) (ii) $a \in G \Rightarrow \exists b \in G \ni b \circ a = e$ (left inverse)

Then $\langle G, \circ \rangle$ is a group.

**Proof:** Let $a$ be any element of $G$. By (i) $e \circ a = a$.

By (i) $e \circ a = a$.

By (ii) $\exists b \in G$ such that $b \circ a = e$.

By (ii) $\exists b' \in G$ such that $b' \circ b = e$.

Now $b' \circ e = b'(b \circ a) = (b' \circ b) \circ a = e \circ a = a$

$a \circ e = (b' \circ e) \circ e = b' \circ (e \circ e) = b' \circ e = a$

Thus $e$ is the identity element of $\langle G, \circ \rangle$. Also $a \circ b = (b' \circ e) \circ b = b' \circ (e \circ b) = b' \circ b = e$. Thus $b$ is the inverse of $a$.

Hence $\langle G, \circ \rangle$ is a group.

**9.11.9 Theorem:** Let $G$ be a group. Let $a, b \in G$. Then the equations $ax = b$ and $ya = b$ have unique solutions in $G$.

**Proof:** Let $e$ be the identity element of $G$.

$x = a^{-1}b \in G$, $y = ba^{-1} \in G$ and

$ax = a(a^{-1}b) = (aa^{-1})b = eb = b$ and
If \( x_1, x_2 \in G \) such that \( ax_1 = b \) and \( ax_2 = b \) then \( ax_1 = ax_2 \). By left cancellation law \( x_1 = x_2 \).

If \( y_1, y_2 \in G \) such that \( y_1a = b \) and \( y_2a = b \) then \( y_1a = y_2a \).

By right cancellation law \( y_1 = y_2 \).

Thus \( a^{-1}b \) and \( ba^{-1} \) are the unique solutions of \( ax = b \) and \( ya = b \) respectively.

**9.11.10 Theorem:** Let \( G \) be a non-empty set. Let \( \cdot \) be an associative binary operation on \( G \) such that the equations \( ax = b \) and \( ya = b \) have solutions in \( G \) for all \( a, b \in G \). Then \( \langle G, \cdot \rangle \) is a group and the solutions are unique.

**Proof:** \( G \neq \emptyset \).

Let \( a \in G \). The equation \( ya = a \) has a solution, say \( e \) in \( G \).

Then \( e \cdot a = a \).

Let \( b \in G \). \( a \cdot x = b \) has a solution say \( c \), in \( G \).

Then \( a \cdot c = b \).

Now \( e \cdot b = e \cdot (a \cdot c) = (e \cdot a) \cdot c = a \cdot c = b \). Thus \( e \) is a left identity w.r.t. the binary operation \( \cdot \).

The equation \( y \cdot b = e \) has a solution, say \( b' \), in \( G \).

Then \( b' \cdot b = e \).

Thus every element \( b \in G \) has a left inverse w.r.t. the left identity \( e \). Thus, by lemma 9.11.8, \( \langle G, \cdot \rangle \) is a group. By theorem 9.11.9, the solutions are unique.

**9.11.11 Theorem:** Let \( G \) be a non-empty finite set. Let \( \cdot \) be an associative binary operation on \( G \) such that for any \( a, b, c \in G \).

(i) \( a \cdot b = a \cdot c \Rightarrow b = c \) and

(ii) \( b \cdot a = c \cdot a \Rightarrow b = c \). Then \( \langle G, \cdot \rangle \) is a group.

**Proof:** Let \( G = \{a_1, a_2, \ldots, a_n\} \) and \( a, b \in G \). If \( a \cdot a_i = a \cdot a_j \) for any \( i \) and \( j \) then, by (i) \( a_i = a_j \).

If \( a_i \cdot a = a_j \cdot a \) for any \( i \) and \( j \) then, by (ii) \( a_i = a_j \). Thus

\[
G = \{a \cdot a_1, a \cdot a_2, \ldots, a \cdot a_n\} = \{a_1 \cdot a, a_2 \cdot a, \ldots, a_n \cdot a\}.
\]
Thus the equations \( a \cdot x = b \) and \( y \cdot a = b \) have solutions in \( G \). By theorem 9.11.10 \( \langle G, \cdot \rangle \) is a group.

9.11.12 Definition: A group \( G \) is said to be an abelian group if the binary operation in the group \( G \) is commutative. (i.e.) if \( ab = ba \) for all \( a, b \in G \).

9.11.13 Example: The set \( \mathbb{Z}^+ \) with the usual addition + of integers is not a group. + in \( \mathbb{Z}^+ \) is associative and commutative. But \( x + y = y, y \in \mathbb{Z}^+ \) implies \( x = 0 \). But \( 0 \notin \mathbb{Z}^+ \). Thus there is no identity in \( \mathbb{Z}^+ \) w.r.t. +.

9.11.14 Example: Consider \( G = \mathbb{Z}^+ \cup \{0\} \) with the usual addition + of integers. Now \( \langle G, + \rangle \) has identity 0 w.r.t. +. But \( \langle G, + \rangle \) is not a group, since for \( 1 \in G \), there is no \( g \in G \) such that \( g + 1 = 0 \).

9.11.15 SAQ: The set \( \mathbb{Z} \) with the usual addition + of integers is a group.

9.11.16 Example: Define \( * \) on \( \mathbb{Q}^+ \) by \( a * b = \frac{ab}{3} \). Then \( \langle \mathbb{Q}^+, * \rangle \) is an abelian group.

Proof: (i) If \( a, b \in \mathbb{Q}^+ \) then \( \frac{ab}{3} \) is a positive rational number.

\[
\therefore a \cdot b \in \mathbb{Q}^+
\]

Thus \( * \) is a binary operation on \( \mathbb{Q}^+ \).

(ii) \((a \cdot b) \cdot c = \left( \frac{ab}{3} \right) \cdot c = \frac{abc}{3^2} = \frac{a \left( \frac{bc}{3} \right)}{3} = \frac{a}{3}(b \cdot c) = a \cdot b \cdot c \)

(iii) \( a \cdot b = \frac{ab}{3} = \frac{ba}{3} = b \cdot a \)

(iv) \( 3 \cdot a = \frac{3a}{3} = a \) \( \forall \ a \in \mathbb{Q}^+ \). Thus 3 is the identity w.r.t. \( * \).

(v) \( \frac{a^2}{3} \in \mathbb{Q}^+ \) for all \( a \in \mathbb{Q}^+ \) and
\[ a \cdot \frac{3^2}{a} = \frac{a \cdot 3^2}{3a} = \frac{3a}{3a} = 3 \]

\[ \frac{3^2}{a} \] is the inverse of \( a \) w.r.t. \( \ast \).

Hence \((Q^+, \ast)\) is an abelian group.

Note that if \( K \) is any positive integer and binary operation \( \ast \) is defined on \( Q^+ \) by
\[ a \ast b = \frac{ab}{K}, \]
once can prove that \((Q^+, \ast)\) is an abelian group with \( K \) as identity and \( \frac{K^2}{a} \) as the inverse of \( a \).

**9.11.17 Example**: \( \mathbb{Z} \) under the usual multiplication is not a group.

**Proof**: The product of any two integers is also an integer.

Multiplication of integers is commutative and associative.

1. \( \in \mathbb{Z} \) and \( 1a = a \forall a \in \mathbb{Z} \).

2. \( \in \mathbb{Z} \) and there is no \( a \in \mathbb{Z} \) \( 2a = 1 \)

Thus \( 2 \) has no multiplicative inverse.

**9.11.18 Example**: Define \( \ast \) on \( \mathbb{R}^+ \) by \( a \ast b = ab \), where \( ab \) is the usual product of real numbers. Then \((\mathbb{R}^+, \ast)\) is an abelian group.

**Proof**: If \( a, b \in \mathbb{R}^+ \) then \( a > 0, b > 0 \Rightarrow ab > 0 \).

Thus \( a \ast b = ab \in \mathbb{R}^+ \)

Also \( (a \ast b) \ast c = (ab) \ast c = (ab)c = a(bc) \)
\[ = a(b \ast c) = a \ast (b \ast c) \]
\[ a \ast b = ab = ba = b \ast a \]
\[ 1 \ast b = 1b = b \forall b \in \mathbb{R}^+ \]

If \( a \in \mathbb{R}^+ \) then \( \frac{1}{a} \in \mathbb{R}^+ \) and \( a \ast \frac{1}{a} = a \left( \frac{1}{a} \right) = 1 \)

Thus \((\mathbb{R}^+, \ast)\) is an abelian group.
9.11.19 Example: $C$ with the usual addition of complex numbers is an abelian group with $0$ as the identity and $-a - ib$ as the inverse of $a + ib, \ a, b$ real.

9.11.20 Example: $C^* = C \setminus \{0\}$ with usual multiplication of complex numbers is an abelian group with $1$ as the identity and $\frac{a - ib}{a^2 + b^2}$ as the inverse of $a + ib, \ a, b$ real.

9.11.21 Definition: If $G$ is a group with a finite number of elements, then we say that $G$ is a finite group. If $G$ has exactly $n$ elements, we say that $G$ is a finite group of order $n$ and we write $|G| = n$. If $G$ is not a finite group then we say that $G$ is an infinite group.

9.11.22 Lemma: If $G$ is a finite group with identity $e$ and $|G| = 2n$ for some integer $n \geq 1$, then $G$ has an element $a \neq e$ such that $aa = e$.

Proof: Let $G^* = G \setminus \{e\}$. With each $a \in G^*$ we write $A_a = \{a, a^{-1}\}$.

Suppose $|A_a| = 2$ for all $a \in G^*$

Let $x \in A_a$. Then $x = a$ or $a^{-1}$.

If $x = a$ then $x^{-1} = a^{-1}$ hence $A_a = A_x$.

If $x = a^{-1}$ then $x^{-1} = a$ hence $A_a = A_x$.

Thus $x \in A_a \Rightarrow A_a = A_x$.

For any $a, b \in G^*$, we have either $A_a = A_b$ or $A_a \cap A_b = \emptyset$.

Thus $G^*$ is the disjoint union of $\{A_a\}_{a \in G^*}$ and $|A_a| = 2$.

Thus $|G^*|$ is even, say $2K$.

$$|G| = |\{e\}| + |G^*| = 1 + 2K$$ is odd which is a contradiction.

Thus $|A_a| = 1$ for some $a \in G^*$

For this $a$ we have $a = a^{-1}$.

$\therefore aa = aa^{-1} = e$ and $a \neq e$.

9.11.23 SAQ: If $*$ is a binary operation on a set $G$, we say that $x \in G$ is an idempotent if $x * x = x$.

Show that any group $G$ has a unique idempotent.
9.11.24 Example: Let $R^* = R \setminus \{0\}$. Define $a \* b = |a|b$ for all $a, b \in R^*$. Then $\ast$ is an associative binary operation on $R^*$.

$$(-3) \ast 2 = 6 \text{ and } 2 \ast (-3) = -6$$

imply $\ast$ is not commutative.

$$I \ast a = a \quad \forall a \in R^*; \quad a \ast \frac{1}{|a|} = I \quad \forall a \in R^*.$$ Thus 1 is a left identity for $\langle R^*, \ast \rangle$ and $a \in R^*$ has a right inverse $\frac{1}{|a|}$ w.r.t. $\ast$ and the left identity 1 \text{ and } -1 and 1 are idempotents w.r.t. $\ast$. By SAQ

9.11.23 $\langle R^*, \ast \rangle$ is not a group.

9.11.25 Example: Table of a group with one element. Let $G = \{e\}$.

<table>
<thead>
<tr>
<th>$e$</th>
<th>$e$</th>
</tr>
</thead>
</table>

9.11.26 Example: Table of a group with two elements. Let $e$ be the identity element. Let $G = \{e, a\}$.

<table>
<thead>
<tr>
<th>$e$</th>
<th>$e$</th>
<th>$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>$a$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$e$</td>
</tr>
</tbody>
</table>

The group is commutative.

9.11.27 Example: Table of a group with 3 elements. Let $e$ be the identity element. Let $G = \{e, a, b\}$.

By left and right cancellation laws and since the group is finite, in each row and in each column all the elements must occur as the product values, each element occurring only once in any column or row. Thus there is only one operation on $G$ which makes $G$ a group. This group is abelian.

<table>
<thead>
<tr>
<th>$e$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>$a$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td>$c$</td>
</tr>
</tbody>
</table>

9.11.28 Example: Group with 4 elements. Let $G = \{e, a, b, c\}$. Let $e$ be the identity. By lemma 9.11.22 there is an element $x$ in $G$ such that $xx = e$. Assume $aa = e$.

Case 1: If $bb = e$ then we must have $cc = e$. The multiplication table is
This group is abelian. This group is called the Klein 4-group and is denoted by $V$. In this group $aa = e$, $bb = e$, $cc = e$, $ab = c$, $bc = a$, $ca = b$.

**Case 2:** The inverse of $b$ is $c$. Thus $bc = cb = e$.

The table is given below:

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>e</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>e</td>
<td>e</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>e</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>a</td>
<td>e</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>e</td>
<td>a</td>
</tr>
</tbody>
</table>

This group is abelian. In this group $aa = e$, $bb = a$, $cc = a$.

Also $bbb = ab = c$,
$ccc = ac = b$

$G = \{e, b, bb, bbb\} = \{e, c, cc, ccc\}$.

**9.11.29 Example:** Let $G = \{1, -1, i, -i\} \subseteq \mathbb{C}$. Then $G$, under the multiplication of complex numbers, is a group whose table is given below:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>-1</th>
<th>i</th>
<th>-i</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>i</td>
<td>-i</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-i</td>
<td>i</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>-i</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>-i</td>
<td>-i</td>
<td>i</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

If we rename 1 as $e$, -1 as $a$, $i$ as $b$ and $-i$ as $c$ then this group table becomes the group table in Example 9.11.28 case 2. Thus these two groups are structurally the same and they differ only in naming their elements.
9.11.30 Quaternion group $Q_8$ : The set $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ under the binary operation defined by the following table.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>-1</th>
<th>i</th>
<th>-i</th>
<th>j</th>
<th>-j</th>
<th>k</th>
<th>-k</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>i</td>
<td>-i</td>
<td>j</td>
<td>-j</td>
<td>k</td>
<td>-k</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-i</td>
<td>i</td>
<td>j</td>
<td>-j</td>
<td>k</td>
<td>j</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>-i</td>
<td>-1</td>
<td>1</td>
<td>k</td>
<td>-k</td>
<td>j</td>
<td>i</td>
</tr>
<tr>
<td>-i</td>
<td>-i</td>
<td>1</td>
<td>-1</td>
<td>-k</td>
<td>k</td>
<td>j</td>
<td>i</td>
<td>-j</td>
</tr>
<tr>
<td>j</td>
<td>j</td>
<td>-j</td>
<td>k</td>
<td>-l</td>
<td>i</td>
<td>-i</td>
<td>-k</td>
<td>l</td>
</tr>
<tr>
<td>-j</td>
<td>-j</td>
<td>k</td>
<td>-l</td>
<td>i</td>
<td>-i</td>
<td>-1</td>
<td>i</td>
<td>-j</td>
</tr>
<tr>
<td>k</td>
<td>k</td>
<td>-k</td>
<td>j</td>
<td>-i</td>
<td>i</td>
<td>-l</td>
<td>1</td>
<td>-l</td>
</tr>
<tr>
<td>-k</td>
<td>-k</td>
<td>j</td>
<td>i</td>
<td>l</td>
<td>1</td>
<td>-l</td>
<td>i</td>
<td>-j</td>
</tr>
</tbody>
</table>

Clearly $\square$ is a binary operation on $Q_8$. Associative : It can be verified that $\square$ is associative in $Q_8$.

Identity : From the table 1 is the identity element in $Q_8$.

Inverse : If $x \in G$ and $x \neq \pm 1$ then $x \square (-x) = 1 \Rightarrow x^{-1} = -x$

If $x = \pm 1$ then $x \square x = 1 \Rightarrow x^{-1} = x$

$\therefore (Q_8, \square)$ is a group.

In $Q_8$, $i^2 = j^2 = k^2 = -1$, $i \square j = k$, $j \square k = i$, $k \square i = j$, $j \square i = -k$, $k \square j = -i$, $i \square k = -j$.

So, $(Q_8, \square)$ is a non-abelian group.

9.11.31 Example : Let $n$ be a positive integer. Let $G_n = \{z \in C, z^n = 1\}$. Then $G_n$ is an abelian group w.r.t. the multiplication of complex numbers.

Proof : Since $1^n = 1$ we have $1 \in G_n$.

If $z_1, z_2 \in G_n$ then $z_1^n = 1, z_2^n = 1$

$$(z_1 z_2)^n = z_1^n z_2^n = 1 \cdot 1 = 1$$

Thus $z_1 z_2 \in G_n$

We know that multiplication of complex numbers is commutative and associative.
If \( z \in G_n \) then \( z^n = 1 \Rightarrow z \neq 0 \). Thus \( \frac{1}{z} \in \mathbb{C} \) and \( \left( \frac{1}{z} \right)^n = \frac{1}{z^n} = \frac{1}{1} = 1 \). So \( \frac{1}{z} \in G_n \) and

\[ z \cdot \frac{1}{z} = 1. \]

Thus \( \frac{1}{z} \) is the inverse of \( z \) in \( G_n \). Thus \( G_n \) is an abelian group under the multiplication of complex numbers. Note that elements of \( G_n \) are the nth roots of unity and \( |G_n| = n \).

### 9.11.32 Exponents:

Let \( G \) be a group. Let \( a \in G \). Let \( n \) be any integer.

If \( n = 0 \), we define \( a^n = e \) where \( e \) is the identity of \( G \).

If \( n > 0 \), then \( a^n \) is defined inductively by \( a^1 = a \) and \( a^n = a^{n-1}a \).

If \( n < 0 \), then \( a^n \) is defined as \( (a^{-1})^{-n} \)

Note that \( a^n = \begin{cases} a \cdot a \cdots a (n \text{ times}) & \text{if } n > 0 \\ e & \text{if } n = 0 \\ a^{-1} a^{-1} \cdots a^{-1} (-n \text{ times}) & \text{if } n < 0 \end{cases} \)

### 9.11.33 Theorem:

Let \( G \) be a group. Let \( a \in G \). Let \( m, n \in \mathbb{Z} \). Then

(i) \( a^{-n} = (a^{-1})^n \)  
(ii) \( a^m a^n = a^{m+n} \)

(iii) \( (a^m)^n = a^{mn} \)  
(iv) \( (ab)^n = a^n b^n \) if \( ab = ba \)

**Proof:** Proof is left as an exercise to the reader.

### 9.12 SUBGROUPS

#### 9.12.1 Definition:

Let \( \langle G, * \rangle \) be a group. Let \( H \) be a non-empty subset of \( G \). We know that \( * : G \times G \to G \) is a mapping. For any \( x, y \in H \), \( (x, y) \in G \times G \). Thus \( x * y \in G \). If \( x * y \in H \) for all \( x, y \in H \), then we say that \( H \) is closed under the group operation \( * \) on \( G \).

#### 9.12.2 Definition:

Let \( G \) be a group. Let \( H \) be a non-empty subset of \( G \) which is closed under the group operation in \( G \). Then we define a binary operation in \( H \) by defining \( ab \) in \( H \) as \( ab \) in the group \( G \) for all \( a, b \in H \). This binary operation in \( H \) is called the induced operation on \( H \) from \( G \).

#### 9.12.3 Definition:

Let \( G \) be a group. Let \( H \) be a non-empty subset of \( G \) which is closed under the group operation of \( G \). If \( H \) is itself a group under the induced operation, then \( H \) is said to be a
subgroup of $G$ and we write $H \leq G$ (or $G \geq H$). We write $H < G$ or $(G > H)$ to mean that $H \leq G$ (or $G \geq H$) and $H \neq G$.

9.12.4 Definition: If $H < G$ then $H$ is called a proper subgroup of $G$. Clearly $G \leq G$. $G$ is called improper subgroup of $G$. $\{e\}$ is a subgroup of $G$ and is called the trivial subgroup of $G$. If $H \leq G$ and $H \neq \{e\}$ then $H$ is called a non-trivial subgroup of $G$.

9.12.5 Example: Let $Z_4 = \{0, 1, 2, 3\}$. Define $+$ on $Z_4$ by the following table.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

It can be easily verified that $\langle Z_4, + \rangle$ is an abelian group. If the elements are renamed as $e = 0$, $a = 1$, $b = 2$, $c = 3$, the group tables of $\langle Z_4, + \rangle$ and the group $G$ of example 9.8.28 case 2 are the same. The subgroups of $\langle Z_4, + \rangle$ are $\{0\}$, $\{0, 2\}$ and $\{0, 1, 2, 3\}$.

9.12.6 Example: Let $V$ be the Klein 4-group $V = \{e, a, b, c\}$ where $a^2 = e$, $b^2 = e$, $c^2 = e$. The subgroups of $V$ are $\{e\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$ and $\{e, a, b, c\} = V$.

In this group the equation $x^2 = e$ has 4 solutions.

9.12.7 Example: (i) $Q^+$ under multiplication is a proper subgroup of $R^+$ under multiplication.

(ii) $\langle Z, + \rangle < \langle Q, + \rangle < \langle R, + \rangle < \langle C, + \rangle$

9.12.8 Example: $\langle 12Z, + \rangle < \langle 6Z, + \rangle < \langle 3Z, + \rangle < \langle Z, + \rangle < \langle R, + \rangle$

9.12.9 Example: $G = \{\pi^n/\pi \in Z\}$ under multiplication is a subgroup of $R^+$ under multiplication.

9.12.10 Example: $G = \{6^n/6 \in Z\}$ under multiplication $< Q^+$ under multiplication $< R^+$ under multiplication.

9.12.11 Theorem: Let $G$ be a group. Let $H$ be a subset of $G$. Then $H$ is a subgroup of $G$ if and only if

(a) $H$ is closed under the binary operation of $G$. 
(b) the identity $e$ of $G$ is in $H$.

(c) $a \in H$ implies $a^{-1} \in H$.

**Proof:** If $H \leq G$, then by definition of a subgroup $H$ is closed w.r.t. the binary operation of $G$.

If $e'$ is the identity element of $H$, then $e'e = e'$ in $H$ and hence in $G$.

But $e'e = e'$ in $G$.

$\therefore e'e = e'$ in $G$

$\therefore e = e'$ by cancellation law in $H$.

If $a \in H$ then $\exists b \in H : ab = e$ in $H$.

But $ab = e$ in $G$ also.

$\therefore a = b = 1$

Conversely suppose that (a), (b) and (c) are satisfied.

Since the operation of $G$ is associative the induced operation of $H$ is also associative.

$e \in H \Rightarrow ae = ea = a \forall a \in H$

$a \in H \Rightarrow a^{-1} \in H \Rightarrow a^{-1}a = aa^{-1} = 1$ in $H$.

Thus $H$ is a subgroup.

**9.12.12 Theorem:** Let $H$ be a non-empty subset of a group $G$. Then $H$ is a subgroup of $G$ if and only if $ab^{-1} \in H \forall a, b \in H$.

**Proof:** Suppose $H$ is a subgroup of $G$. Let $a, b \in H$.

Then $b^{-1} \in H$ and $H$ is closed under the operation from $G$.

$\therefore ab^{-1} \in H$

Conversely suppose that $a, b \in H \Rightarrow ab^{-1} \in H$.

Since $H$ is not empty, let $a \in H$.

Then $a, a^{-1} \in H$

$\therefore aa^{-1} = e \in H$
If \( b \in H \) then \( e, b \in H \)

\[
\therefore eb^{-1} = b^{-1} \in H
\]

If \( b, c \in H \) then \( b, c^{-1} \in H \)

\[
\therefore bc = b\left(c^{-1}\right)^{-1} \in H
\]

Thus \( H \) is closed under the operation from \( G \). Thus \( H \) is a subgroup of \( G \).

**9.12.13 SAQ:** Let \( H \) and \( K \) be subgroups of a group \( G \). Prove that \( H \cup K \) is a subgroup of \( G \), if and only if \( H \subseteq K \) or \( K \subseteq H \).

**9.12.14 Example:** \( 2\mathbb{Z} \) and \( 5\mathbb{Z} \) are subgroups of \( (\mathbb{Z}, +) \) \( 2, 5 \in 2\mathbb{Z} \cup 5\mathbb{Z} \). But \( 2 + 5 = 7 \notin 2\mathbb{Z} \cup 5\mathbb{Z} \).

Thus the union of two subgroups need not be a subgroup.

**9.12.15 SAQ:** Let \( G \) be a finite group with identity \( e \). Let \( a \in G \) Show that there exists \( n \in \mathbb{Z}^+ \) such that \( a^n = e \).

**9.12.16 Result:** Let \( G \) be a group with identity \( e \) and \( S \) be a subset of \( G \).

Then \( H_S = \{ x \in G/\forall s \in S \} \) is a subgroup of \( G \).

**Proof:** \( H_S \neq \emptyset \) since \( xe = es = s \forall s \in S \implies x^{-1} \in H_S \)

\[
x, y \in H_S \implies s(xy) = (sx)y = (sy)x = x(ys) = (xy)s \forall s \in S
\]

\[
\implies xy \in H_S
\]

\[
x \in H_S \implies sx = xs \implies x^{-1}s = sx^{-1} \forall s \in S \implies x^{-1} \in H_S
\]

Thus \( H_S \) is a subgroup of \( G \).

**9.12.17 Definition:** Let \( G \) be a group. Then the set \( H_G = \{ x \in G/xg = gx \forall g \in G \} \) is called the centre of \( G \).

**9.12.18 SAQ:** Show that the centre \( H_G \) of a group \( G \) is an abelian subgroup of \( G \).

**9.12.19 Theorem:** Let \( G \) be a group and \( a \in G \). Then \( H = \{ a^n \mid n \in \mathbb{Z} \} \) is a subgroup of \( G \). If \( K \) is any subgroup of \( G \) such that \( a \in K \) then \( H \subseteq K \).
Proof: \(a^0 = e \Rightarrow e \in H\)

If \(a^n, a^m \in H\) then \(a^n (a^{-m})^{-1} = a^n a^{-m} = a^{n-m} \in H\) where \(n - m \in Z\). Thus \(H\) is a subgroup of \(G\).

If \(a \in K\) then \(a^n \in K\) for all \(n \in Z\). Thus \(H \subseteq K\).

9.12.20 Definition: If \(G\) is a group and \(a \in G\) then the subgroup \(\{a^n/n \in Z\}\) is called the cyclic subgroup of \(G\) generated by \(a\) and is denoted by \(\langle a \rangle\).

9.12.21 Definition: Let \(G\) be a group. \(G\) is said to be a cyclic group if \(G = \langle a \rangle\) for some \(a \in G\). \(a \in G\) is called a generator of \(G\) if \(G = \langle a \rangle\).

9.12.22 Example: In the group \((Z, +)\) the cyclic subgroup \(\langle n \rangle\) generated by \(n \in Z\) is \(\{nx/x \in Z\}\).

9.12.23 Note: Let \(G\) be a group. Let \(a \in G\). Then \(\langle a \rangle = \langle a^{-1} \rangle\) since

\[
\langle a \rangle = \{a^n/n \in Z\} = \{(a^{-1})^{-n} / n \in Z\}
\]

\[
= \{(a^{-1})^m / m = -n \in Z\}
\]

\[
= \langle a^{-1} \rangle
\]

9.12.24 Example: \(Z_4\) is a cyclic group with generators 1 and 3.

\[
\langle 3 \rangle = \{0,3,3+3,3+3+3\} = \{0,3,2,1\} = Z_4
\]

\[
\langle t \rangle = \{0,1,2,3\} = Z_4
\]

9.12.25 Example: The Klein 4-group \(V\) is not a cyclic group since \(\langle e \rangle = \{e\}\).

\[
\langle a \rangle = \{e,a\} \quad \text{since} \quad a^2 = a
\]

\[
\langle b \rangle = \{e,b\} \quad \text{since} \quad b^2 = b
\]

\[
\langle c \rangle = \{e,c\} \quad \text{since} \quad c^2 = c
\]

Thus no element of \(V\) generates \(V\).

9.12.26 SAQ: Prove that a cyclic group with only one generator can have at most 2 elements.

9.12.27 Example: \(\langle Z, + \rangle\) is a cyclic group. Both 1 and -1 are generators of \(\langle Z, + \rangle\).

9.12.28 Example: \(\langle 6Z, + \rangle\) under addition is a cyclic group. 6 and -6 are generators of \(\langle 6Z, + \rangle\).
9.12.29 Example: \( G = \{6^n / n \in \mathbb{Z}\} \) under multiplication is a cyclic group. 6 and \( \frac{1}{6} \) are generators of \( G \).

9.12.30 Example: The group \( \langle Q, + \rangle \) is not cyclic.

Proof: Suppose \( Q \) is cyclic. \( \exists q \in Q : Q = \langle q \rangle \). If \( q = 0 \) then \( \langle q \rangle \neq Q \).

Hence \( q \neq 0 \)

\[ 1 \in Q = \langle q \rangle \Rightarrow 1 = mq \text{ for some } 0 \neq m \in \mathbb{Z} \]

Now, \( \frac{1}{2m} \in Q = \langle q \rangle \Rightarrow \frac{1}{2m} = nq \text{ for some } 0 \neq n \in \mathbb{Z} \).

\[ \frac{1}{2m} = nq = \frac{n}{m} \Rightarrow n = \frac{1}{2} \text{ which is a contradiction since } \frac{1}{2} \notin \mathbb{Z}. \]

Thus \( \langle Q, + \rangle \) is not cyclic.

19.12.31 Example: The group \( \langle Q^+, \cdot \rangle \) is not cyclic.

Proof: Suppose \( Q^+ = \langle a \rangle \) where \( a \in Q^+ \) and \( a \neq 1 \)

\[ 2 \in Q^+ \Rightarrow 2 = a^n \text{ for some } n \in \mathbb{Z} \]

\[ \Rightarrow n = 1 \text{ and } 2 = a \]

\[ \Rightarrow Q^+ = \langle 2 \rangle \]

\[ \frac{1}{3} \in Q^+ \Rightarrow \frac{1}{3} = 2^K \text{ for some } K \in \mathbb{Z} \]

\[ \Rightarrow 1 = 3 \times 2^K \text{ which is a contradiction.} \]

Thus \( \langle Q^+, \cdot \rangle \) is not cyclic.

9.12.32 Example: Let \( G = \{ a + b\sqrt{2} / a, b \in \mathbb{Z}\} \). Then \( G \) is a group under the usual addition operation +. 0 is the identity and \(-a - b\sqrt{2}\) is the inverse of \( a + b\sqrt{2} \). \( \langle G, + \rangle \) is not cyclic.

Proof: Suppose that \( G \) is cyclic with generator \( a + b\sqrt{2} \).

\[ 1 \in G \Rightarrow 1 = n(a + b\sqrt{2}) = na + nb\sqrt{2} \text{ for some } 0 \neq n \in \mathbb{Z}. \]
\[ \Rightarrow na = 1 \text{ and } nb = 0 \]
\[ \Rightarrow na = 1 \text{ and } b = 0 \]
\[ \sqrt{2} \in G \Rightarrow \sqrt{2} = m(a + b\sqrt{2}) = ma + mb\sqrt{2} = ma \text{ for some } 0 \neq m \in Z. \]
But \( ma \in Z \) and \( \sqrt{2} \notin Z \)

Thus \( \langle G, + \rangle \) is not cyclic.

### 9.13 ANSWERS TO SAQs

#### 9.4.14 SAQ:
Let \( a, b \) and \( c \) be integers. Then \( (a, (b, c)) = ((a, b), c) \)

**Proof:** Let \( d_1 = (a, (b, c)) \) and \( d_2 = ((a, b), c) \). Then \( d_1 \mid a \) and \( d_1 \mid (b, c) \). By part 2 of theorem 9.4.4, \( d_1 \mid b \) and \( d_1 \mid c \). Thus \( d_1 \mid (a, b) \) and \( d_1 \mid c \). Hence \( d_1 \mid d_2 \)

Similarly one can prove that \( d_2 \mid d_1 \)
By part 4 of theorem (9.4.4) \( d_1 = d_2 \)

#### 9.4.16 SAQ:
Suppose \( m = ax + by \), where \( x \) and \( y \) are integers. Since \( g \mid a \) and \( g \mid b \), \( g \mid (ax + by) = m \), by part (3) of theorem 9.4.4

Conversely, suppose that \( m \) is a multiple of \( g \). Then \( m = gk \) for some integer \( k \). But \( g = ax_0 + by_0 \) for some integers \( x_0 \) and \( y_0 \). Thus \( m = (ax_0 + by_0)k = a(x_0k) + b(y_0k) \).

#### 9.4.26 SAQ:
In the Euclidian algorithm (theorem 9.4.25), consider the top two equations, prove that \( (b, c) = (r_1, r_2) \).

**Proof:** The top two equations are
\[ b = cq_1 + r_1, \quad 0 < r_1 < c \]
\[ c = r_1q_2 + r_2, \quad 0 < r_2 < r_1 \]

By theorem 9.4.22
\[ (b, c) = (b - cq_1, c) = (r_1, c) = (r_1, c - r_1q_2) = (r_1, r_2) \]

#### 9.4.30 SAQ:
Let \( n \) and \( n - 1 \) be two consecutive natural numbers. Clearly, \( n \mid n(n + 1) \) and \( (n + 1) \mid n(n + 1) \). Suppose \( n \mid x \) and \( (n + 1) \mid x \). Since \( n \mid x \) there exists an integer \( m \) such that \( x = nm \).

Since \( (n + 1) \mid nm \) and \( (n, n + 1) = 1 \), by theorem 9.4.24 \( (n + 1) \mid m \). Thus \( n(n + 1)/nm = x \).

Hence \( [n, n + 1] = n(n + 1) \)

#### 9.4.34 SAQ:
\([ma, mb]\) is a multiple of \( ma \). Since \( ma \) is a multiple of \( m \), it follows that \([ma, mb]\) is a multiple of \( m \). Hence, there exists an integer \( h \) such that \([ma, mb] = mh \). Denote \([a, b]\) by \( d \). Now, \( a \mid d, b \mid d, am \mid dm, bm \mid dm \) and so \( mh \mid dm \).
Thus \( h \mid d \).

Further \( \text{am} \mid \text{mn}, \text{bm} \mid \text{hh}, \text{a} \mid \text{h}, \text{b} \mid \text{h} \) and so \( d \mid h \). Therefore, \( h = d \). Thus

\[
[ma, mb] = mh = md = m[a, b]
\]

9.6.6 SAQ: \( ax \equiv ay \pmod{m} \Rightarrow m \mid (ax - ay) \Rightarrow m \mid a(x - y) \)

\[
\Rightarrow m \mid (x - y) \text{ since } (a, m) = 1
\]

\[
\Rightarrow x \equiv y \pmod{m}
\]

9.6.8 SAQ: If \( x \equiv y \pmod{m_i} \) for \( i = 1, 2, \ldots, r \) then \( m_i \mid (x - y) \) for \( i = 1, 2, \ldots, r \) that is \( x - y \) is a common multiple of \( m_1, m_2, \ldots, m_r \) and therefore \( [m_1, m_2, \ldots, m_r](x - y) \)

\[
\Rightarrow x \equiv y \pmod{[m_1, m_2, \ldots, m_r]}
\]

9.7.13 SAQ: (i) The congruence is \( 4x \equiv 5 \pmod{6} \)

Here \( a = 4, b = 5, m = 6 \)

\[
d = (a, m) = (4, 6) = 2 \text{ does not divide } b = 5.
\]

\( \therefore \) The congruence \( 4x \equiv 5 \pmod{6} \) does not have a solution.

(ii) The congruence is \( 3x \equiv 5 \pmod{7} \)

Here, \( a = 3, b = 5, m = 7 \)

\[
d = (3, 7) = 1
\]

\( \therefore \) The congruence \( 3x \equiv 5 \pmod{7} \) has only one solution.

Now, \( 3x \equiv 5 \pmod{7} \) \( \cdots \cdots \) (1)

\[
0 \equiv 7 \pmod{7} \cdots \cdots \) (2)

Adding (1) and (2), \( 3x \equiv 12 \pmod{7} \)

\[
\Rightarrow x \equiv 4 \pmod{7} \text{ } \left( \because (3, 7) = 1 \right)
\]

\( \therefore x = 4 \) is the solution of \( 3x \equiv 5 \pmod{7} \)

9.10.7 SAQ: \( (1, 0) \in G \times G \). But \( 1 \cdot 0 = 0 \) is not defined. Thus \( (1, 0) \) has no image in \( G \) under \( \cdot \).
\textbf{9.10.10 SAQ :} For any \( x \in \mathbb{R} \), \( f(x) \) and \( g(x) \) are real numbers. So \( f(x) + g(x) \) is a unique real number. Hence \( h \) is a real valued function on \( \mathbb{R} \).

Since \( f(x) + g(x) = g(x) + h(x) \) and

\[
(f(x) + g(x)) + k(x) = f(x) + (g(x) + k(x)) \quad \text{for any} \quad f, g, k \in G \quad \text{and} \quad x \in \mathbb{R}
\]

we get that + in \( G \) is commutative (i.e.) \( f + g = g + f \) and + in \( G \) is associative (i.e.) \( (f + g) + k = f + (g + k) \quad \forall f, g, k \in G \).

\textbf{9.10.11 SAQ :} (i) \( A - B, B - A \) are subsets of \( X \). So \((A - B) \cup (B - A)\) is also a subset of \( X \).

\[
\therefore A + B = (A - B) \cup (B - A) = (B - A) \cup (A - B) = B + A
\]

So, + is commutative.

By using the Demorgans laws and \( S - T = S \cap T' \) for any subsets \( S \) and \( T \) of \( X \) where \( T' = X - T \), we have

\[
(A + B) + C = (A \cap B' \cap C') \cup (B \cap A' \cap C') \cup (C \cap A' \cap B') \cup (A \cap B \cap C) = A + (B + C)
\]

Hence + is associative.

(ii) We know that \( A \cap B = B \cap A \) and

\[
A \cap (B \cap C) = (A \cap B) \cap C \quad \forall A, B, C \in G
\]

\[
\therefore A \triangle B = B \triangle A \quad \text{and} \quad (A \triangle B) \triangle C = A \triangle (B \triangle C)
\]

Thus \( \triangle \) is commutative and associative.

\textbf{9.11.15 SAQ :} We know that + is an associative binary operation in \( \mathbb{Z} \). \( 0 \in \mathbb{Z} \) and \( 0 + a = a \quad \forall a \in \mathbb{Z} \).

If \( a \in \mathbb{Z} \) then \(-a \in \mathbb{Z} \) and \( a + (-a) = a - a = 0 \). Thus \( \langle \mathbb{Z}, + \rangle \) is a group.

\textbf{9.11.23 SAQ :} Let \( e \) be the identity in \( G \).

Then \( ee = e \).

If \( xx = x \) then \( xx = xe \Rightarrow x = e \) by left cancellation law.

\( e \) is the unique idempotent of \( G \).

\textbf{9.12.13 SAQ :} If \( HK \subseteq \text{or} \ K \subseteq H \) then \( H \cup K = K \) or \( H \) which is a subgroup of \( G \).

Suppose \( H \cup K \) is a subgroup of \( G \).

Suppose \( H \nsubseteq K \) then \( \exists h \in H \ni h \notin K \).
Let \( k \in K \). Then \( k, h \in H \cup K \) which is a subgroup.

\[
\therefore k h \in H \cup K
\]

If \( k h \in K \) then \( k^{-1}(k h) = (k^{-1}k)h = eh = h \in K \), which is a contradiction.

\[
\therefore k h \in H. \text{ Then } k = k\left(h h^{-1}\right) = (kh)h^{-1} \in H.
\]

Thus \( K \subseteq H \).

9.12.15 SAQ: If \( a = e \) then \( a = e = e \). Let \( a \neq e \). If \( a^n \neq a^m \) for all \( n, m \in \mathbb{Z}^+ \) with \( n \neq m \) then \( H = \{a^n/n \in \mathbb{Z}^+\} \) would be an infinite subset of the finite set \( G \), which is a contradiction. Thus \( \exists n, m \in \mathbb{Z}^+ \) such that \( n \neq m \) and \( a^n = a^m \). Then \( n - m \) or \( m - n \in \mathbb{Z}^+ \) and \( a^{n-m} = e \) or \( a^{m-n} = e \).

Let \( k = \begin{cases} n - m \text{ if } n > m \\ m - n \text{ if } m > n \end{cases} \). Then \( k \in \mathbb{Z}^+ \) and \( a^k = e \).

9.12.18 SAQ: \( H_G = \{x \in G/gx = xg \forall g \in G\} \)

Let \( e \) be the identity in \( G \).

\[
g = eg = ge \quad \forall \ g \in G \Rightarrow e \in H_G \Rightarrow H_G \neq \emptyset
\]

\[
x \in H_G \Rightarrow gx = xg \Rightarrow gx^{-1} = x^{-1}g \quad \forall \ g \in G
\]

\[
\Rightarrow x^{-1} \in H_G
\]

\[
x, y \in H_G \Rightarrow g(xy) = (gx)y = (xg)y = x(gy)
\]

\[
= x(yg) = (xy)g \quad \forall \ g \in G
\]

\[
\Rightarrow xy \in H_G
\]

Thus \( H_G \) is a subgroup of \( G \).

\[
x, y \in H_G \Rightarrow x \in H_G, y \in G \Rightarrow xy = yx
\]

Thus \( H_G \) is an abelian group.

9.12.26 SAQ: Let \( G = \langle a \rangle \). Then \( \langle a^{-1} \rangle = \{a^{-1} \}^n/n \in \mathbb{Z} \}

\[
= \{a^{-n}/n \in \mathbb{Z}\} = \{a^m/m \in \mathbb{Z}\} = \langle a \rangle
\]
Thus a and $a^{-1}$ are generators. By hypothesis, we have $a = a^{-1}$. Then $a^2 = a^{-1}a = e$.

Hence $a^n = e$ for any $n > 1$

If $a = e$ then $G = \{e\}$

If $a \neq e$ then $G = \langle a \rangle = \{e, a\}$

9.14 EXERCISES

NUMBER THEORY

1. If $n$ is an even integer then show that $2^{2n} - 1$ is divisible by 15.

2. Show that every odd integer is of the form $4n + 1$ or $4n - 1$ where $n \in \mathbb{Z}$.

3. By using Euclidean algorithm find g.c.d. of

   (i) 26, 118  \hspace{1cm} (ii) 2210, 493  \hspace{1cm} (iii) 858, 728, 325  \hspace{1cm} (iv) 7469, 2464.

4. Find g.c.d. g of $a = -427, b = 616$ and then find integers $x$ and $y$ to satisfy $g = ax + by$.

5. Write each of the following numbers in canonical form.

   (i) 2560  \hspace{1cm} (ii) 4950  \hspace{1cm} (iii) 28812

6. If $ab \equiv 0 \pmod{p}$ and $p$ is prime, prove that $a \equiv 0 \pmod{p}$ or $b \equiv 0 \pmod{p}$.

7. Solve the following congruences.

   (i) $3x \equiv 4 \pmod{5}$  \hspace{1cm} (ii) $259x \equiv 5 \pmod{11}$  \hspace{1cm} (iii) $15x \equiv 12 \pmod{21}$

   (iv) $13x \equiv 9 \pmod{25}$  \hspace{1cm} (v) $8x \equiv 3 \pmod{27}$  \hspace{1cm} (vi) $3x \equiv 1 \pmod{125}$

   (vii) $11x \equiv 2 \pmod{317}$

8. Show that $10! - 32 \equiv 0 \pmod{11}$

9. Prove that $28! + 233 \equiv 0 \pmod{899}$

10. If $p$ is a prime, show that $2(p - 3)! + 1$ is a multiple of $p$.

11. If $a, b$ are prime to 1365, prove that $a^{12} - b^{12}$ is divisible by 1365.

12. Show that $n^7 - n$ is divisible by 42, for any integer $n$.  

Group Theory Exercises:

13. Let $G$ be the set of all real valued functions on $\mathbb{R}$. For $f, g \in G$ define $f \ast g = h$ by $h(x) = f(x)g(x)$ for all $x \in \mathbb{R}$. Prove that $\ast$ is an associative and commutative binary operation on $G$.

14. Let $G$ be the set of all real valued functions on $\mathbb{R}$. Define $f * g = h$, by $h(x) = (f * g)(x) = \frac{f(x)}{g(x)} \forall f, g \in G, x \in \mathbb{R}$.

Show that $*$ is not a binary operation on $G$.

[Hint: Take $f(x) = x + 1, g(x) = x - 1$ for all $x \in \mathbb{R}$. Then $f, g \in G$. But $f * g$ is not in $G$ since $(f * g)(1) = \frac{1+1}{1-1} = \frac{2}{0}$ is not defined.]

15. Verify whether the following operations on the given sets are binary operations, if so determine whether they are commutative and associative.

i) On $\mathbb{Z}^+$, define $a \cdot b = a - b$.

ii) On $\mathbb{Z}^+$, define $a \cdot b = a^b$

iii) On $\mathbb{Q}$, define $a \cdot b = a - b$

16. Verify whether the following operations are binary operations on the specified sets, if so whether they are commutative and associative.

(i) On $\mathbb{Z}$, define $a \cdot b = a - b$  (ii) On $\mathbb{Q}$, define $a \cdot b = ab + 1$

(iii) On $\mathbb{Q}$, define $a \cdot b = \frac{ab}{2}$  (iv) On $\mathbb{Z}^+$ define $a \cdot b = ab$  (v) On $\mathbb{Z}^+$, define $a \cdot b = a^b$

17. Show that $\mathbb{Z}^+$ with the operation of multiplication is not a group. [Hint: There is an identity, but any $x \neq 1$ has no inverse]

18. Show that $\mathbb{Q}^+$ with multiplication is an abelian group.

19. Let $G$ be a non-empty set. Let $\ast$ be an associative binary operation satisfying the following properties.

(i) $\exists e \in G \exists a \ast e = a \forall a \in G$

(ii) $a \in G$ implies $\exists b \in G$ such that $a \ast b = e$, show that $\langle G, \ast \rangle$ is a group.

20. Prove that the associativity of $\ast$ is necessary in the above problem.
Hint : \( G = \{ e, a, b \} \).

\[
\begin{array}{c|ccc}
* & e & a & b \\
\hline
e & e & a & b \\
a & a & e & e \\
b & b & e & e \\
\end{array}
\]

21. Prove that \( \langle \mathbb{Z}, \ast \rangle \) where \( \ast \) is not a group.

[Hint : \( \ast \) is not a group because \( \ast \) is not associative, commutative binary operation with 1 as the identity. But \( \ast \) and there is no \( \ast \) since \( \ast \).]

22. Prove that \( \langle G, \ast \rangle \) where \( \ast \) is not a group.

23. Let \( G \). Define \( \ast \) in \( G \) by \( \ast \). Show that \( \ast \) is an abelian group.

[Hint : \( \ast \) is the identity and \( \ast \).]

24. Let \( G \). Define \( \ast \) on \( G \) by \( \ast \). Compare this \( \ast \) with \( \ast \) of example 9.11.24.

25. If \( G \) is a group and \( \ast \), prove that \( \ast \) is a group.

[Hint : use \( \ast \) and induction principle].

26. Let \( n \) be an integer such that \( G \). Let \( G \) be an abelian group with identity \( e \). Show that \( \ast \) is a subgroup of \( G \).

27. Let \( G \) be a group. Let \( H \) be a finite subset of \( G \) which is closed under the induced operation on \( H \) from \( G \). Show that \( H \) is a subgroup of \( G \). [Hint : Theorem 9.11.11].

28. If \( G \) is a family of subgroups of a group \( G \), prove that \( \ast \) is also a subgroup of \( G \).

29. Let \( G \) be a subgroup of a group \( G \). For \( G \), define \( \sim \) if and only if \( \ast \). Show that \( \sim \) is an equivalence relation on \( G \).

30. Let \( H \) and \( K \) be subgroups of a group \( G \). Show that \( \ast \) if and only if \( \ast \).
9.15 ANSWERS TO EXERCISES IN NUMBER THEORY

9.14.3 : (i) 2   (ii) 7   (iii) 13   (iv) 77

9.14.4 :

9.14.5 : (i)   (ii)   (iii)   

9.14.7 : (i) i.e.   (ii) i.e.   

   i.e.   (iv) i.e.   (v) i.e.   (vi) i.e.   (vii) i.e.   

9.16 MODEL EXAMINATION QUESTIONS

Number theory :

1. Find the values of \( x \) and \( y \) that satisfy
   (i)   
   (ii)   

2. If   prove that   Also prove that   

3. Prove that the number of primes is infinite. 

4. Prove that   

5. If   prove that   

6. If   prove that   

7. If \( p \) is a prime, prove that   

Group Theory :

8. Define a binary operation. Give an example of a binary operation which is neither commutative nor associative.

9. Define a group. Prove that the identity element of a group \( G \) is unique and also prove that the inverse of any element of \( G \) is unique.
10. Define a group. If $G$ is a group, $a, b \in G$ prove that $(a^{-1})^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$.

11. Define a group. Let $G$ be a group and let $\circ$ be an associative binary operation in $G$ such that
   (i) $\exists e \in G$ such that $e \circ a = a$.
   (ii) $a \in G \Rightarrow \exists b \in G \ni b \circ a = e$

   Prove that $\langle G, \circ \rangle$ is a group.

12. Let $G$ be a non-empty set. Let $\circ$ be an associative binary operation on $G$ such that the equations $a \circ x = b$ and $y \circ a = b$ have solutions in $G$ for all $a, b \in G$. Show that $\langle G, \circ \rangle$ is a group.

13. Define $\ast$ on $\mathbb{Q}^+$ by $a \ast b = \frac{ab}{3}$. Prove that $\langle \mathbb{Q}^+, \ast \rangle$ is an abelian group.

14. Let $G$ be a finite group consisting of an even number of elements. Show that there is an $e \neq a \in G \ni a \circ a = e$, where $e$ is the identity element of $\langle G, \circ \rangle$.

15. If $G = \{e, a, b, c\}$ find all the binary operations $\sqcup$ on $G$ such that $\langle G, \sqcup \rangle$ is a group. Are these groups abelian.

16. Define a subgroup of a group. Determine all the subgroups of $\langle \mathbb{Z}_4, + \rangle$.

17. Define a subgroup of a group. Find all the subgroups of the Klein 4-subgroup.

18. Let $H$ be a non-empty subset of a group $G$. Prove that $H$ is a subgroup of $G$ if and only if $ab^{-1} \in H$ for all $a, b \in H$.

19. Prove that $\langle \mathbb{Q}^+, \cdot \rangle$ is not a cyclic group.

REFERENCES:

Lesson - 10

PERMUTATIONS; CYCLIC GROUPS

10.1 OBJECTIVE OF THE LESSON

In this lesson the student will be introduced to the concepts of permutations and their classification into even and odd permutations in the case of permutations of a finite set. The student will also be introduced to the idea of group generated by single element, called cyclic group and will learn to classify all cyclic groups.

10.2 STRUCTURE OF THE LESSON

This lesson has the following components.

10.3 Introduction
10.4 Permutations
10.5 Orbits and cycles
10.6 Even and odd permutations
10.7 Cyclic groups
10.8 Generators of finite cyclic groups
10.9 Answers to SAQs (Self-Assessment Questions)
10.10 Exercises
10.11 Model Examination Questions

10.3 INTRODUCTION

In this lesson the concepts of "permutation", and "cyclic group" have been introduced. The set $S_A$ of all permutations of a non-empty set $A$, under the binary operation of "composition of mappings", is proved to be a group. Two important examples $S_3$ and $D_4$ and the lattice diagrams for subgroups of these groups of permutations have been presented. It is also proved that the symmetric group $S_n$ on $n$ symbols is a non-abelian group if $n \geq 3$. The concepts of "orbits", "cycles" and "transpositions" are introduced for permutations. It is proved that any permutation of a finite set $A$ with at least two elements is a product of its disjoint cycles and also is a product of transpositions. A procedure for decomposing a permutation of a finite set $A$ into disjoint cycles and also into transpositions is presented. The procedure is illustrated with examples. It is also proved that no permutation of a finite set with at least two elements can be decomposed simultaneously into an even number of transpositions and also into an odd number of transpositions. The concept of a cyclic group is introduced and it is proved that an infinite cyclic group is structurally the same as the additive group of integers and any finite cyclic group of order $n$ is structurally the same as the group $(\mathbb{Z}_n, +_n)$ of the additive group of integers modulo $n$. The number of generators of a finite
cyclic group of order \( n \) is proved to be \( \phi(n) \) where \( \phi \) is the Euler function. A good number of examples and self assessment questions have been presented.

10.4 PERMUTATIONS

10.4.1 Definition: Let \( A \) and \( B \) be non-empty sets. Let \( f \) be a subset of \( A \times B \) such that

(i) for each \( a \in A \) there exists an element \( b \) in \( B \) such that \( (a,b) \in f \)

(ii) if \( a \in A, b, b' \in B \) such that \( (a,b) \in f \) and \( (a,b') \in f \) then \( b = b' \).

Then \( f \) is called a function (or mapping or a map) from \( A \) into \( B \).

If \( f \) is a function from \( A \) into \( B \), we write \( f : A \rightarrow B \).

10.4.2 Notation: If \( (a,b) \in f \) we write \( af = b \). The element \( b \) is called the image of \( a \) under the mapping \( f \). \( A \) is called the domain of \( f \) and \( B \) is called the codomain of \( f \). The set \( Af = \{af / a \in A\} \) is called the image of \( A \) under \( f \) (or the range of \( f \)).

10.4.3 Definition: Let \( \phi : A \rightarrow B, \psi : B \rightarrow C \) be functions. Then the function \( \phi \psi : A \rightarrow C \) defined by \( a(\phi \psi) = (a\phi)\psi \) is called the composite function of \( \phi \) and \( \psi \) (or the product of \( \phi \) and \( \psi \)).

10.4.4 Definition: Let \( \phi : A \rightarrow B \) be a function.

(i) \( \phi \) is called a one-to-one map (or injection) if \( a\phi = a'\phi \) implies \( a = a' \).

(ii) \( \phi \) is called an onto mapping (or surjective) if \( b \in B \) then there exists \( a \in A \) such that \( a\phi = b \).

(iii) \( \phi \) is called a bijection if it is one-to-one and onto.

10.4.5 Definition: A permutation of a set \( A \) is a bijection of \( A \) onto \( A \).

10.4.6 Theorem: Let \( A \) be a non-empty set, and let \( S_A \) be the set of all permutations of \( A \). Then \( S_A \) is a group under the product (composition) of permutations.

Proof: The function \( I \) defined on \( A \) by \( aI = a \) for all \( a \in A \) is clearly a permutation of \( A \).

So, \( S_A \neq \phi \)

For \( \sigma, \tau \in S_A, a(\sigma\tau) = a(\sigma)\tau \) for all \( a \in A \).

Now, we show that \( \sigma\tau \) is a permutation. \( \sigma, \tau \in S_A \Rightarrow \sigma, \tau \) are bijections from \( A \rightarrow A \).

For \( a, a' \in A, a(\sigma\tau) = a'(\sigma\tau) \Rightarrow (a\sigma)\tau = (a'\sigma)\tau \)
Permutations; Cyclic Groups

10.3

Differential Equation,
Abstract Algebra...

\[ \Rightarrow a = a' \quad (\because \sigma \text{ is one-to-one}) \]

Thus \( \tau \sigma \) is one-to-one.

If \( c \in \mathcal{A} \) then \( \exists b \in \mathcal{A} \) such that \( b \tau = c \) \( (\because \tau \text{ is onto}) \)

Since \( \sigma \) is onto \( \exists a \in \mathcal{A} \) such that \( a \sigma = b \).

Now, \( a (\sigma \tau) = (a \sigma) \tau = b \tau = c \)

Thus \( \sigma \tau \) is onto.

Hence \( \sigma \tau \in S_{\mathcal{A}} \). So \( S_{\mathcal{A}} \) is closed under the composition (multiplication) of permutations.

**Associative Property**: Let \( \sigma, \tau, \mu \in S_{\mathcal{A}} \) and \( a \in \mathcal{A} \).

\[
\begin{align*}
\sigma((\tau \mu))a &= (\sigma(\tau \mu))a = ((a \sigma) \tau)(\mu) = (a \sigma)(\tau \mu) \\
&= a(\tau \mu)
\end{align*}
\]

\[ \therefore (\sigma \tau) \mu = \sigma(\tau \mu) \]

**Identity**: \( a (1 \sigma) = (1 \sigma) a = a \sigma = (a \sigma) I = a (\sigma I) \) for all \( a \in \mathcal{A} \).

\[ \therefore 1 \sigma = \sigma I = \sigma \forall \sigma \in S_{\mathcal{A}} \]

Thus \( I \) is the identity element in \( S_{\mathcal{A}} \).

**Inverse**: Let \( \sigma \in S_{\mathcal{A}} \). If \( a \in \mathcal{A} \) then there exists a unique \( a' \in \mathcal{A} \) such that \( a' \sigma = a \). Define \( \sigma^{-1} : \mathcal{A} \rightarrow \mathcal{A} \) by \( a \sigma^{-1} = a' \). Then since \( \sigma \) is a bijection we get that \( \sigma^{-1} \) is also bijection.

We observe that \( (a \sigma^{-1}) \sigma = a' \sigma = a = a I \ \forall \ a \in \mathcal{A} \) and \( a' (\sigma \sigma^{-1}) = (a' \sigma) \sigma^{-1} = a \sigma^{-1} = a' = a' I, \ \forall a' \in \mathcal{A} \).

\[ \therefore \sigma \sigma^{-1} = \sigma^{-1} \sigma = I \ \forall \ \sigma \in S_{\mathcal{A}} \]

\[ \therefore \sigma^{-1} \text{ is the inverse of } \sigma \text{ in } S_{\mathcal{A}} \]

Thus \( S_{\mathcal{A}} \) is a group under the composition of mappings.

10.4.7 **Definition**: If \( \mathcal{A} \) is the finite set \( \{1, 2, \ldots, n\} \), then the group of all permutations of \( \mathcal{A} \) is called the symmetric group on \( n \) letters (or \( n \) symbols), and is denoted by \( S_n \).
10.4.7(a) Theorem: The symmetric group $S_n$ has $n!$ elements.

Proof: Let $f \in S_n$. How many choices does $f$ have to send 1? Clearly $n$. For we can send 1 under $f$ to any element of \{1, 2, ..., n\}. But now $f$ is not free to send 2 any where, for since $f$ is not free to send 2 anywhere, for since $f$ is one-to-one we must have $f(1) \neq f(2)$. So, we can send 2 any where except onto $f(1)$. Hence $f$ can send 2 into $n-1$ different images. Continuing this way, we see that $f$ can send i into $n-(i-1)$ different images. Hence the number of such functions $f$ in $S_n$ is $n(n-1)(n-2)\cdots 1 = n!$.

10.4.8 Notation: If $\sigma \in S_n$, then $\sigma$ is represented as

$$\begin{pmatrix}
1 & 2 & \cdots & i & \cdots & n \\
\sigma(1) & \sigma(2) & \cdots & \sigma(i) & \cdots & \sigma(n)
\end{pmatrix}$$

10.4.9 Example: $A = \{1, 2, 3, 4, 5, 6, 7\}$.

If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 4 & 3 & 1 & 6 & 7 & 5
\end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
4 & 3 & 1 & 2 & 6 & 7 & 5
\end{pmatrix}$

then $\sigma \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
3 & 2 & 1 & 4 & 7 & 5 & 6
\end{pmatrix}$

Observe that while computing $\sigma \tau$ we first apply the permutation $\sigma$ followed by $\tau$. We now present two important examples.

10.4.10 Example: Let $A = \{1, 2, 3\}$. $S_3$ consists of the following six elements.

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\
1 & 2 & 3
\end{pmatrix}, \rho_1 = \begin{pmatrix} 1 & 2 & 3 \\
2 & 3 & 1
\end{pmatrix}, \rho_2 = \begin{pmatrix} 1 & 2 & 3 \\
3 & 1 & 2
\end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\
1 & 3 & 2
\end{pmatrix}, \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\
3 & 2 & 1
\end{pmatrix}, \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\
2 & 1 & 3
\end{pmatrix}$$
S₃ is the group of symmetries D₃ of an equilateral triangle where ρ₀, ρ₁, ρ₂ are rotations and μ₁, μ₂, μ₃ are reflexions about the angle bisectors.

The multiplication table for S₃ is presented below.

<table>
<thead>
<tr>
<th></th>
<th>ρ₀</th>
<th>ρ₁</th>
<th>ρ₂</th>
<th>μ₁</th>
<th>μ₂</th>
<th>μ₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>ρ₀</td>
<td>ρ₀</td>
<td>ρ₁</td>
<td>ρ₂</td>
<td>μ₁</td>
<td>μ₂</td>
<td>μ₃</td>
</tr>
<tr>
<td>ρ₁</td>
<td>ρ₁</td>
<td>ρ₂</td>
<td>ρ₀</td>
<td>μ₂</td>
<td>μ₃</td>
<td>μ₁</td>
</tr>
<tr>
<td>ρ₂</td>
<td>ρ₂</td>
<td>ρ₀</td>
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<td>μ₁</td>
<td>μ₂</td>
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<td>ρ₀</td>
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<td>μ₂</td>
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<td>ρ₂</td>
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<tr>
<td>μ₃</td>
<td>μ₃</td>
<td>μ₁</td>
<td>μ₂</td>
<td>ρ₂</td>
<td>ρ₁</td>
<td>ρ₀</td>
</tr>
</tbody>
</table>

The table is not symmetric about the main diagonal (μ₂μ₃ = ρ₂ ≠ ρ₁ = μ₃μ₂)

Thus S₃ is not an abelian group.

{ρ₀}, {ρ₀, μ₁}, {ρ₀, μ₂}, {ρ₀, μ₃}, {ρ₀, ρ₁, ρ₂} and S₃ are the subgroups of S₃. The inclusions among these subgroups is presented in the following (lattice) diagram.

10.4.11 Example (The Dihedral group D₄) : If the vertices of a square are numbered 1, 2, 3, 4 then the group of symmetries of the square is called the Dihedral group D₄ (also called the octic group). We use ρᵢ for rotations, μᵢ for mirror images in perpendicular bisectors of sides and δ for diagonal flips. The elements and the multiplication table for D₄ are given below:

ρ₀ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \mu₁ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \\
ρ₁ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad \mu₂ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}
\[ \rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \]

\[ \rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \]

<table>
<thead>
<tr>
<th></th>
<th>\rho_0</th>
<th>\rho_1</th>
<th>\rho_2</th>
<th>\rho_3</th>
<th>\mu_1</th>
<th>\mu_2</th>
<th>\delta_1</th>
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<tr>
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<td>\rho_1</td>
<td>\rho_3</td>
<td>\rho_2</td>
<td>\rho_0</td>
</tr>
</tbody>
</table>

Notice that the table is not symmetric about the main diagonal. So \( D_4 \) is not an abelian group.

The subgroups of \( D_4 \) are \{\rho_0\}, \{\rho_0, \mu_1\}, \{\rho_0, \mu_2\}, \{\rho_0, \rho_2\}, \{\rho_0, \delta_1\}, \{\rho_0, \delta_2\}, \{\rho_0, \rho_2, \mu_1, \mu_2\}, \{\rho_0, \rho_1, \rho_2, \rho_3\}.

The relations among these subgroups are presented in the following diagram.
10.4.12 Example: In $S_3$,

(i) $\rho_1^2 = \rho_2, \rho_1^3 = \rho_0$.

Thus $\langle \rho_1 \rangle = \{\rho_0, \rho_1, \rho_2\}$

(ii) $\rho_2^2 = \rho_1, \rho_2^3 = \rho_0$. Thus $\langle \rho_2 \rangle = \{\rho_0, \rho_1, \rho_2\}$

(iii) $\mu_1^2 = \rho_0$. Thus $\langle \mu_1 \rangle = \{\rho_0, \mu_1\}$

(iv) $\mu_2^2 = \rho_0$. Thus $\langle \mu_2 \rangle = \{\rho_0, \mu_2\}$

(v) $\mu_3^2 = \rho_0$ Thus $\langle \mu_3 \rangle = \{\rho_0, \mu_3\}$

10.4.13 SAQ: Write the multiplication table for the cyclic subgroup of $S_5$ generated by

$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$.

We now present an example of a non-abelian group in which every proper subgroup is abelian.

10.4.14 Example: We know that $S_3$ is a non-abelian group. $\{\rho_0\}, \{\rho_0, \rho_1, \rho_2\}, \{\rho_0, \mu_1\}, \{\rho_0, \mu_2\}, \{\rho_0, \mu_3\}$ are the only proper subgroups of $S_3$ and each of these subgroups is abelian.

10.4.15 Example: $S_n$ is a non-abelian group for $n \geq 3$.

Proof: Since $n \geq 3$, the permutations

\[ f = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n-1 & n \\ 2 & 3 & 4 & 5 & \ldots & n-1 & 1 \end{pmatrix} \]

and

\[ g = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n-1 & n \\ 2 & 1 & 3 & 4 & \ldots & n-1 & n \end{pmatrix} \]

are in $S_n$.

\[ fg = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n-1 & n \\ 1 & 3 & 4 & 5 & \ldots & n & 2 \end{pmatrix} \]

\[ gf = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n-1 & n \\ 3 & 2 & 4 & 5 & \ldots & n & 1 \end{pmatrix} \]

Thus $fg \neq gf$.

Hence $S_n$ is not abelian for $n \geq 3$. 

10.5 ORBITS AND CYCLES

10.5.1 Definition : Let $A$ be a non-empty set. Let $\sigma \in S_A$. Let $a \in A$. Then $\theta_{\sigma}(a) = \{a\sigma^n / n \in \mathbb{Z}\}$ is called the orbit of $a$ under $\sigma$.

10.5.2 Example : Let $
sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 3 & 1 & 6 & 8 & 5 & 7 \end{pmatrix} \in S_8$.

Then $\theta_{\sigma}(1) = \{1, 2, 4\}, \theta_{\sigma}(2) = \{2, 4, 1\}$

$\theta_{\sigma}(3) = \{3\}, \theta_{\sigma}(4) = \{4, 1, 2\}, \theta_{\sigma}(5) = \{5, 6, 8, 7\}$

$\theta_{\sigma}(6) = \{6, 8, 7, 5\}, \theta_{\sigma}(7) = \{7, 5, 6, 8\}$

$\theta_{\sigma}(8) = \{8, 7, 5, 6\}$

Note that $\theta_{\sigma}(1) = \theta_{\sigma}(2) = \theta_{\sigma}(4), \theta_{\sigma}(5) = \theta_{\sigma}(6) = \theta_{\sigma}(7) = \theta_{\sigma}(8)$.

10.5.3 Lemma : Let $A$ be a non-empty set. Let $\sigma \in S_A$. Then any two orbits of $\sigma$ are either disjoint or identical.

Proof : Let $c \in \theta_{\sigma}(a) \cap \theta_{\sigma}(b)$. Then

$c = a\sigma^n$ for some $n \in \mathbb{Z}$ and $c = b\sigma^m$ for some $m \in \mathbb{Z}$.

$x \in \theta_{\sigma}(a) \Rightarrow x = a\sigma^{n_1}$ for some $n_1 \in \mathbb{Z}$.

$x = c\sigma^{-n_1} = c\sigma^{-n_1}$

$x = b\sigma^m \sigma^{-n_1} = b\sigma^{m-n_1} \in \mathbb{Z}$

Hence $x \in \theta_{\sigma}(b)$

Thus $\theta_{\sigma}(a) \subseteq \theta_{\sigma}(b)$

Similarly $\theta_{\sigma}(b) \subseteq \theta_{\sigma}(a)$

$\therefore \theta_{\sigma}(a) = \theta_{\sigma}(b)$

10.5.4 Definition : Let $A$ be a non-empty set. Let $\sigma \in S_A$. If there exists a finite subset $B = \{a_1, a_2, \ldots, a_n\}$ of $A$ such that $a_1 \sigma = a_2$, $a_2 \sigma = a_3$, $\ldots$, $a_n \sigma = a_1$ and $x \sigma = x$ for all $x \in A \setminus B$ then $\sigma$ is called a cycle of length $n$. We denote $\sigma$ by $\sigma = (a_1, a_2, \ldots, a_n)$.
10.5.5 Definition: Let $A$ be a non-empty set. Let $B_1 = \{a_1, \ldots, a_n\}$ and $B_2 = \{b_1, \ldots, b_m\}$ be subsets of $A$. If $B_1 \cap B_2 = \emptyset$ then we say that the cycles $\sigma_1 = (a_1, a_2, \ldots, a_n)$ and $\sigma_2 = (b_1, b_2, \ldots, b_m)$ are disjoint.

10.5.6 Lemma: Any two disjoint cycles commute.

Proof: Let $B_1 = \{a_1, a_2, \ldots, a_n\}$, $B_2 = \{b_1, b_2, \ldots, b_m\}$ be subsets of $A$ such that $B_1 \cap B_2 = \emptyset$. Let $\sigma_1 = (a_1, a_2, \ldots, a_n)$, $\sigma_2 = (b_1, b_2, \ldots, b_m)$

$$a_i \sigma_1 \sigma_2 = (a_i \sigma_1) \sigma_2 = \begin{cases} a_{i+1} & \text{if } i < n \\ a_1 & \text{if } i = n \end{cases} \quad \forall 1 \leq i \leq n$$

Thus $a_i \sigma_2 \sigma_1 = a_i \sigma_1 \sigma_2 \text{ for } 1 \leq i \leq n$.

$$b_j \sigma_1 \sigma_2 = b_j \sigma_2 = \begin{cases} b_{j+1} & \text{for } j < m \\ b_j & \text{for } j = m \end{cases}$$

Thus $b_j \sigma_1 \sigma_2 = b_j \sigma_2 \sigma_1 \text{ for } 1 \leq j \leq m$.

$$x \sigma_1 \sigma_2 = x \sigma_2 = x = x \sigma_2 \sigma_1 \forall x \in A \setminus (B_1 \cup B_2)$$

Thus $\sigma_1 \sigma_2 = \sigma_2 \sigma_1$.

10.5.7 Note: If $\sigma = (a_1, a_2, \ldots, a_n)$ then

$$\sigma = (a_1, a_{i+1}, \ldots, a_n, a_i, a_2, \ldots, a_{i-1}) \text{ for } 1 \leq i \leq n.$$
10.5.9 Lemma: Let $A$ be a finite set. Let $\sigma \in S_A$. Let $x \in A$. Then there exists a non-negative integer $\ell$ such that

$$0_{\sigma}(x) = \{x, x\sigma^0, x\sigma, x\sigma^2, \ldots, x\sigma^{\ell-1}\}$$

Proof: If $x\sigma = x$ then $x = x\sigma^{-1}$ and $x\sigma^n = x \forall n \in Z$

Hence $0_{\sigma}(x) = \{x\}$

Suppose $x \neq x\sigma$

Let $x = \{x, x\sigma, x\sigma^2, \ldots, x\sigma^k, \ldots\}$

Then $X \subseteq A$, $A$ is finite and hence $X$ is finite.

Thus $\exists$ distinct positive integers $n$ and $m$ such that $x\sigma^n = x\sigma^m$.

We may assume that $m > n$.

Let $\ell = m - n$

Then $x\sigma^n = x\sigma^m \Rightarrow x\sigma^{n-m} = x \Rightarrow x\sigma^\ell = x$

Also $x\sigma^{-\ell} = x$

We may assume that $\ell$ is the least positive integer such that $x\sigma^\ell = x$. Clearly $x\sigma^{t\ell} = x$, for all $t \in Z$.

Then $X = \{x, x\sigma, x\sigma^n, \ldots, x\sigma^{\ell-1}\}$

For any $k \in Z, \exists t, r \in Z$ such that $k = t\ell + r, 0 \leq r < \ell$

Then $x\sigma^k = x\sigma^{t\ell + r} = x\sigma^{t\ell}\sigma^r = x\sigma^r \in \{x, x\sigma, \ldots, x\sigma^{\ell-1}\}$

Thus $0_{\sigma}(x) = \{x, x\sigma, x\sigma^2, \ldots, x\sigma^{\ell-1}\}$. 
10.11 Permutations; Cyclic Groups

**10.5.10 Theorem**: Every permutation $\sigma$ of a finite set $A$ is a product of a finite number of disjoint cycles.

**Proof**: Let $\theta_{\sigma}(x_1), \theta_{\sigma}(x_2), \ldots, \theta_{\sigma}(x_\ell)$ be the distinct orbits of $\sigma$ under $\theta$.

Then $A = \bigcup_{i=1}^{\ell} \theta_{\sigma}(x_i)$ and $\theta_{\sigma}(x_i) \cap \theta_{\sigma}(x_j) = \phi$ if $i \neq j$. Let $\theta_{\sigma}(x_i) = \{x_i, x_i\sigma, \ldots, x_i\sigma^{\ell_i}\}$ for $1 \leq i \leq \ell$.

Define the cycles $\sigma_1, \sigma_2, \ldots, \sigma_\ell$ by $\sigma_i = (x_i, x_i\sigma, x_i\sigma^2, \ldots, x_i\sigma^{\ell_i})$ for $1 \leq i \leq \ell$.

We claim that $\sigma = \sigma_1 \sigma_2 \ldots \sigma_\ell$.

Let $a \in A$. Then there exists unique $x_i \ni a \in \theta_{\sigma}(x_i)$.

Now $a = x_i\sigma^k$ for some $0 \leq k \leq \ell_i$.

$$a\sigma_i = \begin{cases} x_i\sigma^{k+1} & \text{if } k < \ell_i \\ x_i & \text{if } k = \ell_i \end{cases}$$

and $a\sigma_j = a$ since $a \not\in \theta_{\sigma}(x_j)$

$$a\sigma_1 \sigma_2 \ldots \sigma_{i-1}\sigma_i\sigma_{i+1} \ldots \sigma_\ell = a\sigma_i \sigma \sigma_2 \ldots \sigma_{i-1}\sigma_i\sigma_{i+1} \ldots \sigma_\ell = a\sigma_i$$

$$= \begin{cases} x_i\sigma^{k+1} & \text{for } k < \ell_i \\ x_i & \text{for } k = \ell_i \end{cases}$$

$$a\sigma = x_i\sigma^k \sigma = \begin{cases} x_i\sigma^{k+1} & \text{for } k < \ell_i \\ x_i & \text{for } k = \ell_i \end{cases}$$

Hence $a\sigma = a$.

Thus $\sigma = \sigma_1 \sigma_2 \ldots \sigma_\ell$.

The cycles $\sigma_1, \sigma_2, \ldots, \sigma_\ell$ are disjoint since

$$\theta_{\sigma}(x_i) \cap \theta_{\sigma}(x_j) = \phi \text{ for } i \neq j.$$}

Since any cycle of length one is the identity permutation, the above theorem can be restated as follows.
10.5.10(a) Theorem: Every non-identity permutation $\sigma$ of a finite set $A$ (with at least two elements) is a product of a finite number of disjoint cycles, each of which has length at least 2.

10.5.11 Definition: A cycle of length 2 is called a transposition.

10.5.12 Example: $\sigma = (3,4)$ in $S_5$ is a transposition.

10.5.13 Theorem: Any permutation of a finite set $A$ with at least two elements is a product of transpositions.

Proof: If $I$ is the identity permutation, then $I = (x_1, x_2) (x_2, x_1)$. Since any non-identity permutation of a finite set is the product of its disjoint cycles of length at least 2, it is enough to prove that every cycle of length at least two is a product of transpositions.

Let $\sigma = (a_1, a_2, \ldots, a_n)$ be a cycle of length $n \geq 2$ in $S_A$. Let $x \in A$.

Suppose $x = a_n$

Then $x\sigma = a_1$

$x(a_1, a_2)(a_1, a_3)\cdots(a_1, a_n) = x(a_1, a_n) = a_1$ \quad (1)

Suppose $x = a_i, i < n$

Then $x\sigma = a_i\sigma = a_{i+1}$

$x(a_1, a_2)\cdots(a_1, a_i)(a_1, a_{i+1})\cdots(a_1, a_n)$

$= a_i(a_1, a_2)\cdots(a_1, a_i)(a_1, a_{i+1})\cdots(a_1, a_n)$

$= a_i(a_1, a_{i+1})\cdots(a_1, a_n)$

$= a_{i+1}(a_1, a_{i+2})\cdots(a_1, a_n) = a_{i+1}$ \quad (2)

If $x \not\in \{a_1, \ldots, a_n\}$ then $x\sigma = x$ and

$x(a_1, a_2)(a_1, a_3)\cdots(a_1, a_n) = x$ \quad (3)

From (1), (2) and (3) we have $\sigma = (a_1, a_2)(a_1, a_3)\cdots(a_1, a_n)$.

10.5.14 Procedure: We now suggest a procedure to decompose a given permutation of a finite set $A$ into a product of transpositions.

Let $\sigma \in S_A$ where $A$ has at least two elements. Choose an arbitrary element $a_1 \in A$.

Let $\theta_\sigma(a_1) = \{a_1, a_1\sigma, a_1\sigma^2, \ldots, a_1\sigma^l\}$. 

Choose an arbitrary element \( a_2 \in A \setminus \theta_\sigma(a_1) \)

Let \( \theta_\sigma(a_2) = \{a_2, a_2\sigma, a_2\sigma^2, \ldots, a_2\sigma^k\} \)

Continuing this process finally we choose

\[
d_k = A \setminus \bigcup_{i=1}^{k-1} \theta_\sigma(a_i) \text{ such that } \theta_\sigma(a_k) = \{a_k, a_k\sigma, \ldots, a_k\sigma^k\} \text{ and } A = \bigcup_{i=1}^k \theta_\sigma(a_i).
\]

Write \( \sigma_i = (a_i, a_i\sigma, \ldots, a_i\sigma^l_i) \) for \( 1 \leq i \leq k \).

Then \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_k \).

For each \( i \), substitute \( \sigma_i = (a_i, a_i\sigma) \cdots (a_i, a_i\sigma^{l_i}) \) to write \( \sigma \) as a product of transpositions. Since a cycle of length 1 is the identity permutation we may omit such cycles.

**Example:** Let \( A = \{1,2,3,4,5,6,7,8\} \)

Let \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} \)

\( \theta_\sigma(1) = \{1,8\}, \theta_\sigma(2) = \{2\}, \theta_\sigma(3) = \{3,6,4\} \quad \theta_\sigma(5) = \{5,7\} \).

Let \( \sigma_1 = (1,8), \quad \sigma_2 = (3,6,4), \quad \sigma_3 = (5,7) \)

\( \sigma_2 = (3,6)(3,4) \)

\( \sigma = \sigma_1 \sigma_2 \sigma_3 = (1,8)(3,6)(3,4)(5,7) \)

**Example:** Let \( A = \{1,2,3,4,5,6,7,8\} \)

\( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix} \)

\( \theta_\sigma(1) = \{1,3,4,7,8,6,5,2\} \)

\( \sigma \) is a cycle.

\( \sigma = (1, 3, 4, 7, 8, 6, 5, 2) \)

\( = (1, 3)(1, 4)(1, 7)(1, 8)(1, 6)(1, 5)(1, 2) \)
10.5.17 Definition: An element $a$ of a group $G$ with identity $e$ is said to be of order $r > 0$ if $a^r = e$ and $a^s \neq e$ for any $0 < s < r$.

Note that the order of $e$ is 1.

10.5.18 Lemma: If $\sigma$ is a cycle of length $n$ in $S_A$, where $A$ is a finite set, then the order of $\sigma$ is $n$.

Proof: Let $\sigma = (a_1, a_2, \ldots, a_n)$. Let $x \in A$.

If $x \neq a_i$ for all $i$ then $x\sigma = x \Rightarrow x\sigma^n = x$. Suppose $x = a_n$. Then

$$a_n\sigma^n = a_1\sigma^{n-1} = a_2\sigma^{n-2} = \ldots = a_n\sigma^{n-n} = a_n \quad \text{(1)}$$

Suppose $x = a_i, i \neq n$. Then

$$a_i\sigma^n = a_1\sigma^{n-i} = a_n\sigma^i = a_1\sigma^{i-1} = \ldots = a_i\sigma^0 = a_i \quad \text{(2)}$$

From (1) and (2) we have $\sigma^n = I$, the identity permutation of $A$. Suppose $k$ is a positive integer $\exists k < n$.

$$a_1\sigma^k = a_2\sigma^{k-1} = \ldots = a_{1+k}$$

$$a_{1+k} \in \{a_2, a_3, \ldots, a_n\}.$$  

Thus $a_1\sigma^k \neq a_i$.

Hence $\sigma^k$ is not the identity permutation of $A$.

Thus the order of $\sigma$ is $n$.

10.5.19 Theorem: The order of the product of two disjoint cycles is the least common multiple of the orders of the individual cycles.

Proof: Let $\sigma = \tau \mu$, where $\tau$ and $\mu$ are disjoint cycles. Since disjoint cycles commute, $\sigma^m = \tau^m \mu^m$ for all $m \in \mathbb{Z}$. Also $\sigma^m = I$ if and only if $\tau^m = I$ and $\mu^m = I$, where $I$ is the identity permutation. For, if $\sigma^m = I$, then $\tau^m \mu^m = I$ and so $\tau^m = \mu^{-m}$. Suppose that $\tau^m \neq I$. Then $a \cdot \tau^m \neq a$ for some $a$ and this implies that $a \mu^{-m} \neq a$ and $a \neq a \mu^{-m}$. Hence $at \neq a$ and $a \mu \neq a$. This is a contradiction, since $\tau$ and $\mu$ are disjoint and both cannot move the same element. Therefore $\sigma^m = I$ if and only if the order of $\tau$ divides $m$ and the order of $\mu$ divides $m$. Since the order of $\sigma$ is the least such positive integer, the conclusion follows.

10.5.19(a) Corollary: The order of a permutation of a finite set $A$ is the least common multiple of the orders of its disjoint cycles.

Proof: Let $\sigma$ be a permutation of $A$ and let $\sigma = \sigma_1 \ldots \sigma_r$, where $\sigma_1, \sigma_2, \ldots, \sigma_r$ are disjoint cycles.

By repeating the proof of the above theorem, we obtain the result.
10.5.20 Example: To find the order of \( \sigma = (1\ 2\ 3) (5\ 6) \) in \( S_6 \).

order of \( (1\ 2\ 3) \) is 3, order of \( (5\ 6) \) is 2. Thus the order of \( \sigma \) is l.c.m. of \( \{2, 3\} = 6 \).

10.5.21 SAQ: Find the order of \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 6 & 7 & 8 & 5 \end{pmatrix} \)

10.5.22 Example: Let \( \sigma = (1\ 3\ 5\ 7\ 6) \) in \( S_8 \).

Then \( \sigma^{-1} = (1\ 6\ 7\ 5\ 3) \)

10.5.23 Example: If \( \sigma = (a_1, a_2, \ldots, a_n) \) is a cycle in \( S_A \) then \( \sigma^{-1} = (a_1, a_n, a_{n-1}, \ldots, a_2) \) is also a cycle.

Thus the order of \( \sigma = \) order of \( \sigma^{-1} \) for any cycle \( \sigma \).

10.5.24 SAQ: Prove that the inverse of a transposition is itself.

10.6 EVEN AND ODD PERMUTATIONS

10.6.1 Theorem: Any permutation \( \sigma \in S_n \) can be expressed as a product of an even number of transpositions or as a product of an odd number of transpositions, but not both.

Proof: Let \( I \) be the identity permutation. Then \( I = (1\ 2) (2\ 1) \) is a product of an even number of transpositions.

Let \( I = \tau_1 \tau_2 \tau_3 \cdots \tau_k \) where each \( \tau_i \) is a transposition.

Choose any integer \( m \) that appears in one of the transpositions in (1) and let \( \tau_j \) be the first transposition counting from left to right in which \( m \) occurs.

If \( j = k \) then \( m \) does not occur in \( \tau_1, \cdots, \tau_{k-1} \) and \( \tau_k \) moves \( m \).

\[ m = mI = m(\tau_i \cdots \tau_k) = m\tau_k. \]

But \( m\tau_k \neq m \).

Thus \( j \neq k \)

Now \( \tau_j \tau_{j+1} \) must have the form on the left hand side of one of the following identities.
\[(m, x) (m, x) = 1\]
\[(m, x) (m, y) = (x, y) (m, x)\]
\[(m, x) (y, z) = (y, z) (m, x)\]
\[(m, x) (x, y) = (x, y) (m, x)\] \quad (2)

If we substitute the correct identity in (2) for \(\tau_j \tau_{j+1}\) in (1), we either reduce the number \(k\) of transpositions in (1) by 2 or shift the first occurrence of \(m\) one step to the right. We repeat this process until \(m\) is eliminated from the expression of (1). Since \(m\) cannot appear for the first time in the final transposition, eventually the situation in the first identity of (2) must occur to eliminate \(m\) completely. Then we choose another integer in \(A\) appearing in our reduced equation (1) and eliminate it from (1). At each step of this process we are replacing a pair of transpositions in (1) by \(I\). We continue this process until R.H.S. of (1) is reduced to a sequence \(I \ I \ I \cdots I\). Since the number \(k\) was reduced by 2 or kept unchanged at each substitution of an identity from (2) we must have \(k\) even.

Let \(\sigma \in S_A\)

\[\sigma = \tau_1 \tau_2 \cdots \tau_r = \tau'_1 \tau'_2 \cdots \tau'_s\] where \(\tau_i, \tau'_j\) are transpositions.

Then \(I = \sigma \sigma^{-1} = \tau'_1 \tau'_2 \cdots \tau'_s \tau_r^{-1} \tau'_r \cdots \tau_1^{-1}\)

\[= \tau'_1 \tau'_2 \cdots \tau'_s \tau_r \tau_{r-1} \cdots \tau_1\]

Thus \(r + s\) is even.

Hence either \(r\) and \(s\) are both even or \(r\) and \(s\) are both odd.

10.6.2 Definition: Let \(\sigma \in S_n\). \(\sigma\) is called an even permutation if it is a product of an even number of transpositions. \(\sigma\) is called an odd permutation if it is a product of an odd number of transpositions.

10.6.3 Lemma: Let \(A_n\) be the set of all even permutations in \(S_n\) and let \(B_n\) be the set of all odd permutations in \(S_n\). Then \(A_n\) and \(B_n\) have \(\frac{n!}{2}\) elements each.

Proof: We know that \(S_n\) has \(n!\) elements.

Let \(\tau = (1, 2)\)

Define \(f : A_n \to B_n\) by \(\sigma f = \tau \sigma\). Since \(\sigma\) is even, we have that \(\tau \sigma\) is odd.

If \(\sigma_1 f = \sigma_2 f\) for \(\sigma_1, \sigma_2 \in A_n\) then \(\tau \sigma_1 = \tau \sigma_2\).
By cancellation laws in $S_n$, we have $\sigma_1 = \sigma_2$

Thus $f$ is one - one.

If $\rho \in B_n$ then $\sigma = \tau^{-1} \rho \in A_n$ and $\sigma f = \tau \sigma = \tau \tau^{-1} \rho = I \rho = \rho$

Thus $f$ is onto $B_n$.

Hence $f$ is a bijection.

$\therefore |A_n| = |B_n|$

But $|A_n| + |B_n| = n!$

$\therefore |A_n| = \frac{n!}{2}$

10.6.4 Theorem: Let $A_n$ be the set of all even permutations in $S_n$, $n \geq 2$. Then $A_n$ is a subgroup of $S_n$.

Proof: $I = (1 \ 2) (2 \ 1) \Rightarrow I \in A_n$.

If $\sigma_1, \sigma_2 \in A_n$ then $\sigma_1 = \tau_1 \tau_2 \cdots \tau_k$, $\sigma_2 = \tau'_1 \tau'_2 \cdots \tau'_\ell$ where $k$ and $\ell$ are even.

Now $\sigma_1 \sigma_2 = \tau_1 \tau_2 \cdots \tau_k \tau'_1 \cdots \tau'_\ell$ and $\ell + k$ is even.

$\therefore \sigma_1, \sigma_2 \in A_n$

$\sigma_1^{-1} = (\tau_1 \tau_2 \cdots \tau_k)^{-1} = \tau_k^{-1} \cdots \tau_1^{-1} \in A_n$

Thus $A_n$ is a subgroup of $S_n$.

10.6.5 Definition: The group $A_n$ of all even permutations is called the alternating group on $n$ letters (symbols).

10.6.6 Example: Let $A = \{1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\}$

(i) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$

$\sigma = (1, 8) (3, 6, 4) (5, 7)$

$= (1, 8) (3, 6) (3, 4) (5, 7)$

$\therefore \sigma$ is an even permutation.
(ii) \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix} \)

\( \sigma = (1, 3, 4, 7, 8, 6, 5, 2) \)

\( = (1,3)(1,4)(1,7)(1,8)(1,6)(1,5)(1,2) \)

\( \sigma \) is an odd permutation.

10.6.7 SAQ: Let \( \sigma \) be a fixed odd permutation in \( S_n \), \( n \geq 2 \). Show that any odd permutation in \( S_n \) is the product of \( \sigma \) and a permutation in \( A_n \).

10.6.8 Example: Let \( \sigma = (a_1, a_2, \ldots, a_{2k+1}) \) be a cycle of odd length in \( S_n \). Then \( \sigma^2 \) is a cycle.

Proof: \( \sigma^2 = (a_1, a_2, \ldots, a_{2k+1})(a_1, a_2, \ldots, a_{2k+1}) \)

\( = (a_1, a_3, a_5, \ldots, a_{2k-1}, a_{2k+1}, a_2, a_4, \ldots, a_{2k-2}, a_{2k}) \)

So \( \sigma^2 \) is a cycle.

10.6.9 SAQ: Let \( \sigma = (a_1, a_2, \ldots, a_{2k}) \) be a cycle of even length in \( S_n \). Show that \( \sigma^2 \) is not a cycle.

10.6.10 Theorem: Let \( \sigma \) be a cycle of length \( n \). Then \( \sigma^r \) is a cycle if, and only if, \( n | r \) or \( (n, r) = 1 \).

Proof: A) Assume that \( n | r \). Then \( \sigma^r = \sigma^{nk} \), \( r = nk \)

\( = (\sigma^n)^k \)

\( = 1^k = I \), where \( I \) is the identity permutation which is a cycle of length 1.

Now, suppose that \( (n, r) = 1 \). Then \( n \nmid r \). Hence \( \sigma^r \neq I \). Thus \( \exists a \in \sigma^r \neq a \). For this \( a \), we have \( a\sigma \neq a \).

\( \therefore \sigma = (a, a\sigma, a\sigma^2, \ldots, a\sigma^{n-1}), \theta_\sigma(a) = \{a, a\sigma, \ldots, a\sigma^{n-1}\} \)

We know that \( n \) is the least positive integer such \( a\sigma^n = a \)

Now, \( a(\sigma^r)^n = a(\sigma^n)^r = a(1)^r = aI = a \)

\( \text{If} \ a(\sigma^r)^k = a \text{ then} \ a\sigma^{rk} = a \). Hence \( n | (r, k) \).
\[(r, n) = 1 \Rightarrow n | k\]

\[\therefore n \text{ is the least positive integer } \exists a (\sigma^r)^n = a\]

\[\therefore \theta_{\sigma^r}(a) = \{a, a\sigma^r, a\sigma^{2r}, \ldots, a(\sigma^r)^{n-1}\}\]

Since \(\theta_{\sigma^r}(a) \subseteq \theta_\sigma(a)\) we have \(\theta_{\sigma^r}(a) = \theta_\sigma(a)\)

If \(b \notin \theta_{\sigma^r}(a)\) then \(b \notin \theta_\sigma(a)\)

This implies \(b_\sigma = b \Rightarrow b\sigma^r = b\)

\[\therefore \sigma^r = (a, a\sigma^r, a(\sigma^r)^2, \ldots, a(\sigma^r)^{n-1})\]

Thus \(\sigma^r\) is a cycle.

B) Conversely suppose that \(\sigma^r\) is a cycle

If \(\sigma^r = 1\) then \(n | r\).

Suppose that \(\sigma^r \neq 1\). Then \(n \nmid r\).

\(\exists a \in a \sigma^r \neq a\). For this \(a\) we have \(a\sigma \neq a\). Let \(m = \text{ length of } \sigma^r\).

Now, \(\sigma = (a, a\sigma, \ldots, a\sigma^{n-1})\) and

\[\sigma^r = (a, a\sigma^r, \ldots, a(\sigma^r)^{m-1})\]

\[\theta_{\sigma^r}(a) \subseteq \theta_\sigma(a) \Rightarrow m \leq n\]

Suppose \(\theta_{\sigma^r}(a) \neq \theta_\sigma(a) \quad \exists b \in \theta_\sigma(a) \mid \theta_{\sigma^r}(a)\)

For this \(b, b_\sigma \neq b\) and \(b\sigma^r = b\).

Then \(\sigma = (b, b\sigma, \ldots, b\sigma^{n-1})\) and \(n\) is the least positive integer such that \(b\sigma^n = b\).

But \(b\sigma^r = b \Rightarrow n | r\) which is a contradiction.

\[\therefore m = n\]
0 \sigma^r (a) = \{a, a\sigma^r, \ldots, \sigma^r (a)^{n-1}\}
\sigma^r = \{\{a, a\sigma^r, \ldots, \sigma^r (a)^{n-1}\}\}

\therefore O(\sigma) = O(\sigma^r) = n
\Rightarrow (n, r) = 1

10.7 CYCLIC GROUPS

We recall that if \( G \) is a group and \( a \in G \), then \( H = \{a^n / n \in \mathbb{Z}\} = \langle a \rangle \) is a subgroup of \( G \) and \( H \) is called the cyclic subgroup of \( G \) generated by \( a \). Also \( G \) is called a cyclic group if \( G = \{a^n / n \in \mathbb{Z}\} \) for some \( a \in G \), \( a \) is called a generator of \( G \) and we write \( G = \langle a \rangle \).

10.7.1 Theorem : Every cyclic group is abelian.

Proof : Let \( G = \langle a \rangle \) be a cyclic group. Let \( x, y \in G \). Then \( \exists \) integers \( n \) and \( m \) such that \( x = a^n \) and \( y = a^m \).

Now \( xy = a^n a^m = a^{n+m} = a^{m+n} = a^m a^n = yx \).

Thus \( G \) is an abelian group.

10.7.2 Example : Let \( +_n \) be addition modulo \( n \) in \( \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\} \). Then \( (\mathbb{Z}_n, +_n) \) is a cyclic group.

Proof : For any \( s, t \in \mathbb{Z}_n \)

\( s +_n t = \) remainder obtained when \( s + t \) is divided by \( n \).

Clearly \( s +_n t \in \mathbb{Z}_n \) for any \( s, t \in \mathbb{Z}_n \).

Now we show that \( +_n \) is associative.

Let \( x, y, z \in \mathbb{Z}_n \),

Let \( x +_n y = r_1 \) and \( (x +_n y) +_n z = r_1 +_n z = s_1 \)
\( x +_n y = r_1 \Rightarrow x + y = np_1 + r_1 \) for some \( p_1 \in \mathbb{Z} \).
\( r_1 +_n z = s_1 \Rightarrow r_1 + z = np_2 + s_1 \) for some \( p_2 \in \mathbb{Z} \).
Now, \( x + y + z = np_1 + n + z = np_1 + np_2 + s_1, 0 \leq s_1 < n \) \quad (1)

Let \( y + _n z = r_2 \) and \( x + _n (y + _n z) = s_2 = x + _n r_2 \)

Then \( y + z = nq_1 + r_2 \) and \( x + r_2 = nq_2 + s_2 \) for some \( q_1, q_2 \in \mathbb{Z} \).

\[
\begin{align*}
x + y + z &= x + nq_1 + r_2 = nq_1 + x + r_2 \\
&= nq_1 + nq_2 + s_2 = n(q_1 + q_2) + s_2, 0 \leq s_2 < n \quad (2)
\end{align*}
\]

From (1) and (2) we get \( s_1 = s_2 \)

\[
(x + _n y) + _n z = x + _n (y + _n z)
\]

**Existence of identity**: \( 0 + _n x = x + _n 0 = x \quad \forall x \in \mathbb{Z}_n \).

Thus 0 is the identity.

**Existence of inverse**: For \( x \in \mathbb{Z}_n \), \( n - x \in \mathbb{Z}_n \) and \( x + _n (n - x) = 0 \).

Thus \( \langle \mathbb{Z}_n, +_n \rangle \) is a group.

Also \( \langle 1 \rangle = \mathbb{Z}_n \) in \( \langle \mathbb{Z}_n, +_n \rangle \)

Thus \( \langle \mathbb{Z}_n, +_n \rangle \) is a cyclic group and it is also an abelian group.

10.7.3 Example: \( \langle \mathbb{Z}, + \rangle \) is a cyclic group. Here 1 is a generator and -1 is also a generator.

10.7.4 Definition: Let \( G \) be a group. Let \( a \in G \). Let \( H = \langle a \rangle \) be the cyclic subgroup of \( G \) generated by \( a \). If \( H \) is finite and \( |H| = |\langle a \rangle| = \text{number of elements in } H \), then \( a \) is said to be of finite order and order of \( a \) is defined as \( |\langle a \rangle| \). If \( \langle a \rangle \) is an infinite group, then \( a \) is said to be of infinite order.

10.7.5 SAQ: Find the orders of 2 and 3 in \( \langle \mathbb{Z}_6, +_6 \rangle \).

10.7.6 Example: Find the orders of 2, 3, 6 and 5 in \( \langle \mathbb{Z}_9, +_9 \rangle \).

Solution:

\[
\begin{align*}
\langle 3 \rangle &= \{0, 3, 6\} \\
\langle 6 \rangle &= \{0, 6, 3\} \\
\langle 5 \rangle &= \{0, 5, 4\} \\
\langle 2 \rangle &= \{0, 2, 4, 6, 8, 1, 3, 5, 7\} = \mathbb{Z}_9
\end{align*}
\]
The order of : 2 is 9.

3, 6, 5 is 3.

10.7.7 Theorem : Every subgroup of a cyclic group is cyclic.

Proof : Let H be a subgroup of a cyclic group G. If $H = \{e\}$, then $H = \langle e \rangle$. So H is cyclic. Suppose $H \neq \{e\}$.

Let $e \neq x \in H$. Then $x \in G$. So $x = a^n$ for some $0 \neq n \in \mathbb{Z}$. Now $x^{-1} = a^{-n} \in H$ since H is a subgroup. Thus $\exists$ a positive integer $n \ni a^n \in H$.

By well ordering principle $\exists$ a least positive integer $n \ni b = a^n \in H$.

Now we show that $H = \langle b \rangle$.

Let $y \in H$. Then $y \in G \Rightarrow y = a^m$ for some $m \in \mathbb{Z}$. There exists unique $t, r \in \mathbb{Z}$ such that $m = tn + r$, $0 \leq r < n$.

$$y = a^m = a^{tn+r} = (a^n)^t a^r \Rightarrow y(a^n)^{-t} = a^r \in H, \text{ since } y, a^n \in H.$$ 

$a^r \in H$, $0 \leq r < n$, $n$ is the least positive integer such $a^n \in H$ imply $r = 0$.

$$\therefore y = (a^n)^t a^0 = b^t \cdot e = b^t$$

$$\Rightarrow y \in \langle b \rangle$$

Thus $H \subseteq \langle b \rangle$ \hspace{1cm} (1)

$b \in H \Rightarrow \langle b \rangle \subseteq H$ \hspace{1cm} (2)

From (1) and (2) we have $H = \langle b \rangle$.

10.7.8 Theorem : Let $G = \langle a \rangle = \{a^n/n \in \mathbb{Z}\}$. If $a^k = a^\ell$ for some $k, \ell \in \mathbb{Z}$ with $k \neq \ell$, then there exists a positive integer m such that $G = \{e, a^0, a^1, a^2, \ldots, a^{m-1}\}$. Thus if G is infinite then $a^k \neq a^\ell$ for $k \neq \ell$, $k, \ell \in \mathbb{Z}$.

Proof : $a^k = a^\ell \Rightarrow a^{k-\ell} = a^{\ell-k} = e$. Thus there is a least positive integer $m$ such that $a^m = e$.

Let $x \in G$. Then $x = a^n$ for some $n \in \mathbb{Z}$. $\exists q, r \in \mathbb{Z} \ni n =mq + r$, $0 \leq r < m$. 

Then \( x = a^n = a^{mq+r} = (a^m)^q \cdot a^r = c^q \cdot a^r = c^r = a^r \)

Thus \( x \in \{ c = a^0, a^1, a^2, \ldots, a^{m-1} \} \)

Now we show that \( e = a^0, a^1, a^2, \ldots, a^{m-1} \) are all distinct.

Suppose \( 0 \leq i < j \leq m-1 \) such that \( a^i = a^j \).

Then \( m > j - i > 0 \) and \( a^{j-i} = a^j a^{-i} = e \).

This contradicts that \( m \) is the least positive integer such that \( a^m = e \).

Thus \( e = a^0, a^1, a^2, \ldots, a^{m-1} \) are distinct.

Thus \( G = \{ e, a, a^2, \ldots, a^{m-1} \} \)

10.7.9 Note: As a consequence of theorem 10.7.8. We have the following:

A. Let \( G = \langle a \rangle \) be an infinite cyclic group. Then \( a^n \neq a^m \) for all \( n \neq m \).

Also \( a^{n+m} = a^n a^m \)

If we identify \( a^n \) by \( n \) and the product \( a^n a^m \) by \( n+m \), \( G \) and \( \langle z, + \rangle \) are just alike except for the names of the elements and the operations. Thus \( G \) and \( \langle z, + \rangle \) are structurally the same.

B. Let \( G = \langle a \rangle \) be finite. Then \( \exists \) a positive integer \( m \ni a^m = e \) and

\[
G = \{ e = a^0, a^1, a^2, \ldots, a^{m-1} \}
\]

For any \( i, j \in \{ 0,1,2,\ldots,m-1 \} \)

\[
a^i a^j = a^{i+j} = a^r \text{ where } i + j = mq + r \text{ for } q, r \in Z, 0 \leq r < m.
\]

That is \( a^i a^j = a^{i+mj} \)

If we identity \( a^i \) by \( i \) and \( a^j a^j \) by \( i+m+j \), the groups \( G = \langle a \rangle \) and \( \langle Z_m, +_m \rangle \) are just alike except for the names of the elements and the operations. Thus \( G \) and \( \langle Z_m, +_m \rangle \) are structurally the same.
10.7.10 Example: \( \langle Z, + \rangle \) and \( \langle 3Z, + \rangle \) are cyclic groups generated by 1 and 3 respectively. If we identify each \( n \in Z \) by \( 3n \) of \( 3Z \) then \( 3(n + m) = 3n + 3m \) implies \( n + m \) in \( Z \) is identified by \( 3n + 3m \) in \( 3Z \).

Thus \( \langle Z, + \rangle \) and \( \langle 3Z, + \rangle \) are structurally the same.

10.7.11 SAQ: Show that

(a) Any two finite cyclic groups of the same order are structurally the same.

(b) Any two infinite cyclic groups are structurally the same.

10.8 GENERATORS OF FINITE CYCLIC GROUPS:

10.8.1 Theorem: Let \( a \) be an element of order \( n \) in a group \( G \). If \( a^m = e \) then \( n \) divides \( m \).

**Proof:** By division algorithm \( \exists \) unique \( q, r \in Z \) \( m = nq + r, 0 \leq r < n \).

\[ e = a^m = a^{nq + r} = (a^n)^q a^r = e^q a^r = a^r \]

Since \( n \) is the least positive integer such that \( a^n = e \), we have that

\[ r = 0 \text{ and } m = nq . \]

Hence \( n \) divides \( m \).

10.8.2 Theorem: Let \( G = \langle a \rangle \) be a cyclic group with \( |G| = n \). Let \( b \in G \) and \( b = a^s \).

Let \( d = \text{g.c.d.}\{s, n\} \). Then \( |b| = \frac{n}{d} \).

Also \( \langle a^s \rangle = \langle a^t \rangle \iff \text{g.c.d.}\{s, n\} = \text{g.c.d.}\{t, n\} \).

**Proof:** Let \( m = |b| \). Then \( m \) is the least +ve integer such that \( b^m = (a^s)^m = a^{sm} = e \).

Since \( |G| = n \), \( n \) is the least positive integer \( \forall a^n = e \)

Thus by theorem 10.8.1 we have \( n \mid sm \). Since \( n \mid sm \), \( sm = nt \) for some \( t \in Z \).

\[ a^{sm} = a^{nt} = (a^n)^t = e^t = e \]

Thus \( m \) is the least +ve integer \( \forall n \mid sm \) \( \text{--------- (1)} \)

\[ d = \text{g.c.d.}\{s, n\} \Rightarrow \exists x, y \in Z \text{ such that } d = xn + ys . \]
Since $d \mid s$, $d \mid n$ we have $1 = x\left(\frac{n}{d}\right) + y\left(\frac{s}{d}\right)$, and $\frac{s}{d}$ are integers.

$$\Rightarrow \text{g.c.d.} \left\{ \frac{n}{d}, \frac{s}{d} \right\} = 1$$

By (1) $m$ is the least positive integer such that $\frac{ms}{n}$ is an integer.

$$\frac{ms}{n} = m\left(\frac{s}{d}\right)$$

is an integer.

$$\Rightarrow m \text{ is the least +ve integer} \exists \left(\frac{n}{d}\right) \text{ divides } m\left(\frac{s}{d}\right)$$

$$\Rightarrow m \text{ is the least +ve integer} \exists \left(\frac{n}{d}\right) \Rightarrow m, \text{ since } \text{g.c.d.} \left\{ \frac{n}{d}, \frac{s}{d} \right\} = 1.$$  

$$\Rightarrow \frac{n}{d} = m$$

Thus $|\langle b \rangle| = |\langle a^s \rangle| = \frac{n}{d} = \frac{n}{\text{g.c.d.} \{s, n\}}$

$$\langle a^s \rangle = \langle a^t \rangle \iff |\langle a^s \rangle| = |\langle a^t \rangle|$$

$$\iff \frac{n}{\text{g.c.d.} \{s, n\}} = \frac{n}{\text{g.c.d.} \{t, n\}}$$

$$\iff \text{g.c.d.} \{s, n\} = \text{g.c.d.} \{t, n\}$$

10.8.3 Corollary: Let $G = \langle a \rangle$ be a cyclic group and $|G| = n$. Then $\langle a^r \rangle = G$ if and only if $\text{g.c.d.} (r, n) = 1$.

Proof: $\langle a^r \rangle = G \iff |\langle a^r \rangle| = n$

$$\iff n = \frac{n}{\text{g.c.d.} \{r, n\}} \text{ by theorem 10.8.2}$$
\( \implies \gcd\{r, n\} = 1 \)

**10.8.4 Note:** If \( G = \langle a \rangle \) is a finite cyclic group of order \( n \), then the number of generators of \( G \) is \( \phi(n) \) where \( \phi \) is the Euler function. \([\phi(n) = \text{number of +ve integers which are less than } n \text{ and relatively prime to } n]\). If \( 1 \leq s < n \) then \( a^s \) is a generator of \( G \), if and only if \( \gcd\{s, n\} = 1 \).

**10.8.5 Note:** Let \( G \) be a finite cyclic group of order \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \alpha_i > 0 \) are integers, \( p_i \) are distinct primes. Then the number of generators of \( G \) is \( \phi(n) = n \left( 1 - \frac{1}{p_1} \right) \left( 1 - \frac{1}{p_2} \right) \cdots \left( 1 - \frac{1}{p_k} \right) \).

**10.8.6 Example:** Find the number of generators of \( \mathbb{Z}_{15} \). Also find the generators.

**Solution:** The number of generators of \( \mathbb{Z}_{15} \) is \( \phi(15) \)

\[
15 = 3 \times 5
\]

\[
\phi(15) = 15 \left( 1 - \frac{1}{3} \right) \left( 1 - \frac{1}{5} \right) = 2 \times 4 = 8
\]

\( s \in \mathbb{Z}_{15} \) is a generator, iff, \( \gcd\{s, 15\} = 1 \).

Hence the set of generators of \( \mathbb{Z}_{15} \) is \( \{1, 2, 4, 7, 8, 11, 13, 14\} \)

**10.8.6 Example:** Find the order of the subgroup generated by 30 in \( \mathbb{Z}_{60} \).

**Answer:** \( [30] = \frac{60}{\gcd\{30, 60\}} = \frac{60}{30} = 2 \)

\( \langle 30 \rangle = \{0, 30\} \)

**10.8.7 Example:** If \( p \) and \( q \) are prime numbers, find the number of generators of \( \mathbb{Z}_{pq} \).

**Answer:** The number of generators of \( \mathbb{Z}_{pq} \) is

\[
\phi(pq) = pq \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{1}{q} \right) = \frac{pq(p-1)(q-1)}{pq} = (p-1)(q-1)
\]

**10.8.8 SAQ:** Find the number of generators of \( \mathbb{Z}_{pr} \), where \( p \) is a prime and \( r \) is an integer \( \geq 1 \).

**10.8.9 Problem:** Let \( G \) be an abelian group. Let \( H \) and \( K \) be finite cyclic subgroups of \( G \) with \( |H| = r, |K| = s \). Show that \( G \) has a subgroup of order \( rs \) if \( (r, s) = 1 \).
Answer: Let $H = \langle a \rangle$ and $K = \langle b \rangle$. Then $O(a) = r$ and $O(b) = s$. Since $G$ is abelian we have $ab = ba$.

Suppose $(ab)^k = e$. Then from $ab = ba$ we have

$$(a \ b)^k = a^k b^k = e \Rightarrow a^k = b^{-k}$$

Since $a$ and $b$ are of finite order, we have $a^k$ and $b^k$ are also of finite order.

$$a^k = b^{-k} \Rightarrow O(a^k) = O(b^{-k}) = O(b^k)$$

Since $O(a) = r$, we have $O(a^k)|r$

Similarly $O(b^k)|s$. Thus $O(a^k) = O(b^k)(r, s)$

Since $(r, s) = 1$ we get $O(a^k) = O(b^k) = 1$

Hence $a^k = e$ and $b^k = e$.

$$O(a) = r, O(b) = s \Rightarrow r|k \text{ and } s/k$$

$\Rightarrow k$ is a common multiple of $r$ and $s$.

Thus $O(ab) = \text{l.c.m.}\{r, s\} = \frac{rs}{(r, s)} = \frac{rs}{1} = rs$.

The subgroup $\langle ab \rangle$ of $G$ generated by $ab$ is a cyclic subgroup of order $rs$.

10.9 ANSWERS TO SELF ASSESSMENT QUESTIONS:

10.5.13 SAQ: $\rho^0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$

$\rho^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{pmatrix}$

$\rho^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix}$
Thus \( \rho = \{\rho^0, \rho^1, \rho^2, \rho^3, \rho^4, \rho^5\} \)

The table is

\[
\begin{array}{cccccc}
\rho^0 & \rho^1 & \rho^2 & \rho^3 & \rho^4 & \rho^5 \\
\hline
\rho^0 & \rho^0 & \rho^1 & \rho^2 & \rho^3 & \rho^4 \\
\rho^1 & \rho^1 & \rho^2 & \rho^3 & \rho^4 & \rho^5 \\
\rho^2 & \rho^2 & \rho^3 & \rho^4 & \rho^5 & \rho^0 \\
\rho^3 & \rho^3 & \rho^4 & \rho^5 & \rho^0 & \rho^1 \\
\rho^4 & \rho^4 & \rho^5 & \rho^0 & \rho^1 & \rho^2 \\
\rho^5 & \rho^5 & \rho^0 & \rho^1 & \rho^2 & \rho^3 \\
\end{array}
\]

We observe that the table is symmetric about the main diagonal. Hence \( \langle \rho \rangle \) is abelian. \( S_3 \) also has 6 elements, but \( S_4 \) is not abelian.

10.5.21 SAQ: \( \sigma = (1, 2) (5, 6, 7, 8) \). Order of \( (1, 2) = 2 \). Order of \( (5, 6, 7, 8) = 4 \). Order of \( \sigma = \text{lcm}\{2, 4\} = 4 \)

10.5.24 SAQ: Let \( \sigma = (a, b) \) in \( S_A \). Let \( x \in A \).

If \( x = a \), then
\[
x\sigma^2 = (a\sigma)\sigma = b\sigma = a = x
\]

If \( x = b \) then \( x\sigma^2 = (b\sigma)\sigma = a\sigma = b = x \)

If \( x \notin \{a, b\} \) then
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\[ x \sigma^2 = (x \sigma) \sigma = x \sigma = x \]

Thus \( \sigma \sigma = I \), the identity permutation. Hence \( \sigma^{-1} = \sigma \)

10.6.7 SAQ: Let \( \sigma = \tau_1 \tau_2 \cdots \tau_{2k+1} \) be an odd permutation in \( S_n \), where then \( \tau_j \) are transactions. Then \( \sigma^{-1} = \tau_{2k+1}^{-1} \tau_{2k}^{-1} \cdots \tau_{1}^{-1} = \tau_{2k+1} \tau_{2k} \cdots \tau_{1} \) is also odd. Let \( \tau \) be any odd permutation in \( S_n \). Then \( \rho = \sigma^{-1} \tau \) is an even permutation in \( S_n \) ad \( \tau = \sigma \rho, \rho \in A_n \).

10.6.9 SAQ: \( \sigma = (a_1, a_2, \cdots, a_{2k}) \)

\[ \sigma^2 = \sigma \sigma = (a_1, a_2, \cdots, a_{2k}) (a_1, a_2, \cdots, a_{2k}) = (a_1, a_3, \cdots, a_{2k-1}) (a_2, a_4, \cdots, a_{2k-2}, a_{2k}) \]

So \( \sigma^2 \) is not a cycle.

10.7.5 SAQ: In \( \langle Z_6, +_6 \rangle \)

\( \langle 2 \rangle = \{0, 2, 4\}, \langle 3 \rangle = \{0, 3\} \)

So, order of 2 is 3 and order of 3 is 2.

10.7.11 SAQ:

(a) Let \( G_1 = \{e_1 = a^0, a, a^2, \cdots, a^{m-1}\} \)

\( G_2 = \{e_2 = b^0, b, b^2, \cdots, b^{m-1}\} \) be two finite cyclic groups of order \( m \).

Identify \( a^r \) by \( b^s \). Then

\[ a^r \cdot a^s = a^{r+s} = b^{r+m} = b^r b^m \]

Thus \( G_1 \) and \( G_2 \) are structurally the same.

(b) Let \( G_1 = \langle a \rangle, G_2 = \langle b \rangle \) be two infinite cyclic groups. Identify \( a^n \) by \( b^n \). Then

\[ a^n = b^m = b^n b^m \]

Thus \( G_1 \) and \( G_2 \) are structurally the same.

10.8.8 SAQ: Number of generators of \( \mathbb{Z}_p \) is \( \phi(p^r) : \)

\[ \phi(p^r) = p^r \left(1 - \frac{1}{p}\right) = p^r - 1 \]
10.10 EXERCISES:

10.10.1: Let \( x \) be a fixed element of a set \( A \). Let \( T_x = \{ \sigma \in S_A / x\sigma = x \} \). Show that \( T_x \) is a subgroup of \( S_A \).

10.10.2: Show that every function from a finite set onto itself is one-to-one.

10.10.3: Let \( A \) be a set, \( B \) a subset of \( A \) and \( x \) be a fixed element of \( B \). Show that \( \{ \sigma \in S_A / x\sigma = x \} \) and \( \{ \sigma \in S_A / B\sigma = B \} \) are subgroups of \( S_A \).

10.10.4: Let \( A = \{1, 2, 3, 4, 5, 6, 7, 8\} \). Compute the permutation given by the following products.

(i) \((1,3,5)(4,7)(6,8)\) \hspace{1cm} (ii) \((1,2,4)(3,5,6)(7,8)\)

(iii) \((3,5,6)(1,7)(2,4,8)\) \hspace{1cm} (iv) \((2,3,4)(1,5)(6,7,8)\)

10.10.5: Determine whether \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix} \) is even or odd.

10.10.6: Write the multiplication table for \( A_3 \).

10.10.7: If a subgroup \( H \) of \( S_n \), \( n \geq 2 \), contains an odd permutation, show that exactly half of the permutations in \( H \) are even [Hint: Let \( \sigma \in H \) be an odd permutation. Then \( \tau \rightarrow \sigma \tau \) is a bijection from the set \( A \) of all odd permutations in \( H \) onto the set \( B \) of all even permutations in \( H \).]

10.10.8: If \( a \) is a generator of a cyclic group \( G \), show that \( a^{-1} \) is also a generator of \( G \).

10.10.9: Let \( G \) be a group. Suppose that \( a \in G \) is the unique element of \( G \) which generates a cyclic subgroup of order 2. Show that \( xa = ax \) for all \( x \in G \).

[Hint: Consider \( (xa^{-1})^2 \)]

10.10.10: Show that \( Z_p \) has no proper subgroup if \( p \) is prime. [Hint: \( \phi(p) = p\left(1 - \frac{1}{p}\right) = p - 1 \). Thus all the non-zero elements of \( Z_p \) are generators of \( Z_p \).]

10.10.11: Find the number of generators of the cyclic groups, whose orders are i) 30, ii) 36, iii) 100, iv) 120, v) 19. Also find the generators in each case.

10.10.12: Find the orders of the different subgroups of the following cyclic groups.

i) \( Z_{16} \) ii) \( Z_{30} \) iii) \( Z_{45} \) iv) \( Z_{60} \) v) \( Z_{17} \)

10.10.13: Give an example of a finite group which is not cyclic [Try Klein's 4-group or \( S_3 \)].
10.11 MODEL EXAMINATION QUESTIONS:

10.11.1: Define a permutation of a non-empty set \( A \). Show that the set \( S_A \) of all permutations of \( A \) is a group under the composition of mappings.

10.11.2: Let \( A = \{1,2,3,4,5,6,7\} \). Let \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 3 & 1 & 6 & 7 & 5 \end{pmatrix} \) and \( \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 2 & 6 & 7 & 5 \end{pmatrix} \). Compute \( \sigma \tau, \tau \sigma, \sigma^{-1} \tau, \tau \sigma^{-1} \).

10.11.3: Write the multiplication table for \( S_3 \).

10.11.4: Prove that \( S_n \) is not an abelian group if \( n \geq 3 \).

10.11.5: Define an orbit of a permutation \( \sigma \) in \( S_A \). Prove that any two orbits of \( \sigma \) are either disjoint or identical.

10.11.6: Define a cycle. Define disjoint cycles. Prove that any two disjoint cycles commute.

10.11.7: Define a transposition. Prove that any permutation of a finite set \( A \) with the at least two elements is a product of transpositions.

10.11.8: Let \( A = \{1,2,3,4,5,6,7,8\} \). Let \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} \). Decompose \( \sigma \) into disjoint cycles and also into transpositions.

10.11.9: If \( \sigma \) is a cycle of length \( n \) in \( S_A \), show that the order of \( \sigma \) is \( n \).

10.11.10: Prove that a permutation of a finite set \( A \) with at least two elements is either even or odd but not both.

10.11.11: Show that for \( n \geq 2 \), the set \( A_n \) of all even permutations in \( S_n \) is a subgroup of \( S_n \).

10.11.12: Let \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} \). Determine whether \( \sigma \) is even or odd.

10.11.13: Prove that \( \langle Z_n, +_n \rangle \) is a cyclic group.

10.11.14: Find the orders of 2, 3, 6 and 5 in \( \langle Z_9, +_9 \rangle \).

10.11.15: Prove that every subgroup of a cyclic group is cyclic.

10.11.16: Show that any cyclic group is structurally either the group \( \langle Z, + \rangle \) or the group \( \langle Z_n, +_n \rangle \) for some \( n \).
10.11.19: Let $a$ be an element of order $n$ in a group $G$. If $a^m = e$, where $e$ is the identity element of $G$, prove that $n$ divides $m$.

10.11.20: Let $G = \langle a \rangle$ be a cyclic group of order $n$. Prove that the order of the subgroup generated by $a^s$ is $\frac{n}{\text{g.c.d.}(s,n)}$.

10.11.21: Prove that the number of generators of a cyclic group of order $n$ is $\phi(n)$ where $\phi$ is the Euler's function.

10.11.22: Find all the generators of $\langle Z_{15}, +_{15} \rangle$.

10.11.23: Find the order of the subgroup generated by 30 in $\langle Z_{60}, +_{60} \rangle$.

REFERENCES:


Lesson Writer

N. Rajani
Lesson - 11

ISOMORPHISM OF GROUPS - GROUPS OF COSETS

11.1 OBJECTIVE OF THE LESSON

In this lesson, the student will be introduced to the idea that two groups may appear different but structurally the same (isomorphic) but for the names of the elements of the underlying sets. The student will also study the famous Langauages theorem for finite groups and some applications of this theorem.

11.2 STRUCTURE OF THE LESSON

This lesson has the following components.

11.3 Introduction
11.4 Isomorphism
11.5 Isomorphic groups
11.6 Non-Isomorphic groups
11.7 Cayley's theorem
11.8 Groups of Cosets
11.9 Applications of Cosets
11.10 Answers to Self Assessment Questions (SAQs)
11.11 Exercises
11.12 Model Examination Questions
11.13 References

11.3 INTRODUCTION

In this lesson we introduce the concept of isomorphism of groups. We present the steps to be followed while trying to show that two groups are isomorphic (11.5.1). When two groups are not isomorphic we suggest some techniques to prove this by considering some structural property of one of the groups which the other group does not possess (11.6). We also prove that every group is isomorphic to a permutation group (11.7 Caley's Theorem). Induced operation of multiplication of partition cells (cosets) of a group is introduced (11.8.2) and we see that if the cosets of a subgroup of a group form a group under the induced operation then every left coset is a right coset (11.8.7). Lagrange's theorem for finite groups (11.9.1) and some applications of this theorem are also presented.

A good number of examples and self assessment questions are also provided to help the student understand the concepts introduced in this lesson.
11.4 ISOMORPHISM

11.4.1 Definition: Let $G$ and $G'$ be groups. A one-to-one function $\phi$ of $G$ onto $G'$ such that $(xy)\phi = (x\phi)(y\phi)$ for all $x, y \in G$ is called an isomorphism of the group $G$ onto the group $G'$. If there is an isomorphism of $G$ onto $G'$ we say that $G$ is isomorphic with $G'$ and use the notation $G \cong G'$.

11.4.2 SAQ: Let $G$ be a non-empty collection of groups. Show that the relation of being "isomorphic" is an equivalence relation in $G$.

We now prove that an isomorphism of groups maps the identity element onto the identity and inverses onto inverses.

11.4.3. Theorem: Let $G$ and $G'$ be groups. Let $\phi: G \rightarrow G'$ be an isomorphism of $G$ with $G'$. Let $e$ and $e'$ be the identity elements of $G$ and $G'$ respectively. Then

(i) $e\phi = e'$

(ii) $a^{-1}\phi = (a\phi)^{-1}$ for all $a \in G$.

Proof:
(i) $e\phi \in G'$, $e'$ is the identity in $G'$. Therefore $e'(e\phi) = e\phi = (ee)\phi = (e\phi)(e\phi)$ in $G'$.

By cancellation law in $G'$, we have $e\phi = e'$

(ii) Let $a \in G$. Then $a^{-1} \in G$, $a\phi \in G'$ and $a^{-1}\phi \in G'$

$e' = e\phi = (aa^{-1})\phi = (a\phi)(a^{-1}\phi)$

$e' = e\phi = (a^{-1}a)\phi = (a^{-1}\phi)(a\phi)$

Thus $a^{-1}\phi$ is the inverse of $a\phi$ in $G'$.

Hence $a^{-1}\phi = (a\phi)^{-1}$

11.4.4 SAQ: Let $\phi: G \rightarrow G'$ be an isomorphism of groups. Let $x \in G$. Show that $x^n\phi = (x\phi)^n$ for all integers $n$.

11.4.5 SAQ: Let $G$ be a cyclic group with a generator $a$, and let $\phi: G \rightarrow G'$ be an isomorphism of groups. Show that for every $x \in G$, $x\phi$ is completely determined by the value $a\phi$. 
11.5 ISOMORPHIC GROUPS

11.5.1: Steps to be followed while showing that two groups \( G \) and \( G' \) are isomorphic (if this is the case).

Step 1: Define the function \( \phi \) that gives the isomorphism of \( G \) with \( G' \).
Step 2: Show that \( \phi \) is one-to-one.
Step 3: Show that \( \phi \) is onto \( G' \).
Step 4: Show that \( (xy)\phi = (x\phi)(y\phi) \) for all \( x, y \in G \)

11.5.2 Example: We show that any group \( G \) of order 4 is isomorphic to either \( Z_4 \) or the Klein 4-group \( V \).

Proof: Let \( e' \) be the identity element of \( G \). By Lemma 9.11.22 there exists \( x \neq e' \) in \( G \) such that \( x^2 = e' \).

Let \( G = \{e', x, y, z\} \)

Now either (i) \( y \) is its own inverse in which case \( zz = e' \) or (ii) \( z \) is the inverse of \( y \).

Case (i):

<table>
<thead>
<tr>
<th>Table for ( G )</th>
<th>Table for Klein 4-group ( V )</th>
</tr>
</thead>
</table>
|                   | \begin{bmatrix}
| e' & x & y & z \\
| e' & e' & x & y & z \\
| x & x & e' & z & y \\
| y & y & z & e' & x \\
| z & z & y & x & e' \\
| \end{bmatrix} | \begin{bmatrix}
| e & a & b & c \\
| e & e & a & b & c \\
| a & a & e & c & b \\
| b & b & c & e & a \\
| c & c & b & a & e \\
| \end{bmatrix} |

Step 1: Define the mapping \( \phi: G \rightarrow V \) by \( e'\phi = e, x\phi = a, y\phi = b, z\phi = c \)

Step 2: It is clear that \( \phi \) is one-to-one.

Step 3: It is clear that \( \phi \) is onto.

Step 4: It can be verified that \( (pq)\phi = (p\phi)(q\phi) \) for all \( p, q \in G \) by considering all the 16 pairs of elements \( (p, q) \) in \( G \times G \).

Thus in this case \( \phi \) is an isomorphism and \( G \) is isomorphic with the Klein 4-group \( V \).

Case (ii):

<table>
<thead>
<tr>
<th>Table for ( G )</th>
<th>Table for ( Z_4 )</th>
</tr>
</thead>
</table>
| \begin{bmatrix}
| e' & y & x & z \\
| e' & e' & y & x & z \\
| y & y & x & z & e' \\
| x & x & z & e' & y \\
| z & z & e' & y & x \\
| \end{bmatrix} | \begin{bmatrix}
| 0 & 1 & 2 & 3 \\
| 0 & 0 & 1 & 2 & 3 \\
| 1 & 1 & 2 & 3 & 0 \\
| 2 & 2 & 3 & 0 & 1 \\
| 3 & 3 & 0 & 1 & 2 \\
| \end{bmatrix} |
Step 1: Define $\phi: G \rightarrow Z_4$ by $e'\phi = 0$, $y\phi = 1$, $x\phi = 2$, $z\phi = 3$

Step 2: Clearly $\phi$ is one-to-one

Step 3: Clearly $\phi$ is onto $Z_4$

Step 4: It can be verified that $(pq)\phi = (p\phi)(q\phi)$ for all $p, q \in G$ by considering all the 16 pairs of elements $(p, q)$ in $G \times G$. Thus $\phi$ is an isomorphism and $G$ is isomorphic to the group $Z_4$.

11.5.3 Theorem: Every finite cyclic group of order $n$ is isomorphic to $Z_n$.

Proof: Let $G$ be a finite cyclic group of order $n$. Let $e$ be the identity element of $G$.

Let $G = \{e = a^0, a^1, a^2, \ldots, a^{n-1}\}$

Step 1: Define $\phi = G \rightarrow Z_n$ by $a^r\phi = r$ for $0 \leq r \leq n - 1$.

Step 2: If $a^r\phi = a^s\phi$ then $r = s$. So $a^r = a^s$. Thus $\phi$ is one-to-one.

Step 3: Let $r \in Z_n$. Then $a^r \in G$ and $a^r\phi = r$. Thus $\phi$ is onto $Z_n$.

Step 4: Let $a^r, a^s \in G$. By division algorithm there exists unique $p, q \in Z$ such that $r + s = np + q$, $0 \leq q \leq n - 1$.

Thus $r + s = q$ in $Z_n$.

$$a^r \cdot a^s = a^{r+s} = a^{np+q} = c^{np} \cdot a^q = (a^n)^p \cdot a^q$$

$$= c^p \cdot a^q = ca^q = a^q$$

$$\left(a^r a^s\right)\phi = a^q \phi = q$$

$$\left(a^r\phi\right) + \left(a^s\phi\right) = r + s = q \in Z_n.$$ 

Thus $\left(a^r a^s\right)\phi = a^r\phi + a^s\phi$.

Hence $\phi$ is an isomorphism of $G$ with $Z_n$. 
11.5.4 Example: Let $\mathbb{R}$ be the set of real numbers. Let $\mathbb{R}^+$ be the set of all real numbers which are greater than zero. We know that $\mathbb{R}$ is a group under addition and $\mathbb{R}^+$ is a group under multiplication. We now prove that these groups are isomorphic.

Proof:

Step 1: Define $\phi : \mathbb{R} \to \mathbb{R}^+$ by $x\phi = e^x$.

$$\left( e^x = \exp(x) \right)$$

Step 2: If $x\phi = y\phi$ then $e^x = e^y$ so $x = y$. Thus $\phi$ is one-to-one.

Step 3: Let $r \in \mathbb{R}^+$. Then $x = \ell_n r \in \mathbb{R} \left( \ell_n r = \log_e r \right)$ and $x\phi = e^{\ell_n r} = r$. Thus $\phi$ is onto $\mathbb{R}^+$

Step 4: For $r, s \in \mathbb{R}$

$$(r + s)\phi = e^{r + s} = e^r e^s = (r\phi)(s\phi)$$

Thus $(\mathbb{R}, +)$ is isomorphic with $(\mathbb{R}^+, \cdot)$.

Now, we prove an important theorem concerning infinite cyclic groups.

11.5.5 Theorem: Any infinite cyclic group is isomorphic with the group $\mathbb{Z}$ of integers under addition.

Proof: Let $G$ be an infinite cyclic group with a generator $a$. Thus

$$G = \{a^n / n \in \mathbb{Z}\}.$$ 

We know that $a^n \neq a^m$ for $n \neq m$, since $G$ is an infinite cyclic group.

Step 1: Define $\phi : G \to \mathbb{Z}$ by $a^n\phi = n$.

Step 2: $a^n\phi = a^m\phi$ implies $n = m$. Hence $a^n = a^m$. Thus $\phi$ is one-to-one.

Step 3: Let $n \in \mathbb{Z}$. Then $a^n \in G$ and $a^n\phi = n$. Thus $\phi$ is onto $\mathbb{Z}$.

Step 4: $(a^n a^m)\phi = a^{n+m}\phi = n + m = a^n\phi + a^m\phi$

Thus $\phi$ is an isomorphism of $G$ with $\mathbb{Z}$.

11.5.6 SAQ: Prove that any two cyclic groups of the same order are isomorphic.
11.6 NON-ISOMORPHIC GROUPS

In this section we discuss some methods to prove that two groups are non-isomorphic, if this is the case.

11.6.1 Note: If the groups $G$ and $G'$ are of finite order and have different numbers of elements then there cannot be any one-to-one function of $G$ onto $G'$.

11.6.2 Example: $Z_6$ and $S_6$ are not isomorphic since there is no one-to-one function from $Z_6$ onto $S_6$.

In the infinite case there may or may not be one-to-one onto functions. We know that there is a one to one onto function of $Z$ onto $Q$, and there can be no one-to-one function of $Z$ onto $R$.

11.6.3 Example: $Z$ under addition is not isomorphic with $R$ under addition, since there is no one-to-one function of $Z$ onto $R$.

In the event there are one-to-one mappings of $G$ onto $G'$, we usually show that the groups are not isomorphic (if this is the case) by showing that one group has some structural property that the other does not possess. We give a formal definition of "structural property".

11.6.4 Definition: A property of a group that must be shared by any isomorphic group is called a structural property.

11.6.5 Examples: Here we list some possible structural properties and we leave it to the reader to verify that these are structural properties.

1. The group is cyclic.
2. The group is abelian.
3. The group has order 10.
4. The group is finite.
5. The group has exactly two elements of order 6.
6. The equation $x^2 = a$ has a solution for each element $a$ in the group.
7. The group has a cyclic subgroup of a given order.

There may be other structural properties of groups.

11.6.6 Examples: We list some non-structural properties of groups.

1. The group contains 7.
2. All the elements of the group are English alphabets.
3. The group operation is called "multiplication".
4. The elements of a group are permutations.
5. The group operation is denoted by juxtaposition.

6. The group is a subgroup of $<R, +>$.

There may be other non-structural properties of groups.

11.6.7 Example: The property of being cyclic is a structural property of groups.

Proof: Let $G$ be a cyclic group. Let $G$ be isomorphic with another group $G'$. We show that $G'$ is also cyclic.

Since $G$ is isomorphic with $G'$, there exists an isomorphism $\phi: G \rightarrow G'$. Let $x$ be a generator of $G$. We show that $x$ is a generator of $G'$. Let $g' \in G'$. There exists $g \in G$ such that $g \phi = g'$. Since $x$ is a generator of $G$, there exists an integer $n$ such that $x^n = g$.

Now $g' = g \phi = x^n \phi = (x \phi)^n$

Thus $G'$ is generated by $x \phi$. Hence $G'$ is also cyclic.

11.6.8 Example: The property of being abelian is a structural property of groups.

Proof: Let the abelian group $G$ be isomorphic with a group $G'$. Let $\phi: G \rightarrow G'$ be an isomorphism. Let $x, y \in G'$

Then there exist $a, b \in G$ such that $a \phi = x$ and $b \phi = y$.

We have $ab = ba$ since $G$ is abelian.

Now $xy = (a \phi)(b \phi) = (ab) \phi = (ba) \phi = (b \phi)(a \phi) = yx$

Hence $G'$ is also an abelian group.

11.6.9 Example: $(\mathbb{Z}, +)$ is not isomorphic to $(\mathbb{Q}, +)$.

Reason: $(\mathbb{Z}, +)$ is cyclic, but $(\mathbb{Q}, +)$ is not cyclic. Observe that there exists a one-to-one function of $\mathbb{Z}$ onto $\mathbb{Q}$.

11.6.10 Example: $Z_6$ is not isomorphic to $S_3$.

Reason: i) $Z_6$ is abelian and $S_3$ is not abelian. (or) ii) $Z_6$ is cyclic and $S_3$ is not cyclic. Observe that $Z_6$ and $S_3$ have the same order.

11.6.11 Example: The group $\mathbb{Q}^*$ of non-zero elements of $\mathbb{Q}$ under multiplication and the group $\mathbb{R}^*$ of non-zero elements of $\mathbb{R}$ under multiplication are not isomorphic.
**Reason:**

i) There is no one-to-one function of $\mathbb{Q}^*$ onto $\mathbb{R}^*$. 

ii) For $a \in \mathbb{R}^*$, the equation $x^3 = a$ has a solution in $\mathbb{R}^*$. This structural property is not shared by $\mathbb{Q}^*$, for example $x^3 = 2$ has no solution in $\mathbb{Q}^*$.

11.6.12 **Example:** The group $\mathbb{R}^*$ of non-zero real numbers under multiplication is not isomorphic to the group $\mathbb{C}^*$ of non-zero complex numbers under multiplication.

**Reason:**

(i) 1 and $-1 \in \mathbb{R}^*$ generate cyclic subgroups \{1\} and \{1, -1\} of orders 1 and 2 respectively. All other elements of $\mathbb{R}^*$ generate infinite cyclic subgroups. But in $\mathbb{C}^*$, $i$ generates the cyclic subgroup \{1, -1, i, -i\} of order 4. $\mathbb{R}^*$ has no cyclic subgroup of order 4.

(ii) For every $a \in \mathbb{C}^*$, the equation $x^2 = a$ has a solution in $\mathbb{C}^*$, but $x^2 = -1$ has no solution in $\mathbb{R}^*$.

11.6.13 **Example:** The group $\mathbb{R}^*$ of non-zero real numbers under multiplication is not isomorphic to the group $\mathbb{R}$ of real numbers under addition.

**Reason:** For every $a \in \mathbb{R}$, the equation $x + x = a$ has a solution in $\langle \mathbb{R}, + \rangle$. Suppose there is an isomorphism $\phi: \mathbb{R} \longrightarrow \mathbb{R}^*$.

There exists an $x \in \mathbb{R}$ such that $x\phi = -1$. Now $y = \frac{x}{2} \in \mathbb{R}$ and $y + y = x$.

$-1 = x\phi = (y + y)\phi = (y\phi)(y\phi) = (y\phi)^2$.

This is a contradiction since the square of no real number is $-1$.

### 11.7 CAYLEY’S THEOREM

In this section we state and prove the well known Cayley's theorem.

11.7.1 **Theorem (Cayley):** Every group is isomorphic to a group of permutations.

**Proof:** Let $G$ be a group. Let $S_G$ be the group of all permutations of the set $G$ given by theorem 10.4.6: With each $a \in G$ define the mapping $\rho_a : G \to G$ by $x\rho_a = xa$ for each $x \in G$.

$\rho_a$ is a permutation of $G$ since (i) the cancellation law in the group $G$ implies that $\rho_a$ is one-to-one and (ii) the solvability of the equation $xa = b$ for all $a, b \in G$ implies that $\rho_a$ is onto. Thus $\rho_a \in S_G$ for each $a \in G$.

Let $G' = \{\rho_a / a \in G\}$.

We prove that $G'$ is a subgroup of $S_G$ and $G$ is isomorphic with $G'$. 
To show that $G'$ is a subgroup of $S_G$:

$$x \rho_a \rho_b = (xa)\rho_b = (xa)b = x(ab) = x \rho_{ab} \text{ for all } x, a, b \in G.$$  

Hence $\rho_a \rho_b = \rho_{ab}$

Thus $G'$ is closed under multiplication. If $e$ is the identity element of $G$, then $x e = xe = x$.

Hence $\rho_e$ is the identity permutation in $S_G$ and is in $G'$. $\rho_a \rho_e = \rho_{ae} = \rho_a = e \rho_a = \rho_e \rho_a$

Thus $\rho_e$ is identity element in $G'$.

Also $\rho_a \rho_{a^{-1}} = \rho_{aa^{-1}} = \rho_e = \rho_{a^{-1} a} = \rho_{a^{-1}} \rho_a$

Thus $\rho_{a^{-1}} = \rho_{a^{-1}} \in G'$

Thus $G'$ is a subgroup of $S_G$.

To show that $G$ is isomorphic with $G'$:

Step 1: Define $\phi : G \rightarrow G'$ by $a \phi = \rho_a$ for $a \in G$.

Step 2: $a \phi = b \phi \Rightarrow \rho_a = \rho_b \Rightarrow e \rho_a = e \rho_b \Rightarrow e a = e b \Rightarrow a = b$

Step 3: From the definition of $G'$ and $\phi$, it is clear that $\phi$ is onto $G'$.

Step 4: $(ab) \phi = \rho_{ab} = \rho_a \rho_b = (a \phi)(b \phi)$.

Thus $\phi$ is an isomorphism of $G$ with $G'$.

11.7.2 Definition: The group $G'$ in the proof of Cayley's theorem is called the right regular representation of $G$.

11.7.3 Definition: For the proof of Cayley's theorem, we could have considered the permutations $\lambda_a$ defined by $x \lambda_a = ax$ for all $x \in G$. Then $\lambda_a \lambda_b = \lambda_{ab}$. The set $G^* = \{\lambda_a / a \in G\}$ is a subgroup of $S_G$ and the mapping $\psi : G \rightarrow G^*$ defined by $a \psi = \lambda_a^{-1}$ is an isomorphism of $G$ with $G^*$. This group $G^*$ is called the left regular representation of $G$.

11.7.4 Example: Let us compute the left regular representation of the three element group given by the following table.
The elements of the left regular representation are

\[ \lambda_e = \begin{pmatrix} e & a & b \\ e & a & b \end{pmatrix}, \]

\[ \lambda_a = \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix}, \lambda_b = \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix} \]

The table for the left regular representation is

<table>
<thead>
<tr>
<th></th>
<th>( \lambda_e )</th>
<th>( \lambda_a )</th>
<th>( \lambda_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_e )</td>
<td>( \lambda_e )</td>
<td>( \lambda_a )</td>
<td>( \lambda_b )</td>
</tr>
<tr>
<td>( \lambda_a )</td>
<td>( \lambda_a )</td>
<td>( \lambda_b )</td>
<td>( \lambda_c )</td>
</tr>
<tr>
<td>( \lambda_b )</td>
<td>( \lambda_b )</td>
<td>( \lambda_c )</td>
<td>( \lambda_a )</td>
</tr>
</tbody>
</table>

11.7.5 Note: For a finite group given by a table, \( \rho_a \) is the permutation of the elements corresponding to their order in the column under \( a \) at the very top, and \( \lambda_a \) is the permutation corresponding to their order in the row opposite to \( a \) at the extreme left. For abelian groups the left regular representation and the right regular representation are the same.

11.8 GROUPS OF COSETS

11.8.1 Definition: By a partition of a non-empty set \( S \) we mean a disjoint collection of non-empty subsets of \( S \) whose union is \( S \). We call the members of the partition as cells or blocks of the partition. We know that an equivalence relation on a non-empty set \( S \) determines and is determined by a partition of \( S \) into disjoint equivalence classes. Each element of a cell is called a representative of that cell. We denote by \( B_a \) the cell represented by \( a \).

11.8.2 Definition: Let \( G \) be a group. Let \( P \) be a partition of \( G \). Let \( B_r \) and \( B_s \) be two cells of \( P \) represented by the elements \( r \) and \( s \) of \( G \). The cell product \( B_r B_s \) is defined as the cell \( B_t \) represented by the product in \( G \) of a representative of \( B_r \) and a representative of \( B_s \). We say that the cell product \( B_r B_s \) is well defined if the product of any representative of \( B_r \) and any representative of \( B_s \) represents the same cell \( B_t \).

The cell multiplication in \( P \), if it is well defined, is called the induced operation on \( P \) by the operation of \( G \)
11.8.3 Theorem: If a group \( G \) can be partitioned into a partition \( P \) with the induced operation on \( P \) well defined, and \( P \) is a group with this induced operation, then the cell represented by the identity element \( e \) of \( G \) is a subgroup of \( G \).

**Proof:** Let \( G \) be partitioned into a partition \( P \) with the induced operation on \( P \) well defined and giving a group.

- Let \( B_c \) be the cell containing \( c \).
- Let \( B_a \) be any cell represented by \( a \in G \).
- Now \( a \in B_a, e \in B_c \) and \( ae = a \in B_a \).
- Since the induced operation is well defined we get \( B_a B_c = B_a \).
- Similarly \( B_c B_a = B_c \).
- Thus \( B_c \) is the identity element of \( P \).
- Therefore \( B_c B_c = B_c \).
- This shows that \( B_c \) is closed under the multiplication of \( G \). By its definition \( e \in B_c \).

Let \( a \in B_c \). Let \( B_{a^{-1}} \) be the cell containing \( a^{-1} \).

\[
a \in B_c \text{ and } a^{-1} \in B_{a^{-1}} \Rightarrow e = aa^{-1} \in B_c B_{a^{-1}} = B_{a^{-1}}
\]

\[
\Rightarrow e \in B_{a^{-1}}
\]

\[
\Rightarrow B_{a^{-1}} = B_c
\]

\[
\Rightarrow a^{-1} \in B_c
\]

Hence \( B_c \) is a subgroup of \( G \).

11.8.4 Definition: Let \( H \) be a subgroup of a group \( G \) and let \( a \in G \). The left coset \( aH \) of \( H \) by \( a \) is defined as the set \( \{ah/h \in H\} \). The right coset \( Ha \) of \( H \) by \( a \) is defined as the set \( \{ha/h \in H\} \).

11.8.5 Example: We now determine the left cosets of \( pZ \) as a subgroup of \( Z \) under addition for any positive integer \( p \).

For any integer \( n \) we have \( pn + pZ = pZ \).

Let \( x \in Z \Rightarrow \exists n, r \in Z \ni x = r + pn, 0 \leq r < p \)

\[
\therefore x + pZ = r + pn + pZ = r + pZ
\]
Thus the set of left cosets is
\[ \{0 + p\mathbb{Z}, 1 + p\mathbb{Z}, 2 + p\mathbb{Z}, \ldots, (p - 1) + p\mathbb{Z}\} . \]

11.8.6 SAQ: Show that if \( H \) is a subgroup of an abelian group then every left coset of \( H \) is also a right coset of \( H \).

11.8.7 Theorem: If a group \( G \) can be partitioned into cells with the induced operation well defined and giving a group, then the cells must be precisely the left (and also the right) cosets of a subgroup of \( G \). In particular, every left coset of this sub-group is a right coset.

Proof: Let \( e \) be the identity of the group. We know that the cell \( B_e \) is a subgroup of \( G \).
(by Theorem 11.8.3)

Let \( a \in G \).

Now \( a \in B_a, e \in B_e \) and \( B_a B_e = B_a \) imply \( ax \in B_a \) for each \( x \in B_e \).

Then \( aB_e \subseteq B_a \) \[ \text{(1)} \]

Let \( x \in B_a \). Now \( a^{-1} \in B_{a^{-1}} \) and \( B_{a^{-1}} B_a = B_{a^{-1}a} = B_e \)

\[ \Rightarrow a^{-1}x \in B_e \Rightarrow a^{-1}x = b \in B_e \Rightarrow x = ab \in aB_e . \]

So \( B_a \subseteq aB_e \) \[ \text{(2)} \]

By (1) and (2) we have \( B_a = aB_e \).

By a similar argument we can show that

\[ B_a = B_e a \]

Hence \( aB_e = B_e a \).

11.8.8 Theorem: Let \( H \) be a subgroup of a group \( G \). The relations, \( a \equiv_r b \) (mod \( H \)) if and only if \( a^{-1}b \in H \) and \( a \equiv_l b \) (mod \( H \)) if and only if \( ab^{-1} \in H \), are equivalence relations on \( G \), left congruence modulo \( H \) and right congruence modulo \( H \) respectively.

Proof - Reflexive: \( a^{-1}a = e \in H \) for all \( a \in G \). Therefore \( a \equiv_r a \) (mod \( H \))

If \( a \equiv_r b \) (mod \( H \)) then \( a^{-1}b \in H \). Since \( H \) is a subgroup, \( (a^{-1}b)^{-1} \in H \).

So, \( (a^{-1}b)^{-1} = b^{-1}(a^{-1})^{-1} = b^{-1}a \in H \Rightarrow b \equiv_l a \) (mod \( H \))
Transitive: If \( a \equiv b \pmod{H} \), \( b \equiv c \pmod{H} \) then \( a^{-1}b \in H \) and \( b^{-1}c \in H \).

Since \( H \) is a subgroup,
\[
(a^{-1}b)(b^{-1}c) = a^{-1}(bb^{-1})c = a^{-1}ec = a^{-1}c \in H.
\]

Therefore \( a \equiv c \pmod{H} \). Hence \( \equiv_{\ell} \pmod{H} \) is an equivalence relation on \( G \).

Similarly we can show that \( \equiv_{r} \pmod{H} \) is an equivalence relation on \( G \).

Let \( B_{c} \) be the cell containing \( c \) w.r.t. \( \equiv_{\ell} \pmod{H} \)
\[
\therefore x \in B_{c} \iff x \equiv_{\ell} c \pmod{H} \iff x^{-1}c = x^{-1}e = x^{-1} \in H \iff x \in H. \text{ Thus } B_{c} = H.
\]

Similarly, for \( \equiv_{r} \pmod{H} \) also we have \( B_{c} = H \).

**11.8.9 Theorem:** Let \( H \) be a subgroup of a group \( G \). The equivalence classes of \( G \) under \( \equiv_{\ell} \pmod{H} \) are the left cosets of \( H \) and the equivalence classes of \( G \) under \( \equiv_{r} \pmod{H} \) are the right cosets of \( H \). There exists a bijection from any coset onto any coset of \( H \).

**Proof:** Let \( \overline{a} \) be the equivalence class of \( a \in G \) under \( \equiv_{\ell} \pmod{H} \).

Now \( \overline{a} = \{ x \in G / a \equiv_{\ell} x \pmod{H} \} = \{ x \in G / a^{-1}x \in H \} = \{ x \in G / a^{-1}x = h \in H \} = \{ x \in G / x = ah \text{ for some } h \in H \} = aH \)

Similarly we can show that the equivalence classes of \( G \) under \( \equiv_{r} \pmod{H} \) are the right cosets of \( H \).

Consider the mapping \( \lambda_{a} : H \to aH \) defined by
\[
h \lambda_{a} = ah \text{ for } h \in H
\]

Clearly \( \lambda_{a} \) is onto.

\( ah_{1} = ah_{2} \Rightarrow h_{1} = h_{2} \) by cancellation law in \( G \).

Thus \( \lambda_{a} \) is a bijection \( \quad (1) \)

Similarly the mapping \( \rho_{a} : H \to Ha \) defined by \( h\rho_{a} = ha \) is a bijection \( \quad (2) \)
From (1) and (2) we conclude that there exists a bijection from any coset of $H$ onto any coset of $H$.

We now present an example to show that cell (Coset) multiplication may not be well defined.

**11.8.10 Example:** Consider the group $S_3$ of example 10.4.10

whose multiplication table is given below

<table>
<thead>
<tr>
<th></th>
<th>$\rho_0$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_0$</td>
<td>$\rho_0$</td>
<td>$\rho_1$</td>
<td>$\rho_2$</td>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
<td>$\mu_3$</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>$\rho_1$</td>
<td>$\rho_2$</td>
<td>$\rho_0$</td>
<td>$\mu_2$</td>
<td>$\mu_3$</td>
<td>$\mu_1$</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>$\rho_2$</td>
<td>$\rho_0$</td>
<td>$\mu_1$</td>
<td>$\mu_3$</td>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
<td>$\mu_0$</td>
<td>$\rho_0$</td>
<td>$\rho_2$</td>
<td>$\rho_1$</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>$\mu_2$</td>
<td>$\mu_1$</td>
<td>$\mu_3$</td>
<td>$\rho_1$</td>
<td>$\rho_0$</td>
<td>$\rho_2$</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>$\mu_3$</td>
<td>$\mu_2$</td>
<td>$\mu_1$</td>
<td>$\rho_2$</td>
<td>$\rho_1$</td>
<td>$\rho_0$</td>
</tr>
</tbody>
</table>

$H = \{\rho_0, \mu_1\}$ is a subgroup of $S_3$ and the left cosets of $H$ are $H = \{\rho_0, \mu_1\}$, $\rho_1 H = \{\rho_1, \mu_2\}$ and $\rho_2 H = \{\rho_2, \mu_3\}$.

Now $\rho_0 \in H$, $\rho_1 \in \rho_1 H$ and $\rho_0 \rho_1 = \rho_1 \in \rho_1 H$.

Also $\mu_1 \in H$, $\rho_1 \in \rho_1 H$ and $\mu_1 \rho_1 = \mu_3 \in \rho_2 H$.

But $\rho_1 H$ and $\rho_2 H$ are distinct left cosets. So the induced operation of left coset multiplication is not well defined.

Now we present an example where the induced operation of coset multiplication is well defined.

**11.8.11 Example:** $H = \{0, 3\}$ is a subgroup of the group $Z_6$. The left cosets of $H$ are $H = \{0, 3\}, 1 + H = \{1, 4\}, 2 + H = \{2, 5\}$.

It can be verified that the induced operation gives

$H + (1 + H) = 1 + H$

$(1 + H) + H = 1 + H$

$(1 + H) + (2 + H) = H,$

$(2 + H) + (1 + H) = H,$

$H + (2 + H) = 2 + H$

$(2 + H) + H = 2 + H$
Thus the induced operation is well defined on the set of left cosets of $H$ and gives a 3-element group.

### 11.9 APPLICATIONS OF COSETS

As an application of cosets, we have the following theorem.

#### 11.9.1 Theorem (Lagrange) :

The order of any subgroup $H$ of a group $G$ of finite order divides the order of $G$.

**Proof**:

Let $G$ be a finite group of order $n$. Let $H$ be a subgroup of $G$.

Suppose that the order of $H$ is $m$ and let $h_1, h_2, \ldots, h_m$ be the $m$ members of $H$. By theorem 11.8.9 the left cosets of $H$ in $G$ are disjoint.

Let $a H \in G$.

\[
ah_i = ah_j \Rightarrow h_i = h_j \text{ by left cancellation law in } G.
\]

Thus the left coset $aH$ has $m$ distinct elements

\[
aH = \{ah_1, ah_2, \ldots, ah_m\}
\]

Since $G$ is a group of finite order, the number of distinct left cosets of $H$ in $G$ is finite.

Let $a_1H, a_2H, \ldots, a_KH$ be the distinct left cosets of $H$ in $G$. Then $G = \bigcup_{i=1}^{K} a_iH$ and

\[
n = O(G) = \sum_{i=1}^{K} O(a_iH) = \sum_{i=1}^{K} m = mK
\]

Thus $n = mK$

Hence $m$ divides $n$.

#### 11.9.2 Corollary :

Every group of prime order is cyclic.

**Proof**:

Let $G$ be a group of prime order, say $p$. Then there exists an element $a$ in $G$ such that $a$ is not the identity element $e$ of $G$. Let $m$ be the order of the element $a$. Since $a \neq e$ we have $m \geq 2$.

If $H$ is the cyclic subgroup of $G$ generated by $a$ then $O(H) = O(a) = m$.

By Lagrange's theorem $m/p$.

Since $p$ is prime we have $m = p$.

Hence $\langle a \rangle = H = G$.

Thus $G$ is cyclic.
11.9.3 SAQ : Show that for a given prime number \( p \), there is only one group of order \( p \) (up to isomorphism).

11.9.4 Theorem : The order of an element of a finite group divides the order of the group.

Proof : Suppose that \( G \) is a finite group of order \( n \). Let \( a \in G \) and \( O(a) = m \).

Let \( H \) be the subgroup of \( G \) generated by \( a \). Let \( e \) be the identity in \( G \).

We know that \( H = \left\{ a^n \mid n \in \mathbb{Z} \right\} \).

If \( r \) and \( s \) are integers such that \( 1 \leq r \leq m \), \( 1 \leq s \leq m \), \( r > s \) and \( a^r = a^s \), then \( a^{r-s} = a^0 = e \) and \( r-s < m \). This contradicts the fact that the order of \( a \) is \( m \). Thus the elements \( a, a^2, \ldots, a^{m-1}, a^m = e \) are all distinct.

If \( t \) is any integer, there exists integers \( q \) and \( r \) such that \( t = mq + r, 0 \leq r < m \).

\[
a^t = a^{mq + r} = (a^m)^q a^r = e^q a^r = ea^r = a^r
\]

Thus \( H = \left\{ e, a, a^2, \ldots, a^{m-1} \right\} \) and is a sub-group of \( G \).

By Lagrange's theorem \( O(H) = m \) divides \( (G) = n \).

11.9.5 Definition : Let \( H \) be a subgroup of a group \( G \). The index \( (G:H) \) of \( H \) in \( G \) is defined as the number of left cosets of \( H \) in \( G \).

11.9.6 Note : \( (G:H) \) may be finite or infinite. If \( G \) is finite then \( (G:H) \) is also finite and \( (G:H) = \frac{|G|}{|H|} \) since each coset of \( H \) has \(|H| \) elements.

11.9.7 Note : It can be shown that \( (G:H) \) is also the number of right cosets of \( H \) in \( G \).

11.9.8 SAQ : Find the index of the subgroup \( \langle 4 \rangle \) of \( Z_{18} \).

11.9.9 Result : In a finite group \( G \), if \( H \) is a subgroup of index 2 then every left coset is also a right coset.

Proof : Since \( (G:H) = 2 \), we have

\[
G = H \bigcup aH = H \bigcup Ha \quad \text{for any} \quad a \in G \quad \text{such that} \quad a \notin H.
\]

Since \( H \cap aH = \phi = H \cap Ha \) we have \( aH = Ha \).
11.9.10 Result:

(a) If \( n \) is odd, any abelian group of order \( 2n \) contains exactly one element of order 2.

(b) This result is not true if \( n \) is even.

Proof:

(a) Let \( G \) be an abelian group of order \( 2n \) when \( n \) is odd.

We know that every group of even order has an element of order 2 (by lemma 9.11.22)

Suppose that there are two distinct elements \( a \) and \( b \) of order 2 in \( G \)

Since \( G \) is abelian the subgroup generated by \( a \) and \( b \) is \( H = \langle a, b \rangle = \{e, a, b, ab\} \) and

\[ O(H) = 4 \]

Suppose \( n = 2K + 1 \)

Then \( 2n = 4K + 2 \)

By Lagrange's theorem 4 divides \( 2n = 4K + 2 \) which is a contradiction.

Thus \( G \) has exactly one element of order 2.

(b) Klein 4-group is a group of order \( 2 \times 2 \) which has three elements of order 2.

11.9.11 Result: A group with at least two elements but with no proper nontrivial subgroups must be finite and of prime order.

Proof: Let \( G \) be a group with at least two elements and having no proper nontrivial subgroups.

Let \( e \) be the identity element of \( G \).

Then \( G \) has an element \( a \) which is different from \( e \). Then \( H = \langle a \rangle \neq \{e\} \) is a non-trivial subgroup of \( G \).

Hence \( H \) cannot be a proper subgroup of \( G \). Therefore \( G = H \) and hence \( G \) is a cyclic group.

If \( G \) is infinite then \( G \cong (\mathbb{Z}, +) \). Since \((\mathbb{Z}, +)\) has non-trivial proper subgroups, it would follow that \( G \) also has non-trivial proper subgroups. Thus \( G \) is a group of finite order, say \( n \). Suppose \( n \) is not prime, let \( n = k \ell \).

Then \( b = a^k \neq e \) and \( O(b) = \ell, 1 < \ell < n \). Thus \( \langle b \rangle \) a proper non-trivial subgroups of \( G \). This is a contradiction to the hypothesis on \( G \).

Thus \( O(G) = n \) is prime.

Now we state and prove a basic theorem concerning indices of subgroups.
11.9.12 Theorem: Suppose $H$ and $K$ are subgroups of a group $G$ such that $K \leq H \leq G$, and suppose $(H : K)$ and $(G : H)$ are both finite. Then $(G : K)$ is finite and $(G : K) = (G : H)(H : K)$.

Proof: Let $(G : H) = r$ and $(H : K) = s$.

Let $\{a_i H / a_i \in G, i = 1, 2, \ldots, r\}$ be the collection of distinct left cosets of $H$ in $G$.

Let $\{b_j K / b_j \in H, j = 1, 2, \ldots, s\}$ be the collection of distinct left cosets of $K$ in $H$.

We show that $X = \{a_i b_j K / i = 1, 2, \ldots, r, j = 1, 2, \ldots, s\}$ is the collection of all distinct left cosets of $K$ in $G$.

Suppose $a_i b_j K = a_{\ell} b_m K$

$\Rightarrow (a_{\ell} b_m)^{-1}(a_i b_j) \in K$

$\Rightarrow b_m^{-1} a_{\ell}^{-1} a_i b_j \in H$ since $K \subseteq H$

$\Rightarrow a_{\ell}^{-1} a_i \in b_m H b_j^{-1} \subseteq H$.

$\Rightarrow a_i H = a_{\ell} H \Rightarrow i = \ell$

Now, $a_i b_j K = a_{\ell} b_m K \Rightarrow b_j K = b_m K \Rightarrow j = m$

$\therefore$ All the elements of $X$ are distinct and $|X| = rs$.

Let $x K$ be a left coset of $K$ in $G$.

Then $x H$ is a left coset of $H$ in $G$. Hence $x H = a_i H$ for some $i$.

Then $a_i^{-1} x \in H$ and $a_i^{-1} x K$ is a left coset of $K$ in $H$.

$\therefore$ There exists $j$ such that $a_i^{-1} x K = b_j K$

This implies $x K = a_i b_j K \in X$.

Thus $X$ is the set of all distinct left cosets of $K$ in $G$. Hence

$$(G : K) = |X| = rs = (G : H)(H : K)$$
11.4.2 SAQ Answer: Define the relation ~ on $G$ by $G_1 \sim G_2$ if and only if $G_1$ is isomorphic with $G_2$: $G_1, G_2 \in G$.

(a) To show that ~ is reflexive.

Let $G \in G$. Let $I_G : G \rightarrow G$ be the identity mapping, given by $g I_G = g, \forall g \in G$. It is clear that $I_G$ is one-to-one and onto $G$.

Also $(g_1 g_2) I_G = g_1 g_2 = (g_1 I_G) (g_2 I_G) \forall g_1, g_2 \in G$. Thus $I_G$ is an isomorphism and hence $G \sim G \forall G \in G$.

(b) To show that ~ is symmetric. Let $G_1, G_2 \in G$ and $G_1 \sim G_2$.

Let $\phi : G_1 \xrightarrow{1-1 \text{ onto}} G_2$ be an isomorphism. Let $\psi : G_2 \rightarrow G_1$ be the inverse of the mapping $\phi$.

We know that $\psi$ is one-to-one and onto $G_1$. Let $x, y \in G_2$. Let $x \psi = g_1$ and $y \psi = g_2$.

Then $x = g_1 \phi$ and $y = g_2 \phi$.

Now $xy = (g_1 \phi) (g_2 \phi) = (g_1 g_2) \phi$

Hence $(xy) \psi = ((g_1 g_2) \phi) \psi = (g_1 g_2) \phi \psi$

$= g_1 g_2 = x \psi y \psi$.

Thus $\psi$ is an isomorphism and hence $G_2 \sim G_1$.

(c) To show that ~ is transitive.

Let $G_1, G_2, G_3 \in G$ such that $G_1 \sim G_2$ and $G_2 \sim G_3$. Let $\phi : G_1 \rightarrow G_2$, $\psi : G_2 \rightarrow G_3$ be isomorphisms. We know that the composition $\phi \psi : G_1 \rightarrow G_3$ is one-to-one and onto.

Also for $x, y \in G_1$, we have

$$(xy)(\phi \psi) = ((xy) \phi) \psi = ((x \phi)(g \phi)) \psi$$

$$= ((x \phi) \psi)(g \phi) \psi = (x (\phi \psi))(g \phi) \psi$$

Thus $\phi \psi$ is an isomorphism and hence $G_1 \sim G_3$. By (a), (b) and (c) it follows that ~ is an equivalence relation on $G$. 
11.4.4 SAQ Answer: We first prove the result by induction for positive integers $n$.

Obviously the result is true for the integer 1. Suppose that the result is true for the positive integer $n$, that is $x^n \phi = (x \phi)^n$.

Now $x^{n+1} \phi = (x^n x) \phi = (x^n \phi) (x \phi) = (x \phi)^n (x \phi) = (x \phi)^{n+1}$.

$\therefore$ The result is true for all positive integers $n$.

If $n$ is negative then $-n$ is positive.

Since $x^{-1} \phi = (x \phi)^{-1}$ by theorem 11.4.4, we have

$$x^n \phi = (x^{-1})^{-n} \phi = (x^{-1} \phi)^{-n} = ((x \phi)^{-1})^{-n} = (x \phi)^n$$

Thus the result is true for all $n$.

11.4.5 SAQ Answer: Since $a$ is a generator of $G$, there exists an integer $n$ such that $x = a^n$.

Now $x \phi = a^n \phi = (a \phi)^n$.

11.5.6 SAQ Answer: Let $G_1$ and $G_2$ be two cyclic groups of the same order.

Then either (a) $G_1$ and $G_2$ are of finite order, say $n$ or (b) $G_1$ and $G_2$ are of infinite order.

In case (a) $G_1$ and $G_2$ are isomorphic to $\mathbb{Z}_n$ for some $n$ by theorem 11.5.3. So $G_1$ and $G_2$ are isomorphic by 11.4.2 SAQ.

In case (b), $G_1$ and $G_2$ are isomorphic to $(\mathbb{Z}, +)$ by theorem 11.5.5. So $G_1$ and $G_2$ are isomorphic by 11.4.2 SAQ.

11.8.6 SAQ Answer: $aH = \{ah/h \in H\} = \{ha/h \in H\} = Ha$.

11.9.3 SAQ Answer: Let $G$ be any group of prime order $p$. By corollary 11.9.2 $G$ is a cyclic group of order $p$. By theorem 11.5.3 $G$ is isomorphic to $\mathbb{Z}_p$.

11.9.8 SAQ Answer: In $\mathbb{Z}_{18}$ we have

$\langle 4 \rangle = \{0, 4, 8, 12, 16, 2, 6, 10, 14\}$

$|\langle 4 \rangle| = 9$
11.21 Isomorphism of Groups -

Groups of Cosets

\[
|Z_{18}| = [\{4\}] (Z_{18}:\{4\})
\]

\[
18 = 9 (Z_{18}:\{4\})
\]

\[
\therefore (Z_{18}:\{4\}) = 2
\]

11.11 EXERCISES

11.11.1 : Give two arguments to show that \(Z_4\) is not isomorphic to the Klein 4-group.

11.11.2 : Let \(\phi: G \rightarrow G'\) be an isomorphism of groups. Let \(H\) be a subgroup of \(G\). Prove that \(H\phi = \{h\phi/h \in H\}\) is a subgroup of \(G'\).

11.11.3 : An isomorphism of a group with itself is called an **automorphism** of the group. For any positive integer \(n\), how many automorphisms are there of \(Z_n\)?

(Hint : Use SAQ 11.4.5). How many automorphisms are there of \(Z\)?

11.11.4 : Let \(\langle G, \cdot \rangle\) be a group. Define a binary operation \(\ast\) on the set \(G\) defined by \(a \ast b = ba\) for \(a, b \in G\). Show that \(\langle G, \ast \rangle\) is a group which is isomorphic to \(\langle G, \cdot \rangle\).

11.11.5 : Let \(G\) be a group and let \(g\) be a fixed element of \(G\). Show that the map \(i_g\), such that \(x i_g = g^{-1}xg\) for \(x \in G\), is an isomorphism of \(G\) with itself, that is, an automorphism of \(G\).

11.11.6 : Let \(R^*\) be the group of all nonzero real numbers under multiplication. Let \(t\) be a fixed real number. Let \(S\) be the set of all real numbers except \(-t\), i.e., \(S = \{x - t/x \in R^*\}\). We observe that the elements of \(S\) can be obtained by renaming each \(x\) in \(R^*\) as \(x - t\). Define \(*\) on \(S\), such that \(\langle S, * \rangle\) is a group which is isomorphic to \(R^*\).

**Hint** : Consider the map \(\phi: R^* \rightarrow S\) defined by \(x\phi = x - t\). Thus \(\phi\) is the function which renames the elements of \(R^*\) as the elements of \(S\). Try to define \(*\) on \(S\) to satisfy \((xy)\phi = (x\phi) \ast (y\phi)\) \(\forall x, y \in R^*\).

Show that \(1 - t\) is the identity in \(\langle S, \ast \rangle\) and for each \(x\) in \(S\), \(\frac{1-tx-t^2}{t+x}\) is the inverse of \(x\).

11.11.7 : How many groups are there of order 19, upto isomorphism?

11.11.8 : Let \(H\) be a subgroup of \(G\). Show that there is a one-to-one and onto mapping of the set of all left cosets of \(H\) in \(G\) onto the set of all right cosets of \(H\) in \(G\). (Hint : Try the rule \(aH \rightarrow Ha^{-1}\))
11.11.9 : Find the index of $A_n$ in $S_n$ for $n > 1$.

11.11.10 : Let $G$ be a group of order $n$. Let $e$ be the identity in $G$. Show that $a^n = e$ for all $a \in G$.

11.11.11 : Show that a finite cyclic group of order $n$ has exactly one subgroup of each order $d$ dividing $n$, and that these are all the sub-groups it has.

11.12 MODEL EXAMINATION QUESTIONS :

11.12.1 : If $\phi : G \rightarrow G'$ is an isomorphism of $G$ with $G'$, prove that $e\phi = e'$ and $a^{-1}\phi = (a\phi)^{-1}$ for all $a \in G$. ($e$ and $e'$ are the identity elements of the groups $G$ and $G'$ respectively).

11.12.2 : Show that $R$ under addition is isomorphic to $\mathbb{R}^+$ under multiplication.

11.12.3 : Show that any two cyclic groups of the same order are isomorphic.

11.12.4 : Show that the group $\mathbb{R}^+$ of nonzero real numbers under multiplication is not isomorphic to the group $R$ of real numbers under addition.

11.12.5 : State and prove Caley's theorem.

11.12.6 : List all the automorphisms of (a) $\mathbb{Z}$ (b) $\mathbb{Z}_\delta$.

11.12.7 : Give an example of a subgroup of $H$ of a group $G$ such that the induced operation on left cosets of $H$ is not well defined.

11.12.8 : Let $H$ be a subgroup of a group $G$. Define the left congruence $\equiv_{(mod H)}$. Show that this relation is an equivalence relation on $G$ which partitions $G$ into the left cosets of $H$. Find the equivalence class of the identity element of $G$.

11.12.9 : State and prove Lagrange's theorem for finite groups.

11.12.10 : If $H$ is a subgroup of index 2 of a group $G$, show that every left coset of $H$ in $G$ is a right coset of $H$ in $G$. Find the index of $A_n$ in $S_n$ for $n > 1$.

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Lesson - 12

NORMAL SUBGROUPS - FACTOR GROUPS

12.1 OBJECTIVE OF THE LESSON

In this lesson, the student will be introduced to the concepts of Factor (quotient) group Homomorphisms of groups. The student will also study the fundamental theorem of homomorphism and will come to understand that Factor groups of a group are nothing but homomorphic images of the group.

12.2 STRUCTURE OF THE LESSON

This lesson has the following components.

12.3 Introduction
12.4 The cosets group
12.5 Inner Automomorphisms, Normal subgroups and Factor (quotient) groups
12.6 Homomorphisms
12.7 Fundamental theorem of homomorphism
12.8 Answers to Self-Assessment Questions (SAQs)
12.9 Exercises
12.10 Model Examination Questions

12.3 INTRODUCTION

In this lesson we prove that the induced operation of multiplication of the cosets of a subgroup H of a group G is well defined if, and only if, every left coset of H in G is a right coset of H in G. If this is the case we prove that the collection of the cosets of H in G is a group under this induced coset multiplication. Also the concepts of automorphism, inner automorphism and normal subgroup are introduced. The factor group G/N of a group G by a normal subgroup N is defined. A simple group is defined and $\Lambda_n$, $n \geq 5$ is shown to be a simple group. Through theorems 12.6.15 and 12.7.1 (Fundamental theorem of homomorphism) it is proved that for a group G, the class of homomorphic images of G and the class of quotient groups of G are the same. A good number of examples and self assessment questions are provided.

12.4 THE COSETS GROUP

12.4.1 Lemma : Let H be a subgroup of a group G such that the induced operation of coset multiplication on left (right) cosets of H is well defined. Then the collection of left (right) cosets of H is a group under this induced coset multiplication.
**Proof:** Suppose that the induced operation of left coset multiplication is well defined. Since \( a \) represents \( aH \) and \( b \) represents \( bH \) and \( ab \) represents \( abH \) and the left coset multiplication is well defined we have

\[
(aH)(bH) = abH \text{ for all } a, b \in G.
\]

Thus the induced operation of coset multiplication is a binary operation on the set of all left cosets of \( H \) in \( G \).

**Associative Law:**

\[
aH(bHcH) = aH(bcH) = (a(bc))H = ((ab)c)H
\]

\[
= (abH)cH
\]

\[
= (aHbH)cH \text{ for all } a, b, c \in G.
\]

**Identity:** \( eH \) \( aH = eaH = aH = aeH = aHeH \)

Since \( ae = ea = a \) in \( G \).

Thus \( eH = H \) is the identity.

**Inverse:** \( aH a^{-1}H = aa^{-1}H = eH = a^{-1}aaH = a^{-1}HaH \), since \( a^{-1}a = aa^{-1} = e \) in \( G \).

Thus \( a^{-1}H \) is the inverse of \( aH \).

Thus if the induced operation of left coset multiplication is well defined, then with this multiplication, the set of all left cosets is a group.

Similarly, if the induced operation of coset multiplication on the set of right cosets is well defined, then with this operation \( HaHb = H_{ab} \), the set of all right cosets is a group.

Now we show that the induced operation on left (right) cosets is well defined if and only if there is no distinction between left cosets and right cosets.

**2.4.2 Theorem:** Let \( H \) be a subgroup of a group \( G \). The operation of induced multiplication is well defined on the left (right) cosets of \( H \) if and only if every left coset is a right coset.

**Proof:** Suppose that the operation of induced multiplication is well defined on the left cosets of \( H \). Then the cell represented by the identity \( e \) of \( G \) is \( eH = H = He \).

For \( a \in G \) the cell represented by \( a \) is \( aH \). By lemma 12.4.1 the left cosets of \( H \) in \( G \) form a group under this induced operation. If we consider \( aH \) for \( B_{a} \) and \( eH = H \) for \( B_{e} \) in theorem 11.8.7. We get \( aH = a \cdot (eH) \) and \( a \cdot (eH) = (eH) \cdot a = H \cdot a = Ha \). 

\[ \therefore aH = Ha \]
Conversely suppose that every left coset \( aH \) is a right coset. We first show that \( aH = Ha \). By assumption \( aH = Hg \) for some \( g \in G \). We know that \( a \in aH = Hg \Rightarrow a \in Hg \Rightarrow Ha = Hg \).

\[ \therefore aH = Ha \forall a \in G \]

Now we show that the induced operation of left coset multiplication is well defined.

Let \( a_1, a_2 \in aH, b_1, b_2 \in bH \)

It is enough to show that \( a_1b_1 \in a_2b_2H \)

\[ a_1, a_2 \in aH \Rightarrow a_1H = aH = a_2H \Rightarrow a_1H = a_2H \]

\[ \Rightarrow a_1 = a_2h_1 \text{ for some } h_1 \in H \]

\[ b_1, b_2 \in bH \Rightarrow b_1H = b_2H \Rightarrow b_1 = b_2h_2 \text{ for some } h_2 \in H \]

Now, \( a_1b_1 = a_2h_1b_2 \)

Since every left coset is a right coset, we have \( Hb_2 = b_2H \).

\[ h_1b_2 \in Hb_2 \Rightarrow h_1b_2 \in b_2H \Rightarrow h_1b_2 = b_2h_3 \text{ for some } h_3 \in H \]

\[ a_1b_1 = a_2b_2h_3h_2 \in a_2b_2H \]

\[ \therefore a_1b_1H = a_2b_2H \]

This shows that the induced operation of left coset multiplication is well defined.

### 12.5 INNER AUTOMORPHISMS, NORMAL SUBGROUPS AND FACTOR (QUOTIENT) GROUPS

#### 12.5.1 Definition:
An isomorphism of a group \( G \) onto itself is called an **automorphism** of \( G \).

#### 12.5.2 Theorem:
Let \( G \) be a group. Let \( g \in G \). Then the mapping \( i_g : G \to G \) defined by \( x_{i_g} = g^{-1}xg \) is an automorphism of \( G \).

**Proof:** The mapping \( i_g \) is defined from \( G \) into \( G \).

\[ x_{i_g} = y_{i_g} \Rightarrow g^{-1}xg = g^{-1}yg \Rightarrow g^{-1}x = g^{-1}y \text{ by right cancellation a law in } G. \]

\[ \Rightarrow x = y \text{ by left cancellation law.} \]

Thus \( i_g \) is one-to-one.
If \( y \in G \) then \( x = g y g^{-1} \in G \) and

\[
\xi_g = g^{-1}(gyg^{-1})g = (g^{-1}g)y(g^{-1}g) = eyey = y
\]

Thus \( i_g \) is onto \( G \).

\[
(xy)i_g = g^{-1}(xy)g = g^{-1}xyg = g^{-1}xgg^{-1}yg = ((g^{-1}xg)(g^{-1}yg)) = x_i_y_i_g
\]

Thus \( i_g \) is a homomorphism.

Thus \( i_g \) is an automorphism of \( G \).

**12.5.3 Definition**: Let \( G \) be a group. Let \( g \in G \). The automorphism \( i_g : G \longrightarrow G \), defined by \( x_i_g = g^{-1}xg \) for all \( x \in G \), is called the **inner automorphism** of \( G \) under conjugation by \( g \).

**12.5.4 Definition**: A subgroup \( H \) of a group \( G \) is a **normal (or invariant) subgroup of** \( G \) if \( a^{-1}Ha = H \) for all \( a \in G \).

**12.5.5 Result**: Let \( H \) be a subgroup of a group \( G \). Then the following statements are equivalent.

(a) \( H \) is a normal subgroup of \( G \).

(b) Every left coset of \( H \) is a right coset of \( H \).

(c) The induced operation is welldefined on the left (right) cosets.

**Proof**: (a) \( \Rightarrow \) (b).

Suppose \( H \) is a normal subgroup of \( G \).

Let \( a \in G \).

Then \( a^{-1}Ha = H \)

The multiplication on the left by \( a \) gives \( Ha = aH \).

Thus every left coset is a right coset.

(b) \( \Rightarrow \) (a).

Suppose that every left coset of \( H \) is a right coset of \( H \).
Let \( a \in G \). \( a^{-1}H \) is a left coset of \( H \). Then \( a^{-1}H = Hb \) for some \( b \in G \).

Now \( a^{-1} \in a^{-1}H \Rightarrow a^{-1} \in Hb \Rightarrow Hb = Ha^{-1} \).

Thus \( a^{-1}H = Ha^{-1} \).

Multiplication on the right by \( a \) gives \( a^{-1}Ha = H \). Thus \( H \) is a normal subgroup of \( G \). (b) and (c) are equivalent by theorem 12.4.2.

12.5.6 Result: Let \( H \) be a subgroup of a group \( G \). Then \( H \) is a normal subgroup of \( G \) if and only if \( a^{-1}Ha \subseteq H \) for all \( a \in G \).

Proof: If \( H \) is a normal subgroup of \( G \) then \( a^{-1}Ha \subseteq H \) for all \( a \in G \).

Then \( (a^{-1})^{-1}Ha^{-1} \subseteq H \).

So \( (a^{-1})^{-1}ha^{-1} = h_1 \) for some \( h_1 \in H \).

\( \Rightarrow h = a^{-1}h_1a \in a^{-1}Ha \)

Therefore \( a^{-1}Ha = H \).

Thus \( H \) is a normal subgroup of \( G \).

12.5.7 SAQ: Show that every subgroup of an abelian group is a normal subgroup.

12.5.8 Example: Show that the subgroup \( \{0, \mu_1\} \) is not a normal subgroup of \( S_3 \) of example 10.4.10.

Answer: \[
\rho_1^{-1} \mu_1 \rho_1 = \rho_2 \mu_1 \rho_1 = \mu_3 \rho_1 = \mu_2 \\
\rho_1^{-1} \rho_0 \rho_1 = \rho_2 \rho_0 \rho_1 = \rho_2 \rho_1 = \rho_0 \\
\rho_1^{-1} H \rho_1 = \{\rho_0, \mu_2\} \neq H.
\]

\( \therefore \) \( H \) is not a normal subgroup of \( S_3 \).

12.5.9 Definition: Two subgroups \( H \) and \( K \) of a group \( G \) are said to be conjugate if \( Hi_g = K \) for some innerautomorphism \( i_g \) of \( G \) under conjugation by \( g \), that is, \( g^{-1}Hg = K \) for some \( g \in G \).
12.5.10 Example: From example 12.5.8, we have \( \{\rho_0, \mu_1\}, \{\rho_0, \mu_2\} \) are conjugate subgroups of \( S_3 \).

12.5.11 SAQ: Show that an intersection of normal subgroups of a group \( G \) is again a normal subgroup of \( G \).

12.5.12 Problem: Let \( a \) be a group containing at least one subgroup of a fixed finite order \( \ell \). Show that the intersection of all subgroups of \( G \) of order \( \ell \) is a normal subgroup of \( G \).

**Answer:** First we prove that if \( H \) is a subgroup of order \( \ell \), then so is \( x^{-1}Hx \) for any \( x \in G \).

\[
\begin{align*}
\exists e & \in H \Rightarrow x^{-1}ex = x^{-1}x = e & \in x^{-1}Hx \\
(x^{-1} h_1 x)(x^{-1} h_2 x)^{-1} & = x^{-1}h_1xx^{-1}h_2^{-1}x = x^{-1}h_1h_2^{-1}x & \in x^{-1}Hx \text{ since } h_1, h_2^{-1} \in H \text{ for any } h_1, h_2 \in H.
\end{align*}
\]

Thus \( x^{-1}Hx \) is a subgroup of \( G \). It can be verified that \( h\phi = x^{-1}hx \) defines a one-to-one mapping \( \phi \) of \( H \) onto \( x^{-1}Hx \).

Hence the subgroups \( H \) and \( x^{-1}Hx \) have the same order \( \ell \).

Let \( A = \{K/K \text{ is a subgroup of order } \ell \text{ of } G\} \)

\( A \) is non-empty by hypothesis.

Let \( A = \bigcap_{K \in A} K \)

We already know that \( H \) is a subgroup of \( G \). Let \( x \in G, h \in H \).

We show that \( x^{-1}hx \) is in every subgroup of order \( \ell \).

Let \( L \) be a subgroup of order \( \ell \) in \( G \).

Now \( (x^{-1})^{-1}Lx^{-1} \in A \).

Hence \( h \in (x^{-1})^{-1}Lx^{-1} \)

\[
\Rightarrow x^{-1}hx \in x^{-1}(x^{-1})^{-1}Lx^{-1}x = eL = L
\]
Thus \( x^{-1}hx \in L \forall h \in H \)

Hence \( H \) is a normal subgroup of \( G \).

12.5.13 \textbf{SAQ} : Show that if a finite group \( G \) has exactly one subgroup \( H \) of a given order, then \( H \) is a normal subgroup of \( G \).

12.5.14 \textbf{SAQ} : Show that the set of all automorphisms of a group \( G \) forms a group under composition of functions and show that the set of all inner automorphisms of \( G \) is a normal subgroup of this group.

12.5.15 \textbf{Problem} : Let \( G \) be a group. Show that the set of all \( g \in G \), such that the inner automorphism \( i_g:G \to G \) is the identity automorphism \( i_e \), is a normal subgroup of \( G \).

\textbf{Answer} : \( i_g = i_e \) iff \( x i_g = x i_e \forall x \in G \)

iff \( g^{-1}xg = e^{-1}xe = exe = x \forall x \in G \)

iff \( xg = gx \forall x \in G \)

Let \( H = \{ g \in G / i_g = i_e \} \)

Then \( H = \{ g \in G / xg = gx \forall x \in G \} \)

Since \( xe = ex \forall x \in G \) we have \( e \in H \).

Let \( g_1, g_2 \in H \).

Then \( g_1x = xg_1 \) and \( g_2x = xg_2 \forall x \in G \)

\( g_2x = xg_2 \Rightarrow g_2xg_2^{-1} = xg_2g_2^{-1} = x \)

\( g_2^{-1}g_2 \times g_2^{-1} = g_2^{-1}x \)

\( \Rightarrow xg_2^{-1} = g_2^{-1}x \forall x \in G \)

Now \( x (g_1g_2^{-1}) = (xg_1)g_2^{-1} = (g_1x)g_2^{-1} \)

\( = g_1(xg_2^{-1}) = g_1(g_2^{-1}x) = (g_1g_2^{-1})x \forall x \in G \)

\( \therefore g_1g_2^{-1} \in H \)
Hence $H$ is a subgroup of $G$.

Let $h \in H, g \in G$

Then we have $hg = gh$.

So, $g^{-1}hg = h \in H$.

Hence $H$ is a normal subgroup of $G$.

12.5.16 Definition: If $N$ is a normal subgroup of a group $G$, the group of cosets of $N$ in $G$ under the induced operation is called the factor (quotient) group of $G$ modulo (by) $N$ and is denoted by $G/N$. The cosets of $N$ in $G$ are called residue classes modulo $N$.

12.5.17 Note: The induced operation in $G/N$ is given by $aNbN = abN$ for all $aN, bN \in G/N$.

12.5.18 SAQ: If $H$ is a subgroup of a group $G$ and $a \in G$, show that $aH = H(aH = H)$ if and only if $a \in H$.

12.5.19 Example: We know that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ under addition for all $n \in \mathbb{Z}^+$. Since $\mathbb{Z}$ is an abelian group $n\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}$.

If $m$ is any integer, there exist integers $p$ and $r$ such that $m = np + r, 0 \leq r < n$.

$$m + n\mathbb{Z} = np + r + n\mathbb{Z} = (np + n\mathbb{Z}) + r + n\mathbb{Z} = n\mathbb{Z} + (r + n\mathbb{Z}) \text{ since } np \in n\mathbb{Z}.$$  

$$= (0 + r\mathbb{Z}) + (r + n\mathbb{Z})$$

$$= (0 + r\mathbb{Z}) + n\mathbb{Z} = r + n\mathbb{Z}$$

Thus $\mathbb{Z}/n\mathbb{Z} = \{0 + n\mathbb{Z}, 1 + n\mathbb{Z}, \ldots, (n-1) + n\mathbb{Z}\}$

The mapping $\phi_n : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}_n$ given by $(m + n\mathbb{Z})\phi_n = m$ for $0 \leq m < n$ is an isomorphism of $\mathbb{Z}/n\mathbb{Z}$ with $\mathbb{Z}_n$.

12.5.20 SAQ: If $H$ is a subgroup of an abelian group $G$, prove that $G/H$ is abelian.

12.5.21 Problem: A group $G$ is called a torsion group if each element of $G$ is of finite order. If $G$ is a torsion group and $H$ is a normal subgroup of $G$ show that $G/H$ is a torsion group.
Answer: Let \( aH \in \frac{G}{H} \), \( a \in G \).

Then there exists \( m \in \mathbb{Z}^+ \) such that \( a^m = e \).

Now \( (aH)^m = a^mH = eH = H \) since \( e \in H \).

Thus \( aH \) is of finite order for each coset \( aH \) of \( H \) in \( G \).

\[ \therefore \frac{G}{H} \text{ is a torsion group.} \]

12.5.22 Problem: Show that \( A_n \) is a normal subgroup of \( S_n \) and compute \( \frac{S_n}{A_n} \).

Answer: We know that \( A_n \) is the set of all even permutations of \( \{1, 2, \ldots, n\} \) and is a subgroup of \( S_n \). To show that \( A_n \) is a normal subgroup of \( S_n \), we prove that \( \rho A_n \rho^{-1} \subseteq A_n \) for all \( \rho \in S_n \).

Let \( \rho \in S_n, \sigma \in A_n \)

If \( \rho \in A_n \) then \( \rho^{-1} \in A_n \) and \( \rho^{-1} \sigma \rho \in A_n \).

Suppose \( \rho \notin A_n \)

Then \( \rho \) and \( \rho^{-1} \) are odd permutations \( \sigma \) is an even permutation.

\[ \therefore \rho^{-1} \sigma \rho \text{ is an even permutation.} \]

So \( \rho^{-1} \sigma \rho \in A_n \). Thus \( \rho^{-1} A_n \rho \subseteq A_n \).

Hence \( A_n \) is a normal subgroup of \( S_n \).

Also if \( \rho \in A_n \) then \( \rho A_n = A_n \).

Suppose \( \rho_1 \) and \( \rho_2 \) are two odd permutations \( \rho_1^{-1} \rho_2 \) is an even permutation.

Thus \( \rho_1^{-1} \rho_2 A_n = A_n \)

Hence \( \rho_1 A_n = \rho_2 A_n \).

Hence there are only two cosets, namely \( A_n \) and the set of all odd permutations. If \( \rho \) is any odd permutation then \( \frac{S_n}{A_n} = \{ A_n, \rho A_n \} \) and the multiplication in \( \frac{S_n}{A_n} \) is given by the following table.
which shows that $\frac{S_n}{A_n}$ is isomorphic to $Z_2$.

12.5.23 **Remark**: Let $G$ be any group with identity element $e$. Then $\{e\}$ is a normal subgroup of $G$ and $\frac{G}{\{e\}}$ is isomorphic with $G$ under the natural mapping $g\{e\} \to g$ for each $g \in G$. $G$ is a normal subgroup of $G$ and $G/G$ is the trivial group with one element.

12.5.24 **Theorem**: A factor group of a cyclic group is cyclic.

**Proof**: Let $G$ be a cyclic group with a generator $a$.

Let $H$ be a normal subgroup of $G$.

Let $xH \in \frac{G}{H}, x \in G$.

Since $a$ generates $G$, $a^n = x$ for some integer $n$.

Now $(aH)^n = a^nH = xH$

Thus $aH$ generates $\frac{G}{H}$.

Hence $\frac{G}{H}$ is a cyclic group.

12.5.25 **Definition**: A **simple group** is a group which has no proper nontrivial normal subgroups.

Now we prove some results which will lead us to prove that the alternating group $A_n$ is a simple group for $n \geq 5$.

12.5.26 **Result**: For $n \geq 3, A_n$ contains every 3-cycle.

**Proof**: $(a,b,c) = (a,b)(a,c)$ is an even permutation for any $a, b, c \in \{1, 2, \cdots, n\}$.

12.5.27 **Result**: For $n \geq 3$, every element of $A_n$ is a product of 3-cycles.

**Proof**: Let $\sigma \in A_n$ for $n \geq 3$.

Then $\sigma = T_1 T_2 \cdots T_{2K-1} T_{2K}$ for some $K$ where the $T_j$ are transpositions for $1 \leq j \leq 2K$. 

\[
\begin{array}{|c|c|}
\hline
A_n & \rho A_n \\
\hline
\rho A_n & A_n \\
\hline
\end{array}
\]
Each product $T_{2j-1} T_{2j}$ is either of the form $(a,b)(c,d)$ or $(a,b)(a,c)$.

Since $(a,b)(c,d) = (a,c,d)(a,c,b)$ and $(a,b)(a,c) = (a,b,c)$ it follows that $T_{2j-1} T_{2j}$ is a product of 3-cycles.

Thus $\sigma$ is a product of 3-cycles.

12.5.28 Result: Let $n \geq 3$. Let $r, s$ be fixed elements of $\{1, 2, \ldots, n\}$. Then for any $i, j, k \in \{1, 2, \ldots, n\}$ we have $(r, s, i)(r, s, j)^2 (r, s, k)(r, s, i)^2 = (i, j, k)$.

Proof: We observe that $(r, s, i)(r, s, j)^2 = (s, i, j)$

$(r, s, k)(r, s, i)^2 = (s, k, i)$

Now $(r, s, i)(r, s, j)^2 (r, s, k)(r, s, i)^2$

$= (s, i, j)(s, k, i) = (i, j, k)$

12.5.29 Result: Let $n \geq 3$. Let $N$ be a normal subgroup of $A_n$. If $N$ contains a 3-cycle then $N = A_n$.

Proof: Suppose $(r, s, i) \in N$.

We first show that $(r, s, j) \in N$ for all $j \in \{1, 2, \ldots, n\}$.

Now $(i, j)(r, s) \in A_n$ and $(r, s, i)^2 \in N$.

Since $N$ is a normal subgroup of $A_n$ we have

$(i, j)(r, s)^{-1} (r, s, i)^2 (i, j)(r, s) \in N$

$(i, j)(r, s)^{-1} (r, s, i)^2 (i, j)(r, s)$

$= (r, s)(i, j)(r, s, i)^2 (i, j)(r, s) = (r, s, j) \in N \quad \text{---------- (1)}$

Let $(k, \ell, m)$ be any three cycle in $S_n$. Then $(r, s, k), (r, s, \ell), (r, s, m) \in N$ by (1).

Since $N$ is a subgroup, by result 12.5.28, we get that $(k, \ell, m) \in N$.

Thus $N$ contains all 3-cycles and hence all products of 3-cycles. By result 12.5.27 $A_n \subseteq N$. 
12.5.30 Theorem: For $n \geq 5$, $A_n$ is a simple group.

Proof: Let $N$ be a normal subgroup of $A_n$. Suppose $\sigma \in N$ and $\sigma$ is not the identity. We know that $\sigma$ can be written as a product of disjoint cycles.

We also know that disjoint cycles commute.

For the decomposition of $\sigma$ into disjoint cycles one of the following cases must hold.

1. $\sigma$ itself is a 3-cycle.
2. $\sigma$ has a cyclic factor $(a_1, a_2, \ldots, a_r)$, $r > 3$ (i.e.) $\sigma = (a_1, a_2, \ldots, a_r)\mu$. $\mu$ is a product of disjoint cycles.
3. There are at least two 3-cycles in $\sigma$. (i.e.) $\sigma = (a_1, a_2, a_3)(a_4, a_5, a_6)\mu$. $\mu$ is a product of disjoint cycles.
4. There is only one 3-cycle in $\sigma$ (i.e.) $\sigma = (a_1, a_2, a_3)\mu$. $\mu$ is a product of disjoint transpositions.
5. All cycles in $\sigma$ are of length $\leq 2$. (i.e.) $\sigma = (a_1, a_2)(a_3, a_4)\mu$ where $\mu$ is a product of an even number of transpositions.

We know that if $N$ has a 3-cycle then $N = A_n$.

Case (1): If $\sigma$ is a 3-cycle then $N = A_n$.

Case (2): Suppose $\sigma = (a_1, a_2, \ldots, a_r)\mu$, $r > 3$

Since $N$ is a normal subgroup of $A_n$ and $\sigma, \sigma^{-1} \in N$ we have

$$
(a_1, a_2, a_3)^{-1}\sigma(a_1, a_2, a_3)\sigma^{-1}
$$

$$
= (a_1, a_3, a_2)(a_1, a_2, \ldots, a_r)\mu(a_1, a_2, a_3)((a_1, a_2, \ldots, a_r)\mu)^{-1}
$$

$$
= (a_1, a_3, a_2)(a_1, a_2, \ldots, a_r)(a_1, a_2, a_3)(a_1, a_1, a_1, \ldots, a_2)\mu \mu^{-1}
$$

$$
= (a_1, a_3, a_4) \in N
$$

Hence $N = A_n$.

Case (3): Suppose $\sigma = (a_1, a_2, a_3)(a_4, a_5, a_6)\mu$. Then $\sigma^{-1}(a_1, a_2, a_4)\sigma^{-1}

$$
= (a_1, a_4, a_2)(a_1, a_2, a_3)(a_4, a_5, a_6)\mu(a_1, a_2, a_4)\mu^{-1}(a_4, a_5, a_6)^{-1}(a_1, a_2, a_3)^{-1}
$$
Normal Subgroups--
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By case (2). N has a 3-cycle.

Hence $N = A_n$.

Case (4): Suppose $\sigma = (a_1, a_2, a_3) \mu$ where $\mu$ is a product of disjoint transpositions.

Since each transposition is its own inverse we have $\mu^2$ is the identity permutation.

$\therefore \sigma^2 = (a_1, a_2, a_3)^2 \mu^2 = (a_1, a_3, a_2) \in N$.

Hence $N = A_n$.

Case (5): Suppose $\sigma = (a_1, a_2) (a_3, a_4) \mu$ where $\mu$ is a product of an even number of transpositions.

Since $n \geq 5$, choose $i \in \{1, 2, \ldots, n\}$ s.t. $i \neq a_1, a_2, a_3, a_4$

Let $\beta = (a_1, a_3, i)$

Now $(a_1, a_2, a_3)^{-1} \sigma (a_1, a_2, a_3) \sigma^{-1}$

$= (a_1, a_3) (a_2, a_4) = \alpha \in N$ and $(\beta^{-1} \alpha \beta) \alpha = (a_1, a_3, i) \in N$.

Hence $N = A_n$.

In all the five cases we have shown that $N = A_n$.

Hence $A_n$ is a simple group.

12.6 HOMOMORPHISMS:

12.6.1 Definition: Let $G$ and $G'$ be groups. Let $\phi : G \to G'$ be a mapping. $\phi$ is said to be a homomorphism if it satisfies the condition $(xy)\phi = (x\phi)(y\phi)$ for all $x, y \in G$.

12.6.2 Definition: A homomorphism $\phi : G \to G'$ of groups $G$ and $G'$ is said to be a nonomorphism if $\phi$ is a one-to-one mapping. $\phi$ is called an epimorphism if it is onto $G'$.

12.6.3 Definition: Let $\phi : G \to G'$ be a homomorphism of groups. Let $e'$ be the identity element of
12.6.4 Example: Let $Z$ be the additive group of integers and $R$ be the additive group of real numbers. Define $\phi: Z \to R$ by $n \phi = n$. Then $\phi$ is a homomorphism since $(n + m) \phi = n + m = n\phi + m\phi$ for all $m, n \in Z$. Ker $\phi = \{ n \in Z / n\phi = 0 \} = \{ n \in Z / n\phi = n = 0 \} = \{ 0 \}$.

Thus $\phi$ is a monomorphism.

12.6.5 Example: Let $R$ be the additive group of real numbers and let $Z$ be the additive group of integers. Define $\phi: R \to Z$ by $x \phi = \text{the greatest integer } \leq x$. This $\phi$ is not a homomorphism.

$$0.5 \phi = 0, \ 0.75 \phi = 0$$

$$(0.5 + 0.75) \phi = (1.25) \phi = 1$$

$$(0.5) \phi + (0.75) \phi = 0 + 0 = 0$$

$$(0.5 + 0.75) \phi \neq 0.5 \phi + 0.75 \phi$$

12.6.6 Example: Let $R^*$ be the multiplicative group of all non-zero real numbers. Define $\phi: R^* \to R^*$ by $x \phi = |x|$. Now $(xy) \phi = |xy| = |x||y| = (x \phi)(y \phi)$ for all $x, y \in R^*$.

Hence $\phi$ is a homomorphism.

$$\text{Ker } \phi = \left\{ x \in R^*/x \phi = 1 \right\} = \left\{ x \in R^*/|x| = 1 \right\} = \{1, -1\}$$

$\phi$ is not onto since $x \phi \neq -1$ for any $x \in R^*$.

12.6.7 SAQ: Show that there always exists a homomorphism of a group $G$ into any group $G'$.

12.6.8 Theorem: Let $\phi$ be a homomorphism of a group $G$ into a group $G'$. Let $e$, $e'$ be the identity elements of $G$ and $G'$ respectively. Then

(i) $e\phi = e'$

(ii) $x^{-1}\phi = (x\phi)^{-1}$ for all $x \in G$

(iii) The Kernel of $\phi$ is a normal subgroup of $G$.

Proof: (i) $e\phi = (e e)\phi = (e\phi)(e\phi)$
But $c\phi = (c\phi)e'$

$\therefore (e\phi)e' = (e\phi)(c\phi)$

By left cancellation law in $G'$ we have $c\phi = e'$.

(ii) $e' = c\phi = (x x^{-1})\phi = (x\phi)(x^{-1}\phi)$ for all $x \in G$.

Also $e' = (x\phi)(x\phi)^{-1}$

$\therefore (x\phi)(x\phi)^{-1} = (x\phi)(x^{-1}\phi)$

(iii) By (i) $e \in \text{Ker} \phi$. Let $x, y \in \text{Ker} \phi$

Then $x\phi = e'$, $y\phi = e'$.

Now $(xy^{-1})\phi = (x\phi)(y^{-1}\phi) = (x\phi)(y\phi)^{-1}$ by (ii)

$\therefore xy^{-1} \in \text{Ker} \phi$

Thus $\text{Ker} \phi$ is a subgroup of $G$.

Let $x \in \text{Ker} \phi$ and $g \in G$

$\left(g^{-1}xg\right)\phi = \left(g^{-1}\phi\right)(x\phi)(g\phi) = (g\phi)^{-1}e'(g\phi)$

$= (g\phi)^{-1}(g\phi) = e'$

$\therefore g^{-1}xg \in \text{Ker} \phi$

Thus $\text{Ker} \phi$ is a normal subgroup of $G$.

12.6.9 Theorem: Let $\phi$ be a homomorphism of $G$ into $G'$. $\phi$ is a monomorphism if and only if $\text{Ker} \phi = \{e\}$, where $e$ is the identity element of $G$.

Proof: Suppose that $\phi$ is a monomorphism.

Let $x \in \text{Ker} \phi$

Then $x\phi = e' = c\phi$ where $e'$ is the identity element of $G'$.

Since $\phi$ is a monomorphism, it is one-to-one.
Thus \( \text{Ker } \phi = \{e\} \)

Conversely suppose that \( \text{Ker } \phi = \{e\} \).

Suppose \( x \phi = y \phi \)

Then \( (x \phi)(y \phi)^{-1} = e' \)

\[
e' = (x \phi)(y \phi)^{-1} = (x \phi)(y^{-1} \phi) = (xy^{-1}) \phi
\]

\( \Rightarrow xy^{-1} \in \text{Ker } \phi = \{e\} \Rightarrow xy^{-1} = e \Rightarrow xy^{-1}y = ey \Rightarrow xc = y \Rightarrow x = y \)

Thus \( \phi \) is one-to-one.

12.6.10 Example : Let \( S_n \) be the group of permutations of \( \{1, 2, \cdots, n\} \). Let \( \{e, a\} \) be the group of order 2. Define \( \phi : S_n \rightarrow \{e, a\} \) by \( \sigma \phi = \begin{cases} e & \text{if } \sigma \text{ is even} \\ a & \text{if } \sigma \text{ is odd} \end{cases} \)

Then \( \phi \) is a homomorphism and \( \text{Ker } \phi = A_n \).

Proof : Let \( \sigma, \psi \in S_n \),

Case (i) : Suppose both \( \sigma \) and \( \psi \) are even.

Then \( \sigma \psi \) is even.

\[
\begin{align*}
\sigma \psi &= (\sigma \psi) \phi, \quad (\sigma \psi)(\psi \phi) = e \quad e = e \\
\therefore (\sigma \psi) \phi &= (\sigma \phi)(\psi \phi)
\end{align*}
\]

Case (ii) : Suppose \( \sigma \) and \( \psi \) are both odd. Then \( \sigma \psi \) is even.

\[
\begin{align*}
(\sigma \psi) \phi &= e \\
\sigma \phi &= \psi \phi = a \\
(\sigma \phi)(\psi \phi) &= aa = e \\
\therefore (\sigma \psi) \phi &= (\sigma \phi)(\psi \phi)
\end{align*}
\]

Case (iii) : Suppose \( \sigma \) is odd and \( \psi \) is even.

Then \( \sigma \psi \) is odd.
\((\sigma \psi) \phi = a\)

\((\sigma \phi)(\psi \phi) = a \epsilon = a\)

\(\therefore (\sigma \psi) \phi = (\sigma \phi)(\psi \phi)\)

Similarly \((\sigma \psi) \phi = (\sigma \phi)(\psi \phi)\) when \(\sigma\) is even and \(\psi\) is odd.

Thus \(\phi\) is a homomorphism.

\[\text{Ker}\phi = \{\sigma \in S_n / \sigma \phi = \epsilon\} = \{\sigma \in S_n / \sigma \text{ is even}\} = A_n\]

Thus the kernel of \(\phi\) is the alternating group \(A_n\).

12.6.11 SAQ : Let \(G\) be a group. Let \(A\) be a non-empty subset of \(G\).

Let \(\langle A \rangle = \left\{ x_1 x_2 \cdots x_n / n \geq 1, \text{ or } x_i or \ x_i^{-1} \in A \right\} \).

Show that \(\langle A \rangle\) is a subgroup of \(G\) and it is the smallest subgroup of \(G\) containing \(A \cdot \langle A \rangle\) is called the subgroup generated by \(A\).

12.6.12 Example : Let \(G\) be a group. Let \(A\) be a non-empty subset of \(G\) such that \(G = \langle A \rangle\). Let \(\phi : G \rightarrow G'\) be a homomorphism. Then \(\phi\) is completely determined by its values at the elements of \(A\).

Proof : Let \(x \in G\). Then \(x = x_1 x_2 \cdots x_n\) where \(x_i\) or \(x_i^{-1} \in A\) for some \(n\).

Now \(x \phi = (x_1 \phi)(x_2 \phi) \cdots (x_n \phi)\)

\[x_i \phi = \left(\left(x_i^{-1}\right)^{-1}\right) \phi = \left(\left(x_i^{-1}\right) \phi\right)^{-1}\]

implies that \(\phi\) is completely determined by the values of \(\phi\) on \(A\).

12.6.13 SAQ : What can you say about homomorphisms of a simple group?

12.6.14 Definition : Let \(f\) be a mapping of a set \(X\) into a set \(Y\). Let \(A \subseteq X\) and \(B \subseteq Y\). The set \(A = \{af / a \in A\}\) is called the image of \(A\) in \(Y\) under \(f\). The set \(Bf^{-1} = \{a \in X / a f \in B\}\) is called the inverse image of \(B\) in \(X\) under \(f\).

12.6.15 Theorem : Let \(G\) be a group and \(N\) be a normal subgroup of \(G\). Then the quotient group
$G/N$ is a homomorphic image of $G$.

**Proof:** Define $\phi : G \rightarrow G/N$ by $g\phi = gN$. For any $g_1, g_2 \in G$ we have

$$(g_1 g_2)\phi = g_1 g_2 N = (g_1 N)(g_2 N) = g_1 \phi g_2 \phi$$

Hence $\phi$ is a homomorphism.

Clearly $G\phi = G/N$.

Thus $\phi$ is onto.

This homomorphism $\phi$ of $G$ onto $G/N$ is called the canonical (or natural) homomorphism of $G$ onto $G/N$.

12.6.16 **SAQ:** Show that the Kernel of the canonical homomorphism of $G$ onto $G/N$ is $N$.

12.6.17 **Example:** Let $Z$ be the additive group of integers. Let $n$ be a positive integer. We know that $nZ$ is a normal subgroup of $Z$. $Z/nZ$ is the homomorphic image of $Z$ under the mapping $\phi$ defined by

$$m\phi = m + nZ \text{ for } m \in Z.$$ 

12.6.18 **Theorem:** Let $\phi : G \rightarrow G'$ be a homomorphism of groups. Let $H$ be a subgroup of $G$ and let $K$ be a subgroup of $G'$. Then

(i) $H\phi$ is a subgroup of $G'$. If $H$ is a normal subgroup of $G$ then $H\phi$ is a normal subgroup of $G\phi$.

(ii) $K\phi^{-1}$ is a subgroup of $G$. If $K$ is a normal sub-group of $G\phi$ then $K\phi^{-1}$ is a normal subgroup of $G$.

**Proof:** Let $e$ and $e'$ be the identity elements of $G$ and $G'$ respectively.

(i) Since $e' = e\phi \in H\phi$, we get that $H\phi$ is a non-empty subset of $G'$.

Let $x, y \in H\phi$.

Then $x = h_1\phi$, $y = h_2\phi$ for some $h_1, h_2 \in H$.

Since $H$ is a subgroup of $G$ we have $h_1 h_2^{-1} \in H$.

$$xy^{-1} = (h_1\phi)(h_2\phi)^{-1} = (h_1\phi)(h_2^{-1}\phi) = (h_1 h_2^{-1})\phi \in H\phi$$
Thus $H_{\phi}$ is a subgroup of $G'$. 

Suppose $H$ is normal in $G$. 

Now $H_{\phi}$ is a subgroup of the group $G_{\phi}$. 

Let $g_{\phi} \in G_{\phi}, \ (g \in G)$. 

Let $h_{\phi} \in H_{\phi}, \ (h \in H)$ 

Now $(g_{\phi})^{-1}(h_{\phi})(g_{\phi}) = (g^{-1}h_{\phi})(g_{\phi}) = (g^{-1}hg_{\phi}) \in H_{\phi}$, since $g^{-1}hg_{\phi} \in H$

Thus $H_{\phi}$ is a normal in $G_{\phi}$.

(ii) \[ e_{\phi} = e' \in K \]

$\therefore e \in K_{\phi}^{-1}$. Thus $K_{\phi}^{-1}$ is non-empty.

Let $x, y \in K_{\phi}^{-1}$. Then $x_{\phi}, y_{\phi} \in K$ 

Since $K$ is a subgroup of $G'$ we have

\[
(\phi(xy^{-1})) = (x_{\phi})(y^{-1}_{\phi}) = (x_{\phi}y^{-1}_{\phi})^{-1} \in K
\]

Thus $xy^{-1} \in K_{\phi}^{-1}$

Suppose that $K$ is a normal subgroup of $G_{\phi}$. 

Let $x \in K_{\phi}^{-1}$. Let $g \in G$. 

Now $x_{\phi} \in K$, $g_{\phi} \in G_{\phi}$ and $K$ is normal in $G_{\phi}$. 

\[
(g^{-1}x_{g})_{\phi} = (g^{-1}_{\phi})(x_{\phi})(g_{\phi}) = (g_{\phi})^{-1}(x_{\phi})(g_{\phi}) \in K
\]

$\therefore g^{-1}x_{g} \in K_{\phi}^{-1}$

Hence $K_{\phi}^{-1}$ is normal in $G$.

12.7 FUNDAMENTAL THEOREM OF HOMOMORPHISM:

Theorem 12.6.15 shows that every quotient group of a group $G$ is a homomorphic image of the group $G$. Now we prove that every homomorphic image of a group $G$ is isomorphic to a quotient group of $G$. 

In Theorem 12.6.8 we proved that the Kernel of a homomorphism $\phi : G \to G'$ of groups is a normal subgroup of $G$.

12.7.1 Theorem (Fundamental Theorem of homomorphism) : Let $\phi : G \to G'$ be a homomorphism of groups with $\ker \phi = N$. Then $G\phi$ is a group and there is a canonical isomorphism of $G/N$ onto $G\phi$.

Proof: By theorem 12.6.18, $G\phi$ is a subgroup of $G'$ and hence $G\phi$ is a group.

Define $\psi : G/N \to G\phi$ by $(aN)\psi = a\phi$ for $aN \in G/N$, we first prove that the definition of $\psi$ does not depend on the representative $a$ of $aN$.

Suppose $aN = bN$ for $a, b \in G$.

Let $e'$ be the identity element of $G'$. Then $ab^{-1} \in N = \ker \phi$

$$(a\phi)(b\phi)^{-1} = (a\phi)(b^{-1}\phi) = (ab^{-1})\phi = e'$$

$\Rightarrow a\phi = b\phi$

Thus $\psi$ is well-defined.

$$(aNbN)\psi = (abN)\psi = (ab)\phi = (a\phi)(b\phi)$$

$$= (aN)\psi(bN)\psi$$

$\therefore \psi$ is a homomorphism ----------- (1)

Suppose $(aN)\psi = e'$

Then $(aN)\psi = a\phi = e' \Rightarrow a \in \ker \phi = N$

$\therefore aN = N$.

$\therefore$ By Theorem 12.6.9 $\psi$ is one-to-one --------- (2)

Let $g \in G$, $g\phi \in G\phi$

Now $(gN)\psi = g\phi$ implies $\psi$ is onto ----------- (3)

From (1), (2) and (3) we conclude that $\psi$ is an isomorphism of $G/N$ onto $G\phi$. 
We note that from this theorem and theorem 12.6.15, the class of homomorphic images and the class of quotient groups of a group G are the same.

**12.7.2 Result**: Let $G$ and $G'$ be groups. Let $\phi : G \to G'$ be a homomorphism. Let $N$ be a normal subgroup of $G$. Let $N'$ be a normal subgroup of $G'$. If $N \phi \subseteq N'$ then there is a natural homomorphism of $G/N$ to $G'/N'$.

**Proof**: Suppose $g_1, g_2 \in G \ni g_1N = g_2N$.

Then $g_1 g_2^{-1} \in N$.

Hence $(g_1 \phi)(g_2 \phi)^{-1} = (g_1 \phi)(g_2^{-1} \phi) = (g_1 g_2^{-1}) \phi \in N\phi$

Since $N\phi \subseteq N'$. We have $(g_1 \phi)(g_2 \phi)^{-1} \in N'$

$\Rightarrow g_1 \phi N' = g_2 \phi N'$

Now define $\psi : G/N \to G'/N'$ by $gN\psi = g\phi N'$.

We have already seen that $\psi$ is well defined.

$$(g_1Ng_2N)\psi = (g_1g_2N)\psi = (g_1g_2)\phi N' = (g_1\phi N')(g_2\phi N')$$

Thus $\psi$ is a homomorphism.

**12.7.3 Problem**: Let $G$ be any group. Let $a$ be any element of $G$. Define $\phi : Z \to G$ by $n\phi = a^n$.

Show that $\phi$ is a homomorphism. Describe the image and possibilities for the Kernel of $\phi$.

**Answer**: Let $n,m \in Z$

Then $(n + m)\phi = a^{n+m} = a^n a^m = n\phi m\phi$

So $\phi$ is a homomorphism.

$Z\phi = \langle a^n/n \in Z \rangle = \langle a \rangle$. the cyclic subgroup of $G$ generated by $a$.

We know that $\text{Ker}\phi$ is a subgroup of $Z$.

.$\text{Ker}\phi = mZ$ for some non-negative integer $m$. If $m = 0$, then $mZ = \{0\}$ and $Z = Z/\{0\} = \langle a \rangle$.

Hence $\langle a \rangle$ is an infinite cyclic group. If $m \neq 0$ then $Z/mZ = \langle a \rangle$. Hence $\langle a \rangle$ is a finite cyclic
group of order m. Also since \( \langle a \rangle = Z_m \) we have \( \frac{Z}{mZ} = Z_m \).

12.7.4 **Example** : Let \( R \) be the additive group of real numbers. Let \( C^* \) be the multiplicative group of non-zero complex numbers. Define \( \phi : R \to C^* \) by \( x\phi = \cos x + i\sin x \)

\[
(x + y)\phi = \cos(x + y) + i\sin(x + y) \quad \text{(1)}
\]

\[
x\phi y\phi = (\cos x + i\sin x)(\cos y + i\sin y)
\]

\[
= \cos x \cos y - \sin x \sin y + i(\cos x \sin y + \cos y \sin x)
\]

\[
= \cos(x + y) + i\sin(x + y) \quad \text{(2)}
\]

From (1) and (2), \( (x + y)\phi = x\phi y\phi \)

Hence \( \phi \) is a homomorphism.

\( \ker \phi = \{ x \in R / \cos x + i\sin x = 1 \} \)

\[
= \{ x \in R / x = 2\pi n, n \in \mathbb{Z} \}
\]

\[
= \langle 2\pi \rangle \text{ in } R .
\]

By fundamental theorem of homomorphism \( \frac{R}{\langle 2\pi \rangle} \) is isomorphic to \( R\phi = \{ c \in C^*/|c| = 1 \} \).

12.7.5 **Definition** : Let \( G \) be a group. Let \( N \) be a normal subgroup of \( G \) which is not equal to \( G \). \( N \) is said to be a **maximal normal subgroup of** \( G \), if it is not properly contained in any proper normal subgroup of \( G \).

12.7.6 **Note** : A proper normal subgroup \( N \) of a group \( G \) is a maximal normal subgroup if and only if \( N \subseteq M \subseteq G \), \( M \) is a normal subgroup of \( G \) implies either \( N = M \) or \( M = G \).

12.7.7 **SAQ** : Show that the maximal normal subgroups of \( \mathbb{Z} \) are the ones generated by prime numbers.

12.7.8 **Theorem** : Let \( N \) be a normal subgroup of a group \( G \). \( N \) is a maximal normal subgroup of \( G \) if and only if \( \frac{G}{N} \) is a simple group.

**Proof** : Assume that \( N \) is a normal subgroup of \( G \). Let \( \phi : G \to \frac{G}{N} \) be the canonical homomorphism as in theorem 12.6.15.
(A) Assume that \( N \) is a maximal normal subgroup of \( G \). Let \( K \) be a normal subgroup of \( \frac{G}{N} \).

Then \( K\phi^{-1} \) is a normal subgroup of \( G \) by (ii) of theorem 12.6.18.

Since \( N\phi = \{ n\phi / n \in N \} = \{ nN / n \in N \} = \{ N \} \),

We have \( N\phi \subseteq K \Rightarrow N \subseteq K\phi^{-1} \)

Since \( N \) is a maximal normal subgroup of \( G \), we have either \( N = K\phi^{-1} \) or \( K\phi^{-1} = G \).

Since \( \phi \) is onto we have either \( N\phi = K \) or \( K = G\phi \) (i.e. \( \{ N \} = K \) or \( K = \frac{G}{N} \)).

Hence \( \frac{G}{N} \) is a simple group.

(B) Conversely suppose that \( \frac{G}{N} \) is a simple group. Suppose that \( M \) is a normal subgroup of \( G \) such that \( N \subseteq M \subseteq G \).

By (i) of theorem 12.6.18, \( M\phi \) is a normal subgroup of \( \frac{G}{N} \).

Then \( M\phi = \{ N \} \) or \( M\phi = \frac{G}{N} \)

Suppose that \( M\phi = \{ N \} \)

Let \( m \in M \). Then \( m\phi = N \).

(i.e.) \( mN = N \Rightarrow m \in N \)

Thus \( M \subseteq N \). So \( M = N \).

Suppose that \( M\phi = \frac{G}{N} \).

Let \( g \in G \). Then \( gN \in \frac{G}{N} = M\phi \)

Hence \( \exists m \in M \) such that \( gN = mN \).

\[ \Rightarrow gm^{-1} \in N \subseteq M \Rightarrow gm^{-1} \in M \Rightarrow gm^{-1}m \in M \Rightarrow g \in M \]

Thus \( G \subseteq M \). Hence \( M = G \).

Thus \( N \) is a maximal normal subgroup of \( G \).
Thus from (A) and (B) we get $N$ is a maximal normal subgroup of $G$, if and only if, $G/N$ is simple.

**12.7.9 Problem**: Let $G$ be a group. Let $I_G$ be the group of inner automorphisms of $G$. Define the mapping $\phi : G \rightarrow I_G$ by $g\phi = i_g$ where $i_g : G \rightarrow G$ is the inner automorphism given by $x_i_g = g^{-1}xg$ for $x \in G$. Show that $\phi$ is a homomorphism of $G$ onto $I_G$. Find the Kernel of $\phi$. Describe when $\phi$ is an isomorphism.

**Answer**: Let $g_1, g_2 \in G$. Let $x \in G$.

\[
x(g_1 g_2)\phi = x i_{g_1 g_2} = (g_1 g_2)^{-1} x g_1 g_2 =
\]

\[
= g_2^{-1} g_1^{-1} x g_1 g_2
\]

\[
x(g_1 \phi)(g_2 \phi) = x (i_{g_1} i_{g_2}) = (g_1^{-1} x g_1) i_{g_2}
\]

\[
= g_2^{-1} g_1^{-1} x g_1 g_2
\]

So $x(g_1 g_2)\phi = x(g_1 \phi)(g_2 \phi)$ for all $x \in G$.

Hence $(g_1 g_2)\phi = g_1 \phi g_2 \phi$

Thus $\phi$ is a homomorphism of $G \rightarrow I_G$. Clearly $\phi$ is onto.

Let $I_G$ be the identity mapping of $G$.

\[
Ker \phi = \{ g \in G / g\phi = I_G \} = \{ g \in G / i_g = I_G \}
\]

\[
= \{ g \in G / x_i_g = x I_G \} = \{ g \in G / g^{-1} x g = x \forall x \in G \}
\]

\[
= \{ g \in G / x g = g x \forall x \in G \}
\]

By Theorem 12.6.9, $\phi$ is one-to-one, iff, $Ker \phi = \{ e \}$. Thus $\phi$ is an isomorphism $\Rightarrow$.

\[
\{ g \in G / x g = g x \forall x \in G \} = \{ e \}
\]

**12.7.10 Definition**: Let $G$ be a group. Then $\{ g \in G / x g = g x \forall x \in G \}$ is called the **centre of $G$**.
12.8 ANSWERS TO SELF ASSESSMENT QUESTIONS (SAQ’s)

12.5.7 SAQ :
Answer : Let \( H \) be a subgroup of an abelian group \( G \).

Since \( G \) is abelian we have \( xy = yx \ \forall \ x, y \in G \). In particular \( ah = ha \ \forall \ a \in G, h \in H \).

Thus \( aH = \{ ah/h \in H \} = \{ ha/h \in H \} = Ha \).

Thus every left coset of \( H \) in \( G \) is a right coset of \( H \) in \( G \).

\( \therefore H \) is a normal subgroup of \( G \).

12.5.11 SAQ :
Answer : Let \( G \) be a group with identity \( e \) and let \( X \) be a non-empty collection of normal subgroups of \( G \).

Let \( N = \bigcap_{H \in X} H \)

Since \( e \in H \) for all \( H \in X \), we have \( e \in N \). Let \( a, b \in N \). Then \( a, b \in H \) for all \( H \in X \) and hence \( ab^{-1} \in H \) for all \( H \in X \).

So, \( ab^{-1} \in \bigcap_{H \in X} H = N \)

Thus \( N \) is a subgroup of \( G \).

Let \( a \in G, n \in N \)

Then \( n \in H \) for all \( H \in X \).

So \( a^{-1}na \in H \) for all \( H \in X \), since \( H \) is a normal subgroup of \( G \).

Therefore \( a^{-1}na \in \bigcap_{H \in X} H = N \)

So, \( a^{-1}N \subseteq N \)

Hence \( N \) is a normal subgroup of \( G \).

12.5.13 SAQ :
Answer : Suppose that \( H \) is the only subgroup of a given order, say \( m \), of \( G \).

In the answer to problem 12.5.12 we have seen that for any subgroup \( H \) of \( G \), \( x^{-1}Hx \) is a subgroup of \( G \) for all \( x \in G \) and \( m = |H| = |x^{-1}Hx| \).
Since $H$ is the only subgroup of order $m$ of $G$. We have $H = x^{-1}Hx$.

\[ \therefore H \text{ is a normal subgroup of } G. \]

12.5.14 SAQ:

**Answer:** Let $G$ be a group. Let $\text{Aut}(G)$ be the set of all automorphisms of $G$. The operation in $\text{Aut}(G)$ is defined by $x \phi \psi = (x \phi) \psi$. With $G_1 = G_2 = G_3 = G$ in part (c) of answer to SAQ 11.4.2, we get $\phi \psi$ is an automorphism of $G$. Thus $\phi \psi \in \text{Aut}(G)$.

We also know that the composition of mappings is associative.

Part (a) of answer to SAQ 11.4.2 we get that the identity mapping $I_G$ of $G$ onto $G$ is an automorphism of $G$.

It is clear that $I_G \phi = \phi I_G$ for all $\phi \in \text{Aut}(G)$. Part (b) of answer to SAQ 11.4.2 with $G_1 = G_2 = G$ gives $\phi^{-1}$ is an automorphism of $G$ for any automorphism $\phi$ of $G$.

Also $\phi \phi^{-1} = \phi^{-1} \phi = I_G$.

Thus $\text{Aut}(G)$ is a group under composition of mappings.

Let $\text{Inaut}(G)$ be the set of all inner automorphisms of $G$.

Let $i_{g_1}, i_{g_2} \in \text{Inaut}(G)$.

\[
x(i_{g_1} i_{g_2}) = (x i_{g_1}) i_{g_2} = (g_1^{-1} x g_1) i_{g_2}
\]

\[
= g_2^{-1} (g_1^{-1} x g_1) g_2 = (g_2^{-1} g_1^{-1}) x (g_1 g_2)
\]

\[
= (g_1 g_2)^{-1} x (g_1 g_2) = x i_{g_1} g_2 \text{ for all } x \in G
\]

\[ \therefore i_{g_1} i_{g_2} \in \text{Inaut}(G) \]

\[
x i_e = e^{-1} x e = exe = x = x I_G \forall x \in G.
\]

\[ \Rightarrow I_G = i_e \in \text{Inaut}(G). \]

\[
x i_g i_g^{-1} = (g^{-1} x g) i_g^{-1} = (g^{-1})^{-1} (g^{-1} x g) g^{-1}
\]
\[ gg^{-1} x gg^{-1} = exe = x = xI_G \]

\[ i_g = I_G. \]

Similarly \( i_g = I_G. \)

Thus \( \text{Iaut}(G) \) is a subgroup of \( \text{Aut}(G). \)

Let \( \phi \in \text{Aut}(G) \) and \( i_g \in \text{Iaut}(G). \)

\[
\begin{align*}
x (\phi^{-1} i_g \phi) &= \left( (x \phi^{-1}) i_g \right) \phi = \left( g^{-1} (x \phi^{-1}) g \right) \\
&= (g^{-1} \phi) (x \phi^{-1}) \phi (g \phi) = (g \phi)^{-1} (x (\phi^{-1} \phi)) (g \phi) \\
&= (g \phi)^{-1} (x I_G) (g \phi) = (g \phi)^{-1} x g \phi = x i_{g \phi} \forall x \in G \\
\therefore \phi^{-1} i_g \phi &\in \text{Iaut}(G).
\end{align*}
\]

\[ \therefore \text{The set of all inner automorphisms of } G \text{ is a normal subgroup of the group of all automorphisms of } G. \]

**12.5.18 SAQ:**

**Answer:** Let \( H \) be a subgroup of a group \( G \) with identity \( e \). Let \( a \in G \).

Suppose that \( aH = H \).

\( a = ae \in aH = H \text{ since } e \in h. \)

\[ \therefore a \in H. \]

Conversely suppose that \( a \in H. \)

Then for any \( h \in H \) we have \( ah \in H. \)

\[ \text{So } aH \subseteq H \quad \text{-------- (1)} \]

Since \( a \in H \) we also have \( a^{-1} \in H \) and \( a^{-1} h \in H \) for all \( h \in H. \)

\[ h = ch = (a a^{-1}) h = a(a^{-1} h) \in aH \text{ for all } h \in H. \]

\[ \therefore H \subseteq aH \quad \text{-------- (2)} \]
From (1) and (2) we have \( aH = H \).

Similarly we can prove that \( Ha = H \), if and only if \( a \in H \).

12.5.20 SAQ :

**Answer:** Let \( H \) be a subgroup of an abelian group \( G \).

Let \( aH, bH \in \frac{G}{H} \), \((a, b \in G)\).

Since \( a, b \in G \) and \( G \) is abelian we have \( ab = ba \).

Now \( aHbH = abH = baH = bHaH \) in \( \frac{G}{H} \).

Hence \( \frac{G}{H} \) is abelian.

12.6.7 SAQ :

**Answer:** Let \( G \) and \( G' \) be groups and \( e' \) be the identity element of \( G' \). Define \( \phi : G \rightarrow G' \) by \( x\phi = e' \forall x \in G \). Then \((xy)\phi = e' \phi = (x\phi)(y\phi) \forall x, y \in G \).

So \( \phi \) is a homomorphism.

12.6.11 SAQ :

**Answer:** Since \( A \) is non-empty, there exists an element \( a \) in \( A \).

Now \( e = aa^{-1} \in \langle A \rangle \).

\( : \langle A \rangle \neq \phi \)

Let \( x = x_1 x_2 \cdots x_n, \ y = y_1 y_2 \cdots y_m \in \langle A \rangle, \ x_i \) or \( x_i^{-1} \in A \) and \( y_j \) or \( y_j^{-1} \in A \).

Then \( xy^{-1} = x_1 x_2 \cdots x_n y_m^{-1} y_{m-1}^{-1} \cdots y_2^{-1} y_1^{-1} \)

Now, \( x_i \) or \( x_i^{-1} \in A \) and \( y_j^{-1} \) or \( (y_j^{-1})^{-1} = y_j \in A \).

Thus \( xy^{-1} \in \langle A \rangle \).

Hence \( \langle A \rangle \) is a subgroup of \( G \) and clearly contains \( A \).

Let \( H \) be any subgroup containing \( A \).
Let $x = x_1 x_2 \cdots x_n \in \langle A \rangle$.

Then $x_i$ or $x_i^{-1} \in A$

$\therefore x_i$ or $\therefore x_i^{-1} \in H$.

In any case $x_i \in H$ since $H$ is a subgroup and $x_i = (x_i^{-1})^{-1}$ for each $i$.

Since $H$ is a subgroup we have $x = x_1 x_2 \cdots x_n \in H$.

Thus $\langle A \rangle \subseteq H$.

Thus $\langle A \rangle$ is the smallest subgroup of $G$ containing $A$.

12.6.13 SAQ:
Answer: Let $\phi: G \to G'$ be a homomorphism of groups. Let $G$ be a simple group. By (iii) of theorem 12.6.8, $\text{Ker} \phi$ is a normal subgroup of $G$. Since $G$ is simple we have either $\text{Ker} \phi = \{e\}$ or $\text{Ker} \phi = G$, where $e$ is the identity of $G$.

If $\text{Ker} \phi = \{e\}$ then $\phi$ is a monomorphism.

If $\text{Ker} \phi = G$, then $g \phi = e' \forall g \in G$ where $e'$ is the identity of $G'$.

12.6.16 SAQ:
Answer: Let $\phi: G \to \frac{G}{N}$ be the canonical homomorphism. We know that $N$ is the identity element of $\frac{G}{N}$.

$\text{Ker} \phi = \{g \in G / g \phi = N\} = \{g \in G / gN = N\} = \{g \in G / g \in N\} = N$

12.7.7 SAQ:
Answer: We know that all subgroups of $Z$ are normal (since $Z$ is an abelian group) and are of the form $nZ$ for some non-negative integer $n$.

(i) Suppose $nZ$ is a maximal subgroup of $Z$. Let $n = k\ell$ be a factorization of $n$ where $k$ and $\ell$ are positive integers.

Then $nZ = kZ \subseteq kZ \subseteq Z$

Since $nZ$ is maximal we have either $nZ = kZ$ or $kZ = Z$. 
If \( n \mathbb{Z} = k \mathbb{Z} \) then \( k = nm \) for some positive integer \( m \).
\[
n = k \ell = nm \ell \Rightarrow m \ell = 1 \Rightarrow m = \ell = 1
\]

If \( k \mathbb{Z} = \mathbb{Z} \), then \( l = km \) for some positive integer \( m \)
\[
\Rightarrow k = l
\]
Thus either \( k = 1 \) or \( l = 1 \).

Conversely suppose that \( n \) is a prime number.

Suppose \( n \mathbb{Z} \subseteq m \mathbb{Z} \subseteq \mathbb{Z} \).

Then \( n \in n \mathbb{Z} \Rightarrow n \in m \mathbb{Z} \Rightarrow n = m \ell \) for some positive integer \( \ell \).
\[
\Rightarrow m = 1 \text{ or } \ell = 1 \text{ since } n \text{ is prime.}
\]

If \( m = 1 \) then \( m \mathbb{Z} = \mathbb{Z} \)

If \( \ell = 1 \) then \( m = n \), and \( n \mathbb{Z} = m \mathbb{Z} \).

Thus \( n \mathbb{Z} \) is a maximal normal subgroup of \( \mathbb{Z} \).

### 12.9 EXERCISES

**12.9.1**: Find the order of the factor group \( \mathbb{Z}_{36} / \langle 5 \rangle \).

**12.9.2**: Find the order of the coset \( 5 + \langle 4 \rangle \) in the factor group \( \mathbb{Z}_{12} / \langle 4 \rangle \).

**12.9.3**: Find all subgroups of \( S_3 \) which are conjugate to \( \{p_0, p_1\} \) (Ref. Example 10.4.10)

**12.9.4**: A group \( G \) is said to be torsion free if no non-identity element of \( G \) is of finite order. In an abelian group \( G \) show that the set \( T \) of all elements of finite order is a normal subgroup of \( G \). This subgroup \( T \) of \( G \) is called the torsion subgroup of \( G \). Show that \( G / T \) is torsion free.

\[\text{[Hint: } (aT)^n = a^n T = T \Rightarrow a^n \in T \Rightarrow (a^n)^m = a^{nm} = e \text{ for some } m. \Rightarrow a \in T \Rightarrow aT = T \text{]}\]

**12.9.5**: Let \( H \) be a normal subgroup of a finite group \( G \) and let \( m = (G : H) \). Show that \( a^m \in H \) for every \( a \in G \).

\[\text{[Hint: If } G \text{ is a finite group and } a \in G \text{ then } a^{o(G)} = e\]
12.9.6: Let $S$ be a subset of a group $G$. Show that the smallest normal subgroup containing $S$ exists.

[Hint: SAQ 12.5.11].

12.9.7: Let $N$ be a normal subgroup and $H$ be a subgroup of a group $G$. Show that $H \cap N$ is a normal subgroup of $H$. Show by an example that $H \cap N$ need not be normal in $G$.

12.9.8: Show that the relation, $A \sim B$ iff $A$ and $B$ are conjugate subgroups on the collection of all subgroups of a group $G$, is an equivalence relation. Characterize those subgroups $G$ whose equivalence classes are singletons.

12.9.9: If $N$ is a normal subgroup and $H$ is a subgroup of a group $G$, prove that $HN = NH$.

12.9.10: If $N$ and $M$ are normal subgroups of a group $G$, show that $NM$ is a normal subgroup of $G$.

12.9.11: If $H$ and $K$ are normal subgroups of a group $G$ such that $\{H \cap K \neq \emptyset\}$, then show that $hk = kh$ for all $h \in H, k \in K$.

12.9.12: Let $\phi: G \to G'$ be a homomorphism of groups. Show that $(x_1 x_2 \cdots x_n)\phi = (x_1 \phi)(x_2 \phi) \cdots (x_n \phi)$ for all $x_1, x_2, \cdots x_n \in G, n \geq 1$ and hence $x^n \phi = (x \phi)^n$.

12.9.13: Determine whether the following functions are homomorphisms. If so, determine the image and the Kernel.

(a) $\phi: \mathbb{Z}_6 \to \mathbb{Z}_2$, given by $x \phi = \text{remainder of } x \text{ when divided by } 2$.

(b) $\phi: \mathbb{Z}_9 \to \mathbb{Z}_2$, given by $x \phi = \text{remainder of } x \text{ when divided by } 2$.

12.9.14: Determine the number of homomorphisms for the following.

(i) $\mathbb{Z}$ onto $\mathbb{Z}$.  (ii) $\mathbb{Z}$ into $\mathbb{Z}_2$.  (iii) $\mathbb{Z}$ onto $\mathbb{Z}_2$.  (iv) $\mathbb{Z}$ into $\mathbb{Z}_8$.  (v) $\mathbb{Z}$ onto $\mathbb{Z}_8$.

(vi) $\mathbb{Z}_{12}$ onto $\mathbb{Z}_5$.  (vii) $\mathbb{Z}_{12}$ onto $\mathbb{Z}_6$.  (viii) $\mathbb{Z}_{12}$ into $\mathbb{Z}_6$.

(ix) $\mathbb{Z}_{12}$ into $\mathbb{Z}_{14}$.  (x) $\mathbb{Z}_{12}$ into $\mathbb{Z}_{16}$.  [Hint: Result 12.6.12].

12.9.15: Let $G_1, G_2$ and $G_3$ be groups. Let $\phi_1: G_1 \to G_2$, $\phi_2: G_2 \to G_3$ be homomorphisms. Show that $\phi_1 \phi_2: G_1 \to G_3$ is a homomorphism.

12.9.16: Let $G_1$ and $G_2$ be groups. Let $\phi_1: G_1 \to G_2$, $\phi_2: G_2 \to G_1$ be homomorphisms such that $\phi_1 \phi_2$ is the identity map of $G_1$ and $\phi_2 \phi_1$ is the identity map of $G_2$. Show that $\phi_1$ and $\phi_2$ are isomorphisms. (Hint: Show that $\phi_1$ and $\phi_2$ are one-to-one and onto).
Let $G$ be a finite abelian group of order $n$. Let $m$ be a positive integer relatively prime to $n$. Show that the map $\phi_m : G \to G$ defined by $a\phi_m = a^m$ is an isomorphism (This implies that the equation $x^m = a$ always has a unique solution in a finite abelian group if $m$ is relatively prime to the order of the group.

**12.10 MODEL EXAMINATION QUESTIONS**

**12.10.1**: If $H$ is a subgroup of a group $G$, show that the operation of induced multiplication is well defined on the left (right) cosets of $H$ if, and only if, every left coset is a right coset.

**12.10.2**: Define a normal subgroup of a group $G$. Show that the following conditions on a subgroup $H$ of $G$ are equivalent.

(i) $H$ is a normal subgroup of $G$.

(ii) $g^{-1}Hg \subseteq H$ for all $g \in G$.

(iii) Every left coset of $H$ in $G$ is a right coset of $H$ in $G$.

**12.10.3**: Show that the alternating group $A_n$ is simple for $n \geq 5$.

**12.10.4**: If a finite group $G$ contains exactly one subgroup $H$ of a given order, show that $H$ is a normal subgroup of $G$.

**12.10.5**: Define a homomorphism of groups. If $\phi : G \to G'$ is a homomorphism of groups, $H$ and $K$ are subgroup of $G$ and $G'$ are respectively, show that $H\phi$ and $K\phi^{-1}$ are subgroups of $G'$ and $G$ respectively. If $H$ is a normal in $G$ show that $H\phi$ in normal in $G\phi$. If $K$ is normal in $G'$ show that $K\phi^{-1}$ is normal in $G$.

**12.10.6**: Show that a group $G'$ is a homomorphic image of a group $G$ iff $G'$ is isomorphic to a quotient group of $G$.

**12.10.7**: Define the Kernel of a homomorphism of groups. Let $R$ be the additive group of real numbers and let $C^\times$ be the multiplicative group of nonzero complex numbers. Show that the mapping $\phi : R \to C^\times$ defined by $x\phi = \cos x + i\sin x$ is a homomorphism and find its kernel.

**12.10.8**: Let $G$ be a group. Let $M$ be a normal subgroup of $G$. Show that $M$ is a maximal normal subgroup of $G$ iff $G/M$ is simple.

**12.10.9**: Show that a homomorphism of groups is a monomorphism of groups iff its kernel is the identity element of the group.

**12.10.10**: Determine the number of homomorphisms of $Z$ onto $Z_8$. 
12.10.11: Show that the composite mapping of two homomorphisms of groups (if defined) is a homomorphism.

12.10.12: Let $G$ be a group. Let $I_G$ be the group of inner automorphisms of $G$. Show that the mapping $\psi: G \to I_G$ defined by $g\psi = i_g$ is a homomorphism and find the Kernel of $\psi$.

12.11 REFERENCE BOOKS

2. Topics in Algebra, by I.N. Herstein; Wiley Eastern Ltd., New Delhi, 1975

Lesson Writer

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Lesson - 13

VECTOR DIFFERENTIATION

13.1 OBJECTIVE OF THE LESSON

This lesson deals with vector function of a single real variable and of several real variables. The differential calculus of a real valued function of a real variable is extended, to some extent, such as limit, continuity, differentiability, to vector function of a single real variable. The concept of partial differentiation of a vector function of several real variables is also introduced. Some results are proved and some problems are discussed.

13.2 STRUCTURE OF THE LESSON

This lesson has the following components.

13.3 Introduction
13.4 Vector functions, Scalar functions, Vector point functions, Scalar point functions
13.5 Limits, Continuity
13.6 Differentiation
13.7 Higher order derivatives
13.8 Partial differentiation
13.9 Answers to SAQ's
13.10 Summary
13.11 Technical Terms
13.12 Exercises
13.13 Answers to Exercises
13.14 Model Examination Questions
13.15 Reference Books

13.3 INTRODUCTION

This lesson deals with vector functions and extends the concepts of differential calculus to these functions. This makes the calculus of these vector functions the natural instrument for the engineer and the physicist in solid mechanics, fluid flow, heat flow, electrostatics and so on.

13.4 VECTOR FUNCTION, SCALAR FUNCTION, VECTOR POINT FUNCTIONS AND SCALAR POINT FUNCTIONS :

13.4.1 Definitions : Let \( \phi \neq S \subseteq \mathbb{R} \). Let \( V \) denote the set of all three-dimensional vectors (\( \mathbb{R} \) is the set of all real numbers).

A function \( \widetilde{f} : S \rightarrow V \) is called a vector function on \( S \).

A function \( f : S \rightarrow \mathbb{R} \) is called a scalar function on \( S \).
13.4.2 Definition: Let $\phi \subseteq D \subseteq \mathbb{R}^3$ and $V$ be as in 13.4.1.

A function $\mathbf{f}: D \to V$ is called a vector point function on $D$.

A function $f: D \to \mathbb{R}$ is called a scalar point function on $D$.

13.4.3 Note: If $\mathbf{t}, \mathbf{j}, \mathbf{k}$ are three mutually perpendicular unit vectors along the three co-ordinate axes, $X, Y, Z$ respectively, then the vector function $\mathbf{f}$ can be written as

$$\mathbf{f}(t) = f_1(t)\mathbf{t} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$$

where $f_1, f_2, f_3$ are real valued functions, which are called the components of $\mathbf{f}$.

13.4.4 Example: If $\mathbf{f}(t) = a\cos t\mathbf{t} + a\sin t\mathbf{j} + 0\mathbf{k}$ for all $t \in [0, 2\pi]$, then the equation represents the circle in the XY-plane with centre at the origin and radius 'a' units, where $a > 0 (a \in \mathbb{R})$.

$$x^2 + y^2 = a^2$$

13.4.5 Example: $\mathbf{f}(t) = at^2 + 2at\mathbf{j} + o\mathbf{k}, \forall t \in \mathbb{R}$ represents a parabola in the XY-plane $\left(y^2 = 4ax\right)$.

13.4.6 Example: The equation $\mathbf{f}(t) = a\cos t\mathbf{t} + b\sin t\mathbf{j}$, $\forall t \in [0, 2\pi]$ is the vector equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a, b \in \mathbb{R}, a \neq b)$.

13.5 LIMITS, CONTINUITY

13.5.1 Definition: Let $a, \delta \in \mathbb{R}$ and $\delta > 0$. The set $\{x \in \mathbb{R} / a - \delta < x < a + \delta\}$ is called the $\delta$-neighbourhood of $a$ or a neighbourhood of $a$ with radius $\delta$ and $\{x \in \mathbb{R} / a - \delta < x < a + \delta, x \neq a\}$ is called the deleted $\delta$-neighbourhood of $a$ or a deleted neighbourhood of $a$.

13.5.2 Definition: Let $a \in \mathbb{R}$ and let $\mathbf{f}$ be a vector function defined in a deleted neighbourhood of $a$. If $\exists$ a vector $\mathbf{L}$ given $\epsilon > 0$, $\delta > 0 \Rightarrow |t - a| < \delta \Rightarrow |\mathbf{f}(t) - \mathbf{L}| < \epsilon$, then we say that the limit of $\mathbf{f}(t)$ as $t$ tends to $a$ is $\mathbf{L}$ and we write $\lim_{t \to a} \mathbf{f}(t) = \mathbf{L}$.

As in the case of real valued functions, we can prove the following results on limits, whenever the limits under consideration exist.

$$\lim_{t \to a} \left[\mathbf{f}(t) \pm \mathbf{g}(t)\right] = \lim_{t \to a} \mathbf{f}(t) \pm \lim_{t \to a} \mathbf{g}(t)$$

$$\lim_{t \to a} k\mathbf{f}(t) = k \lim_{t \to a} \mathbf{f}(t), \text{ where } k \in \mathbb{R}$$
Vector Differentiation

13.3 Differential Equation, Abstract Algebra...

13.3 Definition: Let \( \mathbf{f} \) be a vector function defined in a neighborhood of \( a \in \mathbb{R} \). We say that \( \mathbf{f} \) is continuous at \( a \) if given \( \varepsilon > 0 \), \( \exists \delta > 0 \),

\[
|t - a| < \delta \Rightarrow |\mathbf{f}(t) - \mathbf{f}(a)| < \varepsilon
\]

i.e. \( \mathbf{f} \) is continuous at \( a \) if \( \lim_{t \to a} \mathbf{f}(t) = \mathbf{f}(a) \).

13.4 Definition: Let \( \mathbf{f} \) be a vector function defined in a neighborhood of \( a \). We say that \( \mathbf{f} \) is right continuous at \( a \) if given \( \varepsilon > 0 \), \( \exists \delta > 0 \),

\[
|\mathbf{f}(t) - \mathbf{f}(a)| < \varepsilon \quad \text{whenever} \quad t \in (a, a + \delta)
\]

13.5 Definition: Let \( \mathbf{f} \) be a vector function defined in a neighborhood of \( a \). We say that \( \mathbf{f} \) is left continuous at \( a \) if given \( \varepsilon > 0 \), \( \exists \delta > 0 \),

\[
|\mathbf{f}(t) - \mathbf{f}(a)| < \varepsilon \quad \text{whenever} \quad t \in (a - \delta, a)
\]

It can be proved that if \( \mathbf{f} \) and \( \mathbf{g} \) are continuous at \( a \), then \( \mathbf{f} + \mathbf{g}, \mathbf{f} - \mathbf{g}, \mathbf{f} \cdot \mathbf{g}, \mathbf{f} \times \mathbf{g} \) and \( k\mathbf{f} \), \( (k \in \mathbb{R}) \), are continuous at \( a \) (Here, \( \cdot \) and \( \times \) denote the scalar product and vector product of vectors).

13.6 Note: If \( \mathbf{f} \) is continuous at every \( t \) in the open interval \( (a, b) \), then we say that \( \mathbf{f} \) is continuous in \( (a, b) \). If further, \( \mathbf{f} \) is right continuous at \( a \) and \( \mathbf{f} \) is left continuous at \( b \), then we say that \( \mathbf{f} \) is continuous in the closed interval \( [a, b] \).

13.6 DIFFERENTIATION

13.6.1 Definition: Let \( \mathbf{f} \) be a vector function defined in a neighborhood of \( a \in \mathbb{R} \). If \( \lim_{t \to a} \frac{\mathbf{f}(t) - \mathbf{f}(a)}{t - a} \) exists, then we say that \( \mathbf{f} \) is differentiable (derivable) at \( a \) and the limit is called the derivative or differential coefficient of \( \mathbf{f} \) at \( a \) and is denoted by \( f^{-1}(a) \) or \( \left( \frac{d\mathbf{f}}{dt} \right)_{t=a} \).

13.6.2 Theorem: If \( \mathbf{f} \) is differentiable at \( a \), then \( \mathbf{f} \) is continuous at \( a \).
Proof: \[ \lim_{t \to a} \left[ \frac{\bar{F}(t) - \bar{F}(a)}{t - a} \right] = \lim_{t \to a} \left[ \frac{\bar{F}(t) - \bar{F}(a)}{t - a} \right] (t - a) \]

\[ = \lim_{t \to a} \frac{\bar{F}(t) - \bar{F}(a)}{t - a} \cdot \lim_{t \to a} (t - a) \]

\[ = \bar{F}'(a) \cdot 0 \]

\[ = 0 \]

which implies \( \lim_{t \to a} \bar{F}(t) = \bar{F}(a) \), so that \( \bar{F} \) is continuous to \( a \).

However, the converse of Theorem 13.6.2 is false, which can be seen from:

13.6.3 Example: Define \( \bar{F}(t) = |t| I_{\forall t \in \mathbb{R}} \). Then \( \bar{F} \) is continuous but not differentiable at \( t = 0 \), because

\[ \lim_{t \to 0} \bar{F}(t) = \lim_{t \to 0} |t| I = 0 I = \bar{0} = \bar{F}(0) \] and

\[ \lim_{t \to 0} \frac{\bar{F}(t) - \bar{F}(0)}{t - 0} = \lim_{t \to 0} \frac{|t| I - \bar{0} I}{t - 0} = \lim_{t \to 0} \frac{|t| I}{t}, \] which does not exist, since

\[ \lim_{t \to 0^+} I = 1 \text{ and } \lim_{t \to 0^-} I = -1 \]

It can be observed from the above that continuity is certainly a necessary but not sufficient condition for differentiability.

13.6.4 Note: \( \frac{d \bar{F}}{dt} \) can also be written as

\[ \frac{d \bar{F}}{dt} = \lim_{x \to t} \frac{\bar{F}(x) - \bar{F}(t)}{x - t} \quad \text{(or)} \quad \lim_{\delta t \to 0} \frac{\bar{F}(t + \delta t) - \bar{F}(t)}{\delta t} \]

(\( \because \) If \( u = t + \delta t, \) then \( x \to t \) iff \( \delta t \to 0 \))

13.6.5 Theorem: If \( \bar{F} \) and \( \bar{g} \) are vector functions on \( S(\subseteq \mathbb{R}) \), and \( \bar{F}, \bar{g} \) are differentiable at \( t \in S \), then \( \bar{F} \pm \bar{g} \) is differentiable at \( t \) and

\[ \frac{d}{dt} (\bar{F} \pm \bar{g}) = \frac{d \bar{F}}{dt} \pm \frac{d \bar{g}}{dt} \]

Proof: Let \( \bar{F} = \bar{F} \pm \bar{g} \)
13.5 Vector Differentiation

\[
\lim_{\delta t \to 0} \frac{\vec{F}(t + \delta t) - \vec{F}(t)}{\delta t} = \lim_{\delta t \to 0} \frac{(\vec{f} \pm \vec{g})(t + \delta t) - (\vec{f} + \vec{g})(t)}{\delta t}
\]

\[
= \lim_{\delta t \to 0} \frac{\vec{f}(t + \delta t) \pm \vec{g}(t + \delta t) - [\vec{f}(t) \pm \vec{g}(t)]}{\delta t}
\]

\[
= \lim_{\delta t \to 0} \frac{\vec{f}(t + \delta t) - \vec{f}(t)}{\delta t} \pm \lim_{\delta t \to 0} \frac{\vec{g}(t + \delta t) - \vec{g}(t)}{\delta t}
\]

\[
= \frac{d\vec{f}}{dt} \pm \frac{d\vec{g}}{dt}
\]

\[
\implies \vec{F} \text{ is differentiable at } t \text{ and } \frac{d\vec{F}}{dt} = \frac{d\vec{f}}{dt} \pm \frac{d\vec{g}}{dt}
\]

13.6.6 Theorem: If \( \vec{f} \) and \( \vec{g} \) are vector functions defined in \( S(\subseteq \mathbb{R}) \) and \( \vec{f}, \vec{g} \) are differentiable at \( t \in S \), then \( \vec{f} \cdot \vec{g}, \vec{f} \times \vec{g}, \) and \( k \vec{f} \) are differentiable at \( t \) and

\[
\frac{d}{dt}(\vec{f} \cdot \vec{g}) = \frac{d\vec{f}}{dt} \cdot \vec{g} + \vec{f} \cdot \frac{d\vec{g}}{dt}
\]

\[
\frac{d}{dt}(\vec{f} \times \vec{g}) = \frac{d\vec{f}}{dt} \times \vec{g} + \vec{f} \times \frac{d\vec{g}}{dt} \text{ and } \frac{d}{dt}(k \vec{f}) = k \frac{d\vec{f}}{dt}
\]

Proof: Let \( \vec{F} = \vec{f} \cdot \vec{g} \)

\[
\lim_{\delta t \to 0} \frac{\vec{F}(t + \delta t) - \vec{F}(t)}{\delta t} = \lim_{\delta t \to 0} \frac{(\vec{f} \cdot \vec{g})(t + \delta t) - (\vec{f} \cdot \vec{g})(t)}{\delta t}
\]

\[
= \lim_{\delta t \to 0} \frac{\vec{f}(t + \delta t) \cdot \vec{g}(t + \delta t) - \vec{f}(t) \cdot \vec{g}(t)}{\delta t}
\]

\[
= \lim_{\delta t \to 0} \frac{\vec{f}(t + \delta t) \cdot \{\vec{g}(t + \delta t) - \vec{g}(t)\} + \{\vec{f}(t + \delta t) - \vec{f}(t)\} \cdot \vec{g}(t)}{\delta t}
\]

\[
= \vec{f}(t) \cdot \frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt} \cdot \vec{g} \quad (\because \vec{f} \text{ is differentiable at } t \Rightarrow \vec{f} \text{ is continuous at} \quad t \Rightarrow \lim_{\delta t \to 0} \vec{f}(t + \delta t) = \vec{f}(t))
\]
The other results can be proved on similar lines.

**13.6.7 Theorem:** If \( \vec{f}, \vec{g}, \vec{h} \) are vector functions defined in a domain \( S \subseteq \mathbb{R} \) which are differentiable at \( t \in S \), then \( [\vec{f} \times (\vec{g} \times \vec{h})] \) and \( \vec{f} \times (\vec{g} \times \vec{h}) \) are differentiable at \( t \) and

\[
\begin{align*}
(i) & \quad \frac{d}{dt}[\vec{f} \times (\vec{g} \times \vec{h})] = \vec{f} \cdot (\vec{g} \times \vec{h}) \\
& \quad = \frac{d\vec{f}}{dt} \cdot (\vec{g} \times \vec{h}) + \vec{f} \cdot \frac{d(\vec{g} \times \vec{h})}{dt}
\end{align*}
\]

\[
\begin{align*}
(ii) & \quad \frac{d}{dt} \vec{f} \times (\vec{g} \times \vec{h}) = \frac{d\vec{f}}{dt} \times (\vec{g} \times \vec{h}) + \vec{f} \times \left( \frac{d\vec{g}}{dt} \times \vec{h} + \vec{g} \times \frac{d\vec{h}}{dt} \right)
\end{align*}
\]

**Proof:** \( \vec{f}, \vec{g}, \vec{h} \) are differentiable at \( t \Rightarrow \vec{f}, \vec{g} \times \vec{h} \) are differentiable at \( t \)

\[
\Rightarrow [\vec{f} \times (\vec{g} \times \vec{h})] \text{ is differentiable at } t
\]

and \( \vec{f} \times (\vec{g} \times \vec{h}) \) is differentiable at \( t \).

\[
\therefore [\vec{f} \times (\vec{g} \times \vec{h})] \text{ and } \vec{f} \times (\vec{g} \times \vec{h}) \text{ are differentiable at } t.
\]

\[
(i) \quad \frac{d}{dt}[\vec{f} \times (\vec{g} \times \vec{h})] = \frac{d}{dt} \vec{f} \cdot (\vec{g} \times \vec{h})
\]

\[
= \frac{d\vec{f}}{dt} \cdot (\vec{g} \times \vec{h}) + \vec{f} \cdot \frac{d(\vec{g} \times \vec{h})}{dt}
\]

\[
= \left[ \frac{d\vec{f}}{dt} \frac{\vec{g}}{dt} \frac{\vec{h}}{dt} \right] + \vec{f} \cdot \left( \frac{d\vec{g}}{dt} \times \vec{h} + \vec{g} \times \frac{d\vec{h}}{dt} \right)
\]

\[
= \left[ \frac{d\vec{f}}{dt} \frac{\vec{g}}{dt} \frac{\vec{h}}{dt} \right] + \vec{f} \cdot \left( \frac{d\vec{g}}{dt} \times \vec{h} + \vec{g} \times \frac{d\vec{h}}{dt} \right)
\]

\[
= \left[ \frac{d\vec{f}}{dt} \frac{\vec{g}}{dt} \frac{\vec{h}}{dt} \right] + \vec{f} \cdot \left( \frac{d\vec{g}}{dt} \times \vec{h} + \vec{g} \times \frac{d\vec{h}}{dt} \right)
\]

which proves (i)

\[
(ii) \quad \frac{d}{dt} \vec{f} \times (\vec{g} \times \vec{h}) = \frac{d\vec{f}}{dt} \times (\vec{g} \times \vec{h}) + \vec{f} \times \frac{d(\vec{g} \times \vec{h})}{dt}
\]
\[
\frac{d\mathbf{F}}{dt} = \frac{d\mathbf{f}_1}{dt} \mathbf{i} + \frac{d\mathbf{f}_2}{dt} \mathbf{j} + \frac{d\mathbf{f}_3}{dt} \mathbf{k}
\]

**Proof:**

\[
\frac{d\mathbf{F}}{dt} = \begin{pmatrix}
\frac{df_1}{dt} \\
\frac{df_2}{dt} \\
\frac{df_3}{dt}
\end{pmatrix} = \begin{pmatrix}
\mathbf{i} \\
\mathbf{j} \\
\mathbf{k}
\end{pmatrix} \cdot \begin{pmatrix}
\frac{df_1}{dt} \\
\frac{df_2}{dt} \\
\frac{df_3}{dt}
\end{pmatrix} = \frac{d}{dt} \left[ f_1(t) \mathbf{i} + f_2(t) \mathbf{j} + f_3(t) \mathbf{k} \right]
\]

[From 13.6.5].
13.6.10 Definition: If $f_1, f_2, f_3$ are constant scalar functions of a real variable $t$ in $S(\subseteq \mathbb{R})$, then $\vec{r} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$ is called a constant function in $S$.

13.6.11 Theorem: $\vec{r}$ is a constant function in $S(\subseteq \mathbb{R}) \iff \frac{d\vec{r}}{dt} = 0$ on $S$.

Proof: (i) Suppose $\vec{r}$ is a constant function.

Let $\vec{r}(t) = \vec{c} \quad \forall \ t \in S$

$\therefore \frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\vec{r}(t + \delta t) - \vec{r}(t)}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{c} - \vec{c}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{0}}{\delta t} = \vec{0}, \forall t \in S$

(ii) Suppose $\frac{d\vec{r}}{dt} = \vec{0} \quad \forall t \in S$

Let $\vec{r}(t) = f_1(t) \vec{i} + f_2(t) \vec{j} + f_3(t) \vec{k}$

$\forall t \in S, \frac{d\vec{r}}{dt} = \vec{0} \Rightarrow \frac{df_1}{dt} \vec{i} + \frac{df_2}{dt} \vec{j} + \frac{df_3}{dt} \vec{k} = \vec{0} = 0\vec{i} + 0\vec{j} + 0\vec{k} \forall t \in S$

$\Rightarrow \frac{df_1}{dt} = 0, \frac{df_2}{dt} = 0, \frac{df_3}{dt} = 0 \quad \forall t \in S$

$\therefore f_1, f_2, f_3$ are constant functions on $S$.

$\Rightarrow \vec{r} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$ is constant on $S$.

Hence the Theorem.

13.6.12 Theorem: If $\vec{r}$ is a vector function of a real variable in a domain $S(\subseteq \mathbb{R})$, which is differentiable at $t \in S$, then $\frac{d}{dt} \vec{r}^2 = 2\vec{r} \cdot \frac{d\vec{r}}{dt} = 2|\vec{r}| \frac{d}{dt} |\vec{r}|$

Proof: We know that $\frac{d}{dt} \vec{r}^2 = \vec{r} \cdot \frac{d\vec{r}}{dt}$

$\therefore \frac{d}{dt} (\vec{r} \cdot \vec{r}) = \frac{d}{dt} |\vec{r}|^2 = \frac{d\vec{r}}{dt} \cdot \vec{r} + \frac{d\vec{r}}{dt} \cdot \vec{r} = 2\vec{r} \cdot \frac{d\vec{r}}{dt}$

Also $\vec{r}^2 = \vec{r} \cdot \vec{r} = |\vec{r}|^2$
\[ \frac{d}{dt} (\vec{r}^2) = 2|\vec{r}| \frac{d}{dt} |\vec{r}| \]

Hence the Theorem.

13.6.13 Note: \[ \vec{r} \cdot \frac{d\vec{r}}{dt} = |\vec{r}| \frac{d}{dt} |\vec{r}| \]

13.6.14 Theorem: Let \( \vec{r} \) be a vector function of a real variable in a domain \( S \subseteq \mathbb{R} \). Then \( \vec{r} \) is constant in magnitude iff \[ \vec{r} \cdot \frac{d\vec{r}}{dt} = 0. \]

Proof: (i) Suppose \( \vec{r} \) is a vector function of constant magnitude

\[ \therefore \vec{r}(t) \cdot \vec{r}(t) - |\vec{r}(t)|^2 = \text{constant} \]

\[ \therefore \frac{d}{dt} (\vec{r} \cdot \vec{r}) = 0 \]

\[ \Rightarrow 2\vec{r} \cdot \frac{d\vec{r}}{dt} = 0 \]

\[ \Rightarrow \vec{r} \cdot \frac{d\vec{r}}{dt} = 0 \]

(ii) Suppose \( \vec{r} \cdot \frac{d\vec{r}}{dt} = 0 \)

\[ \Rightarrow 2\vec{r} \cdot \frac{d\vec{r}}{dt} = 0 \]

\[ \Rightarrow \frac{d}{dt} |\vec{r}(t)|^2 = 0 \quad \forall \ t \in S \]

\[ \Rightarrow |\vec{r}(t)|^2 = \text{constant} \quad \forall \ t \in S \]

\[ \Rightarrow \vec{r} \] has constant magnitude in \( S \).

Hence the theorem.

13.6.15 Theorem: Let \( \vec{r} \) be a vector function of a real variable in a domain \( S \subseteq \mathbb{R} \). Then \( \vec{r} \) is of constant direction iff \[ \vec{r} \times \frac{d\vec{r}}{dt} = 0. \]
Proof: Let \( \vec{r}(t) = f(t)\vec{F}(t) \), where \( f(t) = |\vec{r}(t)| \) and \( \vec{F}(t) \) is a vector function with unit magnitude for every \( t \in S \).

\[
\vec{r} = f \vec{F} \quad \text{so that} \quad \vec{F} \quad \text{is constant in magnitude and direction.}
\]

\[
\frac{d\vec{r}}{dt} = \frac{df}{dt} \vec{F} + f \frac{d\vec{F}}{dt}
\]

\[
\therefore \vec{r} \times \frac{d\vec{r}}{dt} = f\vec{F} \times \left( \frac{df}{dt}\vec{F} + f \frac{d\vec{F}}{dt} \right)
\]

\[
= f \frac{df}{dt} \vec{F} \times \vec{F} + f^2 \vec{F} \times \frac{d\vec{F}}{dt}
\]

\[
= f^2 \vec{F} \times \frac{d\vec{F}}{dt} \quad \text{-------- (1)}
\]

(i) Suppose \( \vec{r} \) has constant direction.

\[
\vec{F} = \text{constant (in magnitude and direction)} \Rightarrow \frac{d\vec{F}}{dt} = 0
\]

\[
\therefore \text{From (1),} \quad \vec{r} \times \frac{d\vec{r}}{dt} = \vec{0}
\]

(ii) Suppose \( \vec{r} \times \frac{d\vec{r}}{dt} = \vec{0} \)

\[
\therefore f^2 \vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}, \text{ from (1)}
\]

\[
\Rightarrow \vec{F} \times \frac{d\vec{F}}{dt} = \vec{0} \quad \text{-------- (2)} \quad (\because f^2 \neq 0)
\]

Since \( \vec{F} \) has constant magnitude, by Theorem 13.6.14,

we have \( \vec{F} \cdot \frac{d\vec{F}}{dt} = 0 \quad \text{-------- (3)} \)

From (2) & (3), it follows that \( \frac{d\vec{F}}{dt} = \vec{0} \)
\[ \Rightarrow \vec{F} \text{ is constant} \]
\[ \Rightarrow \vec{F} \text{ has constant direction} \]
\[ \Rightarrow \vec{F} \text{ has constant direction} \quad (\therefore \vec{F} = f \vec{F}) \]

**13.6.16 Definition (Composite function) :** Let \( \phi \neq S \subseteq \mathbb{R} \). If \( \phi \) is a scalar function on \( S \), then the range \( \phi(S) \) of \( \phi \) is a subset of \( \mathbb{R} \). If \( \vec{F} \) is a vector function on \( \phi(S) \), then \( \vec{F} \circ \phi \) is a vector function on \( S \) defined as:
\[ (\vec{F} \circ \phi)(t) = \vec{F}(\phi(t)), \forall t \in S \]

**13.6.17 Theorem :** Let \( \phi \) be a scalar function on \( S \) and \( \vec{F} \) be a vector function on \( \phi(S) \). If \( \phi \) is differentiable at \( t \) and \( \vec{F} \) is differentiable at \( \phi(t) \), then \( \vec{F} \circ \phi \) is differentiable at \( t \) and
\[ \left( \vec{F} \circ \phi \right)'(t) = \vec{F}'(\phi(t))\phi'(t) \]

**Proof :**
\[
\lim_{\delta t \to 0} \frac{(\vec{F} \circ \phi)(t + \delta t) - (\vec{F} \circ \phi)(t)}{\delta t}
= \lim_{\delta t \to 0} \frac{\vec{F}[\phi(t + \delta t)] - \vec{F}(\phi(t))}{\delta t}
= \lim_{\delta t \to 0} \frac{\vec{F}[\phi(t + \delta t)] - \vec{F}(\phi(t))}{\delta t} \cdot \frac{\phi(t + \delta t) - \phi(t)}{\delta t}
\]
This limit exists and \( = \vec{F}'(\phi(t)) \phi'(t) \), since \( \phi \) is differentiable at \( t \) implies that \( \phi \) is continuous at \( t \) and hence \( \phi(t + \delta t) \to \phi(t) \) as \( \delta t \to 0 \).

Hence \( \vec{F} \circ \phi \) is differentiable at \( t \) and \( (\vec{F} \circ \phi)'(t) = \vec{F}'(\phi(t))\phi'(t) \).

**13.7 HIGHER ORDER DERIVATIVES**

**13.7.1 Definition :** If \( \vec{F} \) is a vector function defined in an interval \( I(\subseteq \mathbb{R}) \) and \( \vec{F} \) is differentiable on \( I \), then its derivative \( \vec{F}' \) is also a vector function on \( I \). If \( \vec{F}' \) is differentiable on \( I \), its derivative is called the second derivative of \( \vec{F} \) and is denoted by \( \vec{F}'' \) or \( \vec{F}^{(2)} \) or \( \frac{d^2 \vec{F}}{dt^2} \). If \( \vec{F} \) is differentiable \( n \) times, the \( n^{th} \) derivative of \( \vec{F} \) is \( \frac{d^n \vec{F}}{dt^n} \) or \( \vec{F}^{(n)} = \frac{d}{dt} \left( \frac{d^{n-1} \vec{F}}{dt^{n-1}} \right) \).
Solved Problems :

13.7.2 : If \( \mathbf{A} = 5t^2 \mathbf{i} + t^3 \mathbf{j} - t \mathbf{k} \), \( \mathbf{B} = 2 \sin t \mathbf{i} - \cos t \mathbf{j} + 5t \mathbf{k} \) find \( \frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) \) at \( t = 0 \)

Solution : \( \mathbf{A} \cdot \mathbf{B} = 5t^2 (2 \sin t) + t^3 (- \cos t) + (-t)5t \)

\[ = 10t^2 \sin t - t^3 \cos t - 5t^2 \]

\[ \frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = 10t^2 \cos t + 20t \sin t + t^3 \sin t - 3t^2 \cos t - 10t \]

At \( t = 0 \), \( \frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = 0 \)

13.7.3 : If \( \mathbf{r} \) is a vector function, show that

(a) \( [\mathbf{r} \mathbf{r} \mathbf{r}] = [\mathbf{r} \mathbf{r} \mathbf{r}] + [\mathbf{r} \mathbf{r} \mathbf{r}] + [\mathbf{r} \mathbf{r} \mathbf{r}] \)

(b) \( [\mathbf{r} \times (\mathbf{r} \times \mathbf{r})] = \mathbf{r} \times (\mathbf{r} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{r} \times \mathbf{r}) \)

Solution : (a) \( [\mathbf{r} \mathbf{r} \mathbf{r}] = [\mathbf{r} \mathbf{r} \mathbf{r}] + [\mathbf{r} \mathbf{r} \mathbf{r}] + [\mathbf{r} \mathbf{r} \mathbf{r}] = 0 + 0 + [\mathbf{r} \mathbf{r} \mathbf{r}] = [\mathbf{r} \mathbf{r} \mathbf{r}] \)

(b) \( [\mathbf{r} \times (\mathbf{r} \times \mathbf{r})] = \mathbf{r} \times (\mathbf{r} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{r} \times \mathbf{r}) \)

\[ = \mathbf{r} \times (\mathbf{r} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{r} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{r} \times \mathbf{r}) \]

\[ = \mathbf{r} \times (\mathbf{r} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{r} \times \mathbf{r}) \quad (\because \mathbf{r} \times \mathbf{r} = 0) \]

13.7.4 : If \( \mathbf{r} = \mathbf{a} \cos wt + \mathbf{b} \sin wt \), where \( \mathbf{a} \) and \( \mathbf{b} \) are constant vectors, then show that \( \mathbf{r} \times \frac{d\mathbf{r}}{dt} = w\mathbf{a} \times \mathbf{b} \)

and \( \frac{d^2 \mathbf{r}}{dt^2} = -w^2 \mathbf{r} \).

Solution : \( \frac{d\mathbf{r}}{dt} = -\mathbf{a} w \sin wt + \mathbf{b} w \sin wt \)

\[ \frac{d^2 \mathbf{r}}{dt^2} = -\mathbf{a} w^2 \cos wt - \mathbf{b} w^2 \cos wt \]

\[ = -w^2 (\mathbf{a} \cos wt + \mathbf{b} \sin wt) \]
\[ \frac{d\vec{r}}{dt} = (a \cos wt + b \sin wt) \times (-aw \sin wt + bw \cos wt) \]

\[ = -a \times aw \cos wt \sin wt + a \times bw \cos^2 wt - b \times aw \sin^2 wt + b \times bw \sin wt \cos wt \]

\[ = \vec{0} + \vec{a} \times bw \cos^2 wt + \vec{a} \times bw \sin^2 wt + \vec{0} \]

\[ = (\vec{a} \times \vec{b}) w (\cos^2 wt + \sin^2 wt) = w\vec{a} \times \vec{b} \quad \text{--------- QED} \]

13.7.5 SAQ : If \( \vec{r} = a \cos t \vec{i} + b \sin t \vec{j} + bk \), find \( \frac{d\vec{r}}{dt} \).

13.7.6 SAQ : If \( \vec{r} = \cos nt \vec{i} + \sin nt \vec{j} \), where \( n \in \mathbb{R} \), then find \( \vec{r} \times \frac{d\vec{r}}{dt}, \vec{r} \cdot \frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2} \).

13.7.7 SAQ : If \( \vec{r} = x \vec{i} + y \vec{j} + zk, x = 2\sin 3t, y = 2\cos 3t, z = 8t \) then show that \( |\vec{r}| = 10 \) and \( |r'| = 18 \).

13.8 PARTIAL DIFFERENTIATION

If \( \phi = \phi(x, y, z) \) is a scalar point function, then the concept of partial derivative of \( \phi \) with respect to \( x \) (or \( y \) or \( z \)) has been introduced at the Intermediate level and \( \frac{\partial \phi}{\partial x} \) is defined as the limit \( \lim_{h \to 0} \frac{\phi(x + h, y, z) - \phi(x, y, z)}{h} \), if it exists. Similarly \( \frac{\partial \phi}{\partial y} \) and \( \frac{\partial \phi}{\partial z} \) can be defined.

13.8.1 Definition : Let \( \vec{r} = \vec{r}(x, y, z) \) be a vector point function. If \( \lim_{h \to 0} \frac{\vec{r}(x + h, y, z) - \vec{r}(x, y, z)}{h} \) exists, then we say that \( \vec{r} \) is partially differentiable with respect to \( x \) and the limit is called partial derivative of \( \vec{r} \) w.r.t. \( x \) and is denoted by \( \frac{\partial \vec{r}}{\partial x} \) or \( \vec{r}_x \). Similarly \( \frac{\partial \vec{r}}{\partial y} \) and \( \frac{\partial \vec{r}}{\partial z} \) can be defined.

We now state the following theorem (without proof).

13.8.2 Theorem : If \( \vec{r} \) and \( \vec{g} \) are vector functions of \( (x, y, z) \) and \( \phi \) is a scalar function of \( (x, y, z) \), then

\begin{align*}
(a) \quad \frac{\partial}{\partial x}(\vec{r} \pm \vec{g}) &= \frac{\partial \vec{r}}{\partial x} \pm \frac{\partial \vec{g}}{\partial x} \\
(b) \quad \frac{\partial}{\partial x}(\vec{r} \cdot \vec{g}) &= \frac{\partial \vec{r}}{\partial x} \cdot \vec{g} + \vec{r} \cdot \frac{\partial \vec{g}}{\partial x}
\end{align*}
13.8.3 Definition: Let \( \vec{f} \) be a vector function of \((x_1, x_2, x_3)\). Then \( \frac{\partial \vec{f}}{\partial x_i} \) is a vector function of \((x_1, x_2, x_3)\) for \(i = 1, 2, 3\). If the partial derivative of \( \frac{\partial \vec{f}}{\partial x_i} \) w.r.t. \(x_j\) exists, then it is called the second partial derivative of \( \vec{f} \) and it is denoted by \( \frac{\partial^2 \vec{f}}{\partial x_j \partial x_i} \) and when \( j = i \), it is denoted by \( \frac{\partial^2 \vec{f}}{\partial x_i^2} \) for \(i, j = 1, 2, 3\). When \((x_1, x_2, x_3)\) is written as \((x, y, z)\), the second order partial derivatives are \( \frac{\partial^2 \vec{f}}{\partial x \partial y}, \frac{\partial^2 \vec{f}}{\partial y \partial x}, \frac{\partial^2 \vec{f}}{\partial x \partial z}, \frac{\partial^2 \vec{f}}{\partial z \partial x}, \frac{\partial^2 \vec{f}}{\partial y \partial z} \) and \( \frac{\partial^2 \vec{f}}{\partial z \partial y} \).

13.8.4 Note (i): If \( \vec{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} \), then \( \frac{\partial \vec{f}}{\partial x} = \frac{\partial f_1}{\partial x} \hat{i} + \frac{\partial f_2}{\partial x} \hat{j} + \frac{\partial f_3}{\partial x} \hat{k} \) where \( f_1, f_2, f_3 \) are scalar functions and \( \vec{f} \) is a vector function of \((x_1, x_2, x_3)\).

(ii): In general \( \frac{\partial^2 \vec{f}}{\partial x \partial y} \neq \frac{\partial^2 \vec{f}}{\partial y \partial x} \) but under certain conditions on \( \vec{f} \), equality holds.

The following is an example for \( \frac{\partial^2 \vec{f}}{\partial x \partial y} \neq \frac{\partial^2 \vec{f}}{\partial y \partial x} \)

13.8.5 Example: Define \( \vec{T}(x, y) = \begin{cases} xy \left( \frac{x^2 - y^2}{x^2 + y^2} \right) \hat{T}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases} \)
\[
\vec{T}_x(0,y) = \left( \frac{\partial \vec{T}}{\partial x} \right)_{(0,y)} = \lim_{x \to 0} \frac{\vec{T}(x,y) - \vec{T}(0,y)}{x} = \lim_{x \to 0} \frac{xy(x^2 - y^2)}{x} = \lim_{x \to 0} \frac{y(x^2 - y^2)}{x^2 + y^2} = -y \vec{I}
\]

\[
\vec{T}_{yx}(0,0) = \left( \frac{\partial^2 \vec{T}}{\partial y \partial x} \right)_{(0,0)} = \left[ \frac{\partial}{\partial y} \vec{T}_x(0,y) \right]_{(0,0)} = \lim_{y \to 0} \frac{\vec{T}_x(0,y) - \vec{T}_x(0,0)}{y} = \lim_{y \to 0} \frac{\vec{T}_x(0,y)}{y} = -\vec{T}
\]

\[
\vec{T}_y(x,0) = \left( \frac{\partial \vec{T}}{\partial y} \right)_{(x,0)} = \lim_{y \to 0} \frac{\vec{T}(x,y) - \vec{T}(x,0)}{y} = \lim_{y \to 0} \frac{x y \left( \frac{x^2 - y^2}{x^2 + y^2} \right)}{y} = x \vec{I}
\]

\[
\vec{T}_{xy}(0,0) = \left( \frac{\partial^2 \vec{T}}{\partial x \partial y} \right)_{(0,0)} = \left[ \frac{\partial}{\partial x} \left( \frac{\partial \vec{T}}{\partial y} \right) \right]_{(0,0)} = \left[ \frac{\partial}{\partial x} \vec{T}_y(x,0) \right]_{(0,0)} = \lim_{x \to 0} \frac{\vec{T}_y(x,0) - \vec{T}_y(0,0)}{x} = \lim_{x \to 0} \frac{\vec{T}_y(x,0)}{x} = -\vec{T}
\]

So, \( \vec{T}_{yx}(0,0) \neq \vec{T}_{xy}(0,0) \) i.e. \( \frac{\partial^2 \vec{T}}{\partial y \partial x} \neq \frac{\partial^2 \vec{T}}{\partial x \partial y} \)

SOLVED PROBLEMS:

13.8.6 : If \( \vec{T} = (2x^2y - x^4)\vec{i} + (e^{xy} - y \sin x)\vec{j} + x^2 \cos y \vec{k} \) find \( \frac{\partial^2 \vec{T}}{\partial x^2}, \frac{\partial^2 \vec{T}}{\partial x \partial y}, \frac{\partial^2 \vec{T}}{\partial y \partial x}, \frac{\partial^2 \vec{T}}{\partial y^2} \)

Solution:

\[
\frac{\partial \vec{T}}{\partial x} = (4xy - 4x^3)\vec{i} + \left( \frac{e^{xy}}{x} - y \cos x \right)\vec{j} + 2x \cos y \vec{k}
\]

\[
\frac{\partial^2 \vec{T}}{\partial x^2} = \left( \frac{\partial}{\partial x} \left( \frac{\partial \vec{T}}{\partial x} \right) \right) = (4y - 12x^2)\vec{i} + \left( \frac{y^2 e^{xy}}{x} + y \sin x \right)\vec{j} + 2 \cos y \vec{k}
\]

\[
\frac{\partial^2 \vec{T}}{\partial y \partial x} = \left( \frac{\partial}{\partial y} \left( \frac{\partial \vec{T}}{\partial x} \right) \right) = 4x \vec{i} + \left( e^{xy} + xye^{xy} - \cos x \right)\vec{j} - 2x \sin y \vec{k}
\]
\[
\frac{\partial \vec{F}}{\partial y} = 2x^2\vec{i} + (xe^{xy} - \sin x)\vec{j} - x^2\sin y\vec{k}
\]

\[
\frac{\partial^2 \vec{F}}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial \vec{F}}{\partial y} \right) = 4x\vec{i} + (e^{xy} + xye^{xy} - \cos x)\vec{j} - 2x\sin y\vec{k}
\]

\[
\frac{\partial^2 \vec{F}}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial \vec{F}}{\partial y} \right) = 0\vec{i} + x^2e^{xy}\vec{j} - x^2\cos y\vec{k}
\]

\[
= x^2e^{xy}\vec{j} - x^2\cos y\vec{k}
\]

In this problem, we observe that \( \frac{\partial^2 \vec{F}}{\partial x \partial y} = \frac{\partial^2 \vec{F}}{\partial y \partial x} \)

13.8.7 : If \( \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \) and \( \vec{a} \) is a constant vector, then show that

\[
\frac{\partial}{\partial x} (\vec{a} \cdot \vec{r}) = \frac{\partial}{\partial y} (\vec{a} \cdot \vec{r}) + \frac{\partial}{\partial z} (\vec{a} \cdot \vec{r}) = \vec{a}
\]

**Solution**: Let \( \vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \) Given that \( \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \)

\[
\therefore \vec{a} \cdot \vec{r} = a_1x + a_2y + a_3z
\]

\[
\frac{\partial}{\partial x} (\vec{a} \cdot \vec{r}) = a_1, \quad \frac{\partial}{\partial y} (\vec{a} \cdot \vec{r}) = a_2, \quad \frac{\partial}{\partial z} (\vec{a} \cdot \vec{r}) = a_3
\]

L.H.S. = \( a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = \vec{a} \) = R.H.S.

13.8.8 : If \( \phi = 2xz^4 - x^2y \), then find \( \left| \frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k} \right| \) at \( (2, -2, -1) \).

**Solution**: \( \frac{\partial \phi}{\partial x} = 2z^4 - 2xy, \quad \frac{\partial \phi}{\partial y} = -x^2, \quad \frac{\partial \phi}{\partial z} = 8xz^3 \)

\[
\therefore \vec{r} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} = (2z^4 - 2xy)\vec{i} - x^2\vec{j} + 8xz^3\vec{k}
\]

At \( (2, -2, -1) \), \( \vec{r} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} = 10\vec{i} - 4\vec{j} - 16\vec{k} \)
At \((2,-2,-1)\), \(\left| \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z} \right| = \sqrt{100+16+256} = \sqrt{372} = 2\sqrt{93}

13.8.9 SAQ: If \(\vec{r} = y\vec{i} + x\vec{j} + xyz\vec{k}\), then show that \(i \times \frac{\partial \vec{r}}{\partial x} + j \times \frac{\partial \vec{r}}{\partial y} + k \times \frac{\partial \vec{r}}{\partial z} = 0\)

13.8.10 SAQ: If \(\vec{r} = x\vec{i} + y\vec{j} + zk\) and \(\vec{a}\) is a constant vector, then show that

(a) \(\frac{\partial}{\partial x}(\vec{a} \times \vec{r}) \cdot \vec{i} + \frac{\partial}{\partial y}(\vec{a} \times \vec{r}) \cdot \vec{j} + \frac{\partial}{\partial z}(\vec{a} \times \vec{r}) \cdot \vec{k} = 0\) and

(b) \(\frac{\partial}{\partial x}(\vec{a} \times \vec{r}) \times \vec{i} + \frac{\partial}{\partial y}(\vec{a} \times \vec{r}) \times \vec{j} + \frac{\partial}{\partial z}(\vec{a} \times \vec{r}) \times \vec{k} = -2\vec{a}\)

13.9 ANSWERS TO SAQ’s

13.7.5 SAQ: \(\sqrt{a^2 + b^2} \left( \vec{r} \cdot \frac{d\vec{r}}{dt} = a \cos t \vec{i} + b \vec{k} \Rightarrow \left| \frac{d\vec{r}}{dt} \right| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2} \right)\)

13.7.6 SAQ: \(\frac{d\vec{r}}{dt} = -n \sin nt \vec{i} + n \cos nt \vec{j}\)

\(\therefore \vec{r} \times \frac{d\vec{r}}{dt} = (\cos nt \vec{i} + \sin nt \vec{j}) \times (-n \sin nt \vec{i} + n \cos nt \vec{j}) = n\vec{k}

\(\vec{r} \cdot \frac{d\vec{r}}{dt} = 0, \quad \frac{d^2\vec{r}}{dt^2} = -n^2\vec{r}\)

13.7.7 SAQ: \(\vec{r} = 6\cos 3t \vec{i} - 6\sin 3t \vec{j} + 8\vec{k}\)

\(|\vec{r}| = \sqrt{36\cos^2 3t + 36\sin^2 3t + 64} = 10\)

\(\vec{r}' = -18\sin 3t \vec{i} - 18\cos 3t \vec{j}\)

\(|\vec{r}'| = \sqrt{324(\sin^2 3t + \cos^2 3t)} = 18\)

13.8.9 SAQ: \(\frac{\partial \vec{r}}{\partial x} = z\vec{j} + y\vec{k}\)

\(\vec{r} \times \frac{\partial \vec{r}}{\partial x} = z\vec{k} - y\vec{j}\)

\(\vec{j} \times \frac{\partial \vec{r}}{\partial y} = x\vec{i} - z\vec{x}, \quad \vec{x} \times \frac{\partial \vec{r}}{\partial z} = y\vec{j} - x\vec{i}\)

Adding we get LHS = 0

13.8.10 SAQ: (a) LHS = \(\sum (\vec{a} \times \vec{i}) \cdot \vec{r} = 0\)
(b) \[ \text{LHS} = \sum (\vec{a} \times \vec{t}) \times \vec{t} = \sum (\vec{a} \cdot \vec{t}) \vec{t} - (\vec{t} \cdot \vec{t}) \vec{a} \]
\[ = \vec{a} - 3\vec{a} = -2\vec{a} \]

13.10 SUMMARY

In this lesson, we discussed vector functions, scalar functions, vector point functions, scalar point functions, their limits, continuity and differentiation, higher order derivatives and partial differentiation of scalar and vector point functions of several real variables and related problems.

13.11 TECHNICAL TERMS

Vector functions, Scalar functions, Vector point functions, Scalar point functions, derivatives, partial derivatives.

13.12 EXERCISES

13.12.1 : If \[ \vec{r} = \sin t \vec{i} + \cos t \vec{j} + t \vec{k}, \]
find \[ \frac{d^2 \vec{r}}{dt^2}, \frac{d^3 \vec{r}}{dt^3}, \frac{d^3 \vec{r}}{dt^3} \]

13.12.2 : If \[ \vec{r} = a \cos t \vec{i} + a \sin t \vec{j} + at \tan \theta \vec{k}, \]
then find
\[ \left| \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \right| \]
and \[ \left| \frac{d\vec{r}}{dt} \frac{d^2\vec{r}}{dt^2} \frac{d^3\vec{r}}{dt^3} \right| \]

13.12.3 : If \[ \vec{a} = 3t^2 \vec{i} - (t + 4) \vec{j} + (t^2 - 2t) \vec{k} \]
\[ \vec{b} = \sin t \vec{i} + 3e^{-t} \vec{j} - 3 \cos t \vec{k}, \]
then find \[ \frac{d^2}{dt^2} (\vec{a} \times \vec{b}) \] at \( t = 0 \)

13.12.4 : Prove that \( (\vec{A} \times \vec{B} - \vec{A}' \times \vec{B}')' = \vec{A} \times \vec{B} - \vec{A}' \times \vec{B} \), where \( \vec{A} \) and \( \vec{B} \) are differentiable vector functions of \( t \) and primes (dashes) denote derivatives w.r.t. \( t \).

13.12.5 : If \[ \frac{d\vec{a}}{dt} = \vec{b} \times \vec{a}, \frac{d\vec{b}}{dt} = \vec{b} \times \vec{c}, \]
then show that \[ \frac{d}{dt} (\vec{a} \times \vec{c}) = \vec{b} \times (\vec{a} \times \vec{c}) \).

13.12.6 : If \[ \vec{A} = 5t^2 \vec{i} + t^3 \vec{j} - 3 \vec{k} \]
and \( \vec{B} = \sin t \vec{i} - \cos t \vec{j} \) then find \[ \frac{d}{dt} (\vec{A} \cdot \vec{B}) \] and \[ \frac{d}{dt} (\vec{A} \times \vec{B}) \]

13.12.7 : If \[ \vec{r} = x^2yz \vec{i} - 2xz^2 \vec{j} + xz^2 \vec{k} \]
and \( \vec{g} = 2z \vec{i} + y \vec{j} - x^2 \vec{k} \) then find \[ \frac{\partial^2}{\partial x \partial y} (\vec{r} \times \vec{g}) \] at \((1,0,-2)\).

13.12.8 : If \[ \vec{r} = 2x^2 \vec{i} - 3yz \vec{j} - xz^2 \vec{k} \]
and \( \phi = 2z - x^3 \), then show that
\[ \frac{\partial \vec{r}}{\partial x} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} \right) = 14 \]

13.12.9 : If \[ \vec{r} = xz \vec{i} - xy \vec{j} + yz \vec{k} \]
and \( \phi = \frac{x}{y} e^z \), then find
\[ \frac{\partial^3}{\partial x^2 \partial z} (\phi \vec{r}) \] at \((2,-2,-1)\).
13.13 MODEL EXAMINATION QUESTIONS

13.13.1 : If \( \mathbf{r} = n \cos nt \mathbf{i} + \sin nt \mathbf{j} \) where \( n \) is constant, then find \( \mathbf{r} \times \frac{d\mathbf{r}}{dt}, \mathbf{r} \cdot \frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2} \)

13.13.2 : If \( \mathbf{r} = \alpha \cos wt + \beta \sin wt \) where \( \alpha, \beta \) are constant vectors then show that \( \mathbf{r} \times \frac{d\mathbf{r}}{dt} = w\alpha \times \beta \)
and \( \frac{d^2\mathbf{r}}{dt^2} - w^2 \mathbf{r} \).

13.13.3 : If \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) and \( \alpha \) is a constant vector, then show that \( \frac{\partial}{\partial x} (\alpha \cdot \mathbf{r}) \mathbf{i} + \frac{\partial}{\partial y} (\alpha \cdot \mathbf{r}) \mathbf{j} + \frac{\partial}{\partial z} (\alpha \cdot \mathbf{r}) \mathbf{k} = \alpha \)

13.14 ANSWERS TO EXERCISES

13.12.1 : \( \cos t \mathbf{i} - \sin t \mathbf{j} + \mathbf{k}; -\sin t \mathbf{i} - \cos t \mathbf{j}; \sqrt{2}; 1 \)

13.12.2 : \( a^2 \sec \theta; a^3 \tan \theta \)

13.12.3 : \(-30 \mathbf{i} + 14 \mathbf{j} + 20 \mathbf{k} \)

13.12.6 : \( 5t^2 \cos t + 11t \sin t - \cos t \)
\( (t^3 \sin t - 3t^2 \cos t) \mathbf{i} - (t^3 \cos t + 3t^2 \sin t) \mathbf{j} + (5t^2 \sin t - 11t \cos t - \sin t) \mathbf{k} \)

13.12.7 : \(-4 \mathbf{i} - 8 \mathbf{j} \)

13.12.9 : \(-6 \mathbf{i} + 16 \mathbf{j} \)

13.15 REFERENCES

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4. Advanced Engineering Mathematics by Erwin Kreyszig, Published by John Wiley&Sons Inc.

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Lesson - 14

DIFFERENTIAL OPERATORS

14.1 OBJECTIVE OF THE LESSON

In this lesson, the concepts of gradient of a scalar point function, the directional derivative of a scalar point function at a point P in the direction of a given vector through a given point P and divergence, curl of a vector point function are introduced. Some results are proved and some problems are discussed.

14.2 STRUCTURE OF THE LESSON

This lesson has the following components.

14.3 Introduction
14.4 Vector equation of a curve, tangent, normal
14.5 Gradient of a Scalar point function
14.6 Level Surface
14.7 Directional Derivative
14.8 Divergence, Curl of a Vector point function
14.9 Answers to SAQ's
14.10 Summary
14.11 Technical Terms
14.12 Exercises
14.13 Answers to Exercises
14.14 Model Examination Questions
14.15 Reference Books

14.3 INTRODUCTION

This lesson deals with gradient, divergence, curl, directional derivative and results on these topics.

14.4 VECTOR EQUATION OF A CURVE, TANGENT, NORMAL

14.4.1 Definition: Let \( \mathbf{r} = \mathbf{r}(t) \) be a vector function defined in \( D \subseteq \mathbb{R} \). For each \( t \in D \), \( \mathbf{r} = \mathbf{r}(t) \) represents the position vector \( \overrightarrow{OP} \) of a point P (with origin 0 as initial point). As \( t \) varies in \( D \), the point P moves on a curve \( C \) in the space. So \( \mathbf{r} = \mathbf{r}(t) \) is called the vector equation of a curve \( C \) (in \( D \)).
14.4.2 Definition: Let \( \overrightarrow{T} = \overrightarrow{T}(t) \) be a curve \( C \) and \( P \) be a point on \( C \). If \( Q \) is a point on the curve \( C \) and \( Q \neq P \), then \( \overrightarrow{PQ} \) is called a secant line. As \( Q \) approaches \( P \) along the curve \( C \), then the limiting position of \( \overrightarrow{QP} \) is called the tangent line to the curve at \( P \).

14.4.3 Definition: Any vector whose line of support is the tangent line is called a tangent vector to the curve at \( P \) [curve, \( P \) are as in Def. 14.4.2].

14.4.4 Theorem: If \( \overrightarrow{T} = \overrightarrow{T}(t) \) is a differentiable vector function representing a curve \( C \) and \( P \) is a point on \( C \), where \( \overrightarrow{OP} = \overrightarrow{T}(t) \), then \( \frac{d\overrightarrow{T}}{dt} \) is a tangent vector to the curve at \( P \).

Proof: Let \( P \) be a point on \( C \) with position vector \( \overrightarrow{T} \) i.e. \( \overrightarrow{OP} = \overrightarrow{T} \). Let \( Q \) be a point on \( C \), \( Q \neq P \) so that \( \overrightarrow{OQ} = \overrightarrow{T} + \delta\overrightarrow{T} = \overrightarrow{T}(t + \delta t) \).

\[
\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \delta\overrightarrow{T}
\]

\[
\therefore \delta\overrightarrow{T} = \overrightarrow{T}(t + \delta t) - \overrightarrow{T}(t)
\]

\[
\frac{\delta\overrightarrow{T}}{\delta t} = \frac{\overrightarrow{T}(t + \delta t) - \overrightarrow{T}(t)}{\delta t}
\]

As \( Q \) approaches \( P \), \( \delta t \to 0 \) \( \therefore \overrightarrow{PQ} \) becomes tangent at \( P \).

\[
\therefore \frac{d\overrightarrow{T}}{dt} = \lim_{\delta t \to 0} \frac{\delta\overrightarrow{T}}{\delta t} = \lim_{\delta t \to 0} \frac{\overrightarrow{T}(t + \delta t) - \overrightarrow{T}(t)}{\delta t}
\]

\[
\therefore \frac{d\overrightarrow{T}}{dt} \text{ is a vector parallel to the tangent line to the curve } C \text{ at } P
\]

i.e. \( \frac{d\overrightarrow{T}}{dt} \) is a tangent vector to the curve \( C \) at \( P \).

14.4.5 Definition: Let \( \overrightarrow{T} = \overrightarrow{T}(t) \) is a differentiable vector point function which represents a curve \( C \) and \( P \) is a point on \( C \) such that \( \overrightarrow{OP} = \overrightarrow{T}(t) \). Then \( \frac{d\overrightarrow{T}}{dt} \) is a tangent vector to the curve \( C \) at \( P \) and \( \frac{d\overrightarrow{T}}{dt} \) is called a unit tangent vector at \( P \), where \( \left| \frac{d\overrightarrow{T}}{dt} \right| \) is the magnitude (length) of the vector \( \frac{d\overrightarrow{T}}{dt} \).
14.4.6 Notation : Let \( C \) be a space curve. Fix a point A on \( C \), then P is uniquely determined by the arc length \( s \) from A to P. Arc length from A to P is denoted by arc AP. Thus the position vector of P is a function of the arc length \( s \). So \( \overrightarrow{OP} \) may be written as \( \overrightarrow{OP} = \overrightarrow{T}(s) \). If \( \overrightarrow{AB} \) is any vector, then \( \overrightarrow{AB} = \overrightarrow{AB} \) (unit vector in the direction of \( \overrightarrow{AB} \)) and \( |\overrightarrow{AB}| = \overrightarrow{AB} \).

14.4.7 Theorem : Unit tangent vector to the curve \( \overrightarrow{T} = \overrightarrow{T}(s) \) at P is \( \overrightarrow{T} = \frac{d\overrightarrow{T}}{ds} \).

Proof : Let \( C \) be a curve. Let A be a fixed point on \( C \) from which arc distances of points on \( C \) are measured. Let \( \text{arc } AP = s \). Let \( \overrightarrow{OP} = \overrightarrow{T}(s) \). Let Q be a point on \( C \) and \( \overrightarrow{OQ} = \overrightarrow{T}(s + \delta s) \).

As \( Q \rightarrow P \), the vector \( \overrightarrow{PQ} \) becomes the tangent vector at P to \( C \). Let \( \overrightarrow{T} \) be the unit tangent vector at P to \( C \).

\[
\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{T}(s + \delta s) - \overrightarrow{T}(s)
\]

\[
\frac{\overrightarrow{PQ}}{\delta s} = \frac{\overrightarrow{T}(s + \delta s) - \overrightarrow{T}(s)}{\delta s}
\]

Hence the unit tangent vector to the curve at P is \( \overrightarrow{T} = \frac{d\overrightarrow{T}}{ds} \).

14.4.8 Definition : Unit vector perpendicular to \( \overrightarrow{T} \) is called principal unit normal vector to the curve at P. It is denoted by \( \overrightarrow{N} \).
14.4.9 Theorem: \[ \mathbf{N} = \frac{d^2\mathbf{r}}{ds^2} + \left| \frac{d^2\mathbf{r}}{ds^2} \right| \]

Proof: We know that \( \mathbf{T} \cdot \mathbf{T} = 1 \Rightarrow \frac{d\mathbf{r}}{ds} \cdot \frac{d\mathbf{r}}{ds} = 1 \)

Differentiating with respect to \( s \), we get
\[ 2 \frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} = 0 \Rightarrow \frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} = 0 \Rightarrow \mathbf{T} \cdot \frac{d^2\mathbf{r}}{ds^2} = 0 \]

\( \therefore \) The vector \( \frac{d^2\mathbf{r}}{ds^2} \) is perpendicular (normal/orthogonal) to \( \mathbf{T} \).

So, \( \frac{d^2\mathbf{r}}{ds^2} \) is normal to the curve at \( P \).

\[ \therefore \text{ Unit normal vector } \mathbf{N} = \frac{d^2\mathbf{r}}{ds^2} + \left| \frac{d^2\mathbf{r}}{ds^2} \right| \]

14.4.10 Notation: If \( \phi \neq \mathbb{D} \subseteq \mathbb{R} \), we defined a scalar function as a function \( \phi : \mathbb{D} \rightarrow \mathbb{R} \) and a vector function as \( \mathbf{r} : \mathbb{D} \rightarrow \mathbb{V} \), where \( \mathbb{V} \) is the set of all 3-dimensional vectors. Also we defined a scalar point function in \( \phi \neq \mathbb{D} \subseteq \mathbb{R}^3 \) as a function \( \phi : \mathbb{D} \rightarrow \mathbb{R} \) and a vector point function as a function \( \mathbf{r} : \mathbb{D} \rightarrow \mathbb{V} \). If \( P = (x,y,z) \in \mathbb{D} \subseteq \mathbb{R}^3 \), we write \( \phi(P) \) as \( \phi(x,y,z) \) and \( \mathbf{r}(P) \) as \( \mathbf{r}(x,y,z) \). If \( \mathbf{r} \) is the position vector of \( P \) (with reference to the origin \( O \)), then \( \phi(P) \) and \( \mathbf{r}(P) \) are also written as \( \phi(\mathbf{r}) \) and \( \mathbf{r}(\mathbf{r}) \) respectively.

14.5 GRADIENT OF A SCALAR POINT FUNCTION

14.5.1 Definition (Gradient of a scalar point function): Let \( \phi \) be a scalar point function having partial derivatives \( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \). The gradient of \( \phi \) denoted by grad \( \phi \) is defined as
\[ \text{grad} \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} \]

14.5.2 Notation: If the operator \( \nabla \) (which is read as del or nabla) stands for \( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \),
then \( \text{grad} \phi \) is written as \( \text{grad} \phi = \nabla \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} \), which is simply written as \( \sum \mathbf{i} \frac{\partial \phi}{\partial x} \).
14.5.3 Theorem: If \( f \) and \( g \) are scalar point functions, then

(i) \( \text{grad} (f \pm g) = \text{grad} f \pm \text{grad} g \)

(ii) \( \text{grad} (fg) = (\text{grad} f)g + f(\text{grad} g) \)

Proof: (i) \( \text{grad} (f \pm g) = \sum \frac{\partial}{\partial x} (f \pm g) \)

\[
= \sum \left( \frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x} \right)
= \sum \frac{\partial f}{\partial x} \pm \sum \frac{\partial g}{\partial x}
= \text{grad} f \pm \text{grad} g
\]

(ii) \( \text{grad} (fg) = \sum \frac{\partial}{\partial x} (fg) = \sum \left( \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x} \right) \)

\[
= \left( \sum \frac{\partial f}{\partial x} \right) g + f \sum \frac{\partial g}{\partial x}
= (\text{grad} f)g + f(\text{grad} g)
\]

14.5.4 Theorem: A necessary and sufficient condition for a scalar point function \( f \) to be constant is that \( \nabla f = \overline{0} \).

Proof: (i) Suppose \( f \) is constant.

\[
\therefore \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0
\]

\[
\therefore \nabla f = \overline{1} \frac{\partial f}{\partial x} + \overline{j} \frac{\partial f}{\partial y} + \overline{k} \frac{\partial f}{\partial z} = \overline{1}0 + \overline{j}0 + \overline{k}0 = \overline{0}
\]

(ii) Suppose \( \nabla f = \overline{0} \Rightarrow \overline{1} \frac{\partial f}{\partial x} + \overline{j} \frac{\partial f}{\partial y} + \overline{k} \frac{\partial f}{\partial z} = \overline{j}0 + \overline{j}0 + \overline{k}0 \)

\[
\Rightarrow \frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \Rightarrow f \text{ is independent of } x, y, z
\]

\[
\Rightarrow f \text{ is a constant.}
\]
14.5.5 Theorem: If \( \phi \) is a scalar point function and \( c \in \mathbb{R} \), then
\[
\text{grad}(c\phi) = c \text{grad}\phi.
\]

Proof: 
\[
\text{grad}(c\phi) = \sum_i \frac{\partial}{\partial x_i} (c\phi) = \sum_i c \frac{\partial \phi}{\partial x_i} = c \sum_i \frac{\partial \phi}{\partial x_i} = c \text{grad}\phi
\]

SOLVED PROBLEMS:

14.5.6: Show that \( \nabla r = \frac{r}{|r|} \) if \( r = x\hat{i} + y\hat{j} + z\hat{k} \) and \( r = |r| \).

Solution: 
\[ r = |r| = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2 \Rightarrow 2r \frac{dr}{dx} = 2x \Rightarrow \frac{dr}{dx} = \frac{x}{r}. \]
Similarly, \( \frac{dy}{dr} = \frac{y}{r}, \frac{dz}{dr} = \frac{z}{r} \).

\[
\nabla r = \sum_i \frac{dr}{dx} \hat{i} = \sum_i \frac{x}{r} \hat{i} + \frac{y}{r} \hat{j} + \frac{z}{r} \hat{k} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{r}{r} \hat{r}
\]

14.5.7: Show that \( \nabla r^m = mr^{m-2}r \), if \( r = x\hat{i} + y\hat{j} + z\hat{k} \) and \( r = |r| \).

Solution: 
\[
\nabla r^m = \sum_i \frac{\partial}{\partial x_i} (r^m) = \sum_i mr^{m-1} \frac{\partial r}{\partial x_i} = \sum_i mr^{m-1} \frac{x}{r} = \sum_i mr^{m-2} \hat{i} = mr^{m-2}r
\]

14.5.8: In 14.4.17, if \( m = 1 \), the \( \nabla r = r^{-1} \hat{r} = \frac{\hat{r}}{r} \) and if \( m = 2 \), the \( \nabla r^2 = 2r\hat{r} \).

14.5.9: If \( \phi = x^3 - y^3 + x^2z \), then find \( \nabla \phi \) at \((1,1,-2)\).

Solution: 
\[
\frac{\partial \phi}{\partial x} = 3x^2 + 2xz, \quad \frac{\partial \phi}{\partial y} = -3y^2, \quad \frac{\partial \phi}{\partial z} = x^2
\]

\[
\therefore \text{grad}\phi = \sum_i \frac{\partial \phi}{\partial x_i} = (3x^2 + 2xz)\hat{i} - 3y^2\hat{j} + x^2\hat{k}
\]

At \((1,1,-2)\), \(\text{grad}\phi = -3\hat{j} + 3\hat{k}\).

14.6 LEVEL SURFACE

14.6.1 Definition (Level Surface): Let \( \phi \) be a scalar point function over the domain \( S \subseteq \mathbb{R}^3 \) and \( c \in \mathbb{R} \). The set of all points \( P \in S \) such that \( \phi(P) = c \) is called a level surface. If \( P = (x,y,z) \), then the level
surface is also denoted by $\phi(x, y, z) = c$. If there is a point $P_0 \in S \implies \phi(P_0) = c$, then the surface through $P_0$ is the set of all points $Q$ in $S \implies \phi(Q) = \phi(P_0)$.

14.6.2 Note: If $P, Q$ are points on a level surface of $\phi$, then $\phi(P) = \phi(Q)$.

14.6.3 Definition: Let $P$ be a point in space and $\delta > 0$. Then the set of all points $Q : PQ < \delta$ is called a $\delta$-neighbourhood of $P$. If $P$ is deleted from this $\delta$-neighbourhood, then remaining set is called deleted $\delta$-neighbourhood of $P$.

14.6.4 Definition: Let $P$ be a point and let $\ell \in \mathbb{R}$. Let $\phi$ be a scalar point function defined in a deleted neighbourhood of $P$. If to each $\epsilon > 0, \exists \delta > 0 : 0 < PQ < \delta \implies |\phi(Q) - \epsilon| < \epsilon$, then $\ell$ is called the limit of $\phi$ at $P$ and is written as $\lim_{Q \to P} \phi(Q) = \ell$.

14.6.5 Definition: Let $P$ be a point and let $\overrightarrow{V} \in V$ (the set of all 3-dimensional vectors). Let $\overrightarrow{\phi}$ be a vector point function defined in a deleted neighbourhood of $P$. If to each $\epsilon > 0, \exists \delta > 0 : 0 < PQ < \delta \implies |\overrightarrow{\phi}(Q) - \overrightarrow{\ell}| < \epsilon$, then $\overrightarrow{\ell}$ is called the limit of $\overrightarrow{\phi}$ at $P$ and is written as $\lim_{Q \to P} \overrightarrow{\phi}(Q) = \overrightarrow{\ell}$.

14.6.6 Definiton: Let $P$ be a point and $\phi$ be a scalar point function defined in a neighbourhood of $P$. We say that $\phi$ is continuous at $P$ if $\lim_{Q \to P} \phi(Q) = \phi(P)$.

14.7 DIRECTIONAL DERIVATIVE

14.7.1 Definition (Directional derivative): Let $\phi$ be a scalar point function defined in a neighbourhood $D$ of a point $P$. Let $L$ be a ray through $P$ in the direction of a unit vector $\overrightarrow{e}$ through $P$.

Let $Q \in L \cap D, Q \neq P$. If $\lim_{Q \to P} \frac{\phi(Q) - \phi(P)}{QP}$ exists, then the limit is called the directional derivative of $\phi$ at $P$ in the direction of $\overrightarrow{e}$.

It is denoted by $\frac{\partial \phi}{\partial \overrightarrow{e}}$ or $\frac{\partial \phi}{\partial s}$ if $s = PQ$.

14.7.2 Note: If $\overrightarrow{e} = \overrightarrow{1}$, unit vector along $\overrightarrow{OX}$, then $\frac{\partial \phi}{\partial \overrightarrow{e}} = \frac{\partial \phi}{\partial \overrightarrow{1}} = \frac{\partial \phi}{\partial x}$.

If $\overrightarrow{e} = \overrightarrow{j}$, unit vector along $\overrightarrow{OY}$, then $\frac{\partial \phi}{\partial \overrightarrow{e}} = \frac{\partial \phi}{\partial \overrightarrow{j}} = \frac{\partial \phi}{\partial y}$.

If $\overrightarrow{e} = \overrightarrow{k}$, unit vector along $\overrightarrow{OZ}$, then $\frac{\partial \phi}{\partial \overrightarrow{e}} = \frac{\partial \phi}{\partial \overrightarrow{k}} = \frac{\partial \phi}{\partial z}$.
14.7.3 **Note**: In the definition 14.7.1, \( \phi \) can be replaced by a vector point function \( \mathbf{\tau} \).

14.7.4 **Note**: If \( \mathbf{\tau} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k} \) when \( f_1, f_2, f_3 \) have directional derivatives at \( P \) along \( \mathbf{e} \), then

\[
\frac{\partial \mathbf{\tau}}{\partial s} = \frac{\partial f_1}{\partial s} \mathbf{i} + \frac{\partial f_2}{\partial s} \mathbf{j} + \frac{\partial f_3}{\partial s} \mathbf{k}.
\]

14.7.5 **Theorem**: \( \nabla \phi \) is a vector normal to the level surface \( \phi(x, y, z) = c \), where \( c \) is a constant.

**Proof**: Let \( P(x, y, z) \) be a point on the level surface and \( \mathbf{\tau} \) be the unit tangent vector at \( P \). The position vector of \( P \) is \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \).

\[
\frac{\partial \mathbf{r}}{\partial s} = \frac{\partial x}{\partial s} \mathbf{i} + \frac{\partial y}{\partial s} \mathbf{j} + \frac{\partial z}{\partial s} \mathbf{k}.
\]

\[
\phi(x, y, z) = c \Rightarrow \frac{\partial \phi}{\partial s} = 0 \Rightarrow \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial s} = 0
\]

\[
\Rightarrow \left( \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} \right) \cdot \left( \frac{\partial x}{\partial s} \mathbf{i} + \frac{\partial y}{\partial s} \mathbf{j} + \frac{\partial z}{\partial s} \mathbf{k} \right) = 0
\]

\[
\Rightarrow \nabla \phi \cdot \frac{\partial \mathbf{r}}{\partial s} = 0
\]

\[
\Rightarrow \nabla \phi \cdot \mathbf{\tau} = 0
\]

\[\therefore \nabla \phi \text{ is normal to the surface } \phi \text{ at } P.\]

14.7.6 **Theorem**: If \( \mathbf{N} \) is a unit vector normal to the level surface \( \phi(x, y, z) = c \) at \( P(x, y, z) \) in the increasing direction of \( \phi \), then \( \nabla \phi = \frac{\partial \phi}{\partial \mathbf{N}} \mathbf{N} \).

**Proof**: Since \( \nabla \phi \) is normal to the surface \( \phi(x, y, z) = c \), there exists a scalar \( k \) \( \nabla \phi = k \mathbf{N} \), where \( \mathbf{N} \) is unit normal vector to the surface at \( P \).

\[\therefore \text{Directional derivative of } \phi \text{ in the direction of } \mathbf{N} \text{ is } \frac{\partial \phi}{\partial \mathbf{N}} = \nabla \phi \cdot \mathbf{N} \]}

\[=\mathbf{K}\mathbf{N} \cdot \mathbf{N} \]

\[= K \cdot \mathbf{N} \]

\[\therefore \nabla \phi = K \mathbf{N} = \frac{\partial \phi}{\partial \mathbf{N}} \mathbf{N} \]
14.7.7 Note: \( |\operatorname{grad} \phi| = |K \cdot N| = K = \frac{\partial \phi}{\partial N} \)

14.7.8 Theorem: \( \operatorname{grad} \phi \) is a vector in the direction in which the maximum value of \( \frac{\partial \phi}{\partial s} \) occurs.

Proof: The directional derivative of \( \phi \) at \( P \) in the direction of the unit vector \( \vec{e} \) is

\[
\frac{\partial \phi}{\partial s} = \vec{e} \cdot \operatorname{grad} \phi = \vec{e} \cdot \frac{\partial \phi}{\partial N} N = (\vec{e} \cdot N) \frac{\partial \phi}{\partial N} = \frac{\partial \phi}{\partial N} \cos (\vec{e}, N),
\]

where \( (\vec{e}, N) \) is the angle between \( \vec{e} \) and \( N \) and this will be maximum, when \( \cos (\vec{e}, N) = 1 \) i.e. when \( (\vec{e}, N) = 0 \) i.e. \( \vec{e} \) is along the normal \( N \). Hence directional derivative is maximum along the normal to the surface and the maximum value of DD is \( = |\nabla \phi| \).

14.7.9 Theorem: Let \( \vec{r} \) be a vector point function defined in a neighbourhood of a point \( P \) and \( \vec{e} \) be a unit vector through \( P \), then \( \frac{\partial \vec{r}}{\partial \vec{e}} = \vec{e} \).

Proof: Let \( P \) be a point in the domain of \( \vec{r} \) and \( \phi \) be a point on a ray \( L \) though \( P(Q \neq P) \) in the direction of \( \vec{e} \).

Then \( \lim_{Q \to P} \frac{\vec{r}(Q) - \vec{r}(P)}{PQ} = \lim_{Q \to P} \frac{OQ - OP}{PQ} = \lim_{Q \to P} \frac{PQ}{QP} \) (where \( \vec{r}(P) = \overrightarrow{OP} \))

\[
\therefore \frac{\partial \vec{r}}{\partial \vec{e}} = \lim_{Q \to P} \frac{PQ \cdot \vec{e}}{QP} = \lim_{Q \to P} \vec{e} = \vec{e}.
\]

14.7.10 Note: \( \frac{\partial \vec{r}}{\partial i} = \frac{\partial \vec{r}}{\partial x} = \vec{i}, \frac{\partial \vec{r}}{\partial j} = \frac{\partial \vec{r}}{\partial y} = \vec{j}, \frac{\partial \vec{r}}{\partial k} = \frac{\partial \vec{r}}{\partial z} = \vec{k} \)

14.7.11 Note: If \( \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \), then \( \frac{\partial \vec{r}}{\partial \vec{e}} = \frac{\partial x}{\partial \vec{e}} \vec{i} + \frac{\partial y}{\partial \vec{e}} \vec{j} + \frac{\partial z}{\partial \vec{e}} \vec{k} = \vec{e} \).

SOLVED PROBLEMS

14.7.12: If \( r = |\vec{r}| \), then show that \( \frac{\partial \vec{r}}{\partial \vec{e}} = \frac{\vec{r} \cdot \vec{e}}{r} \)

Solution: We know that \( r^2 = \vec{r} \cdot \vec{r} \)

\[
\therefore \frac{\partial}{\partial \vec{e}} (\vec{r} \cdot \vec{r}) = \frac{\partial}{\partial \vec{e}} (r^2) \]
\[ 2r \frac{\partial r}{\partial \varepsilon} = 2r \frac{\partial r}{\partial \varepsilon} \]

\[ r \cdot \frac{\partial r}{\partial \varepsilon} = r \frac{\partial r}{\partial \varepsilon} \]

\[ r \cdot \varepsilon = r \frac{\partial r}{\partial \varepsilon} \]

\[ \frac{\partial r}{\partial \varepsilon} = \frac{r \cdot \varepsilon}{r} \]

**14.7.13 Note:**
\[
\frac{\partial r}{\partial x} = \frac{1}{r} \cdot \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{1}{r} \cdot \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{1}{r} \cdot \frac{z}{r}
\]

**14.7.14 Theorem:** The directional derivative of a scalar point function \( \phi \) at a point \( P \) in the direction of a unit vector \( \varepsilon \) is \( (\text{grad} \phi)_p \cdot \varepsilon \).

**Proof:** Directional derivative of \( \phi \) at \( P \) along \( \varepsilon \) is
\[
\frac{\partial \phi}{\partial \varepsilon} = \frac{\partial \phi}{\partial s}
\]

\[
= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial s} = \left( \frac{\partial \phi}{\partial x} x + \frac{\partial \phi}{\partial y} y + \frac{\partial \phi}{\partial z} z \right) \cdot \left( \frac{\partial x}{\partial s} + \frac{\partial y}{\partial s} + \frac{\partial z}{\partial s} \right)
\]

\[
= (\text{grad} \phi)_p \cdot \varepsilon
\]

**14.7.15 Definition (Angle between two surfaces):** Let \( P \) be a point of intersection of the level surfaces \( \phi_1 (x, y, z) = 0 \) and \( \phi_2 (x, y, z) = 0 \). The angle between the two surfaces at \( P \) is defined as the angle between the normals to the two surfaces at \( P \).

**14.7.16 Note:** If \( \theta \) is the angle between the surfaces at \( P \), then
\[
\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|}.
\]

(since \( \nabla \phi_1 \) is normal to the surface \( \phi_1 \) at \( P \)).

**SOLVED PROBLEMS**

**14.7.17:** Show that \( \nabla f(r) = f'(r) \frac{\mathbf{r}}{r} \).

**Solution:**
\[
\nabla f(r) = \sum \mathbf{r} \frac{\partial}{\partial x} f(r)
\]
\[
\sum i f'(r) \frac{\partial r}{\partial x}.
\]

\[
= \sum i f'(r) \frac{X}{r}
\]

\[
= \frac{f'(r)}{r} \sum x i
\]

\[
= f'(r) \frac{r}{r}
\]

14.7.18: Find the directional derivative of \( f = x^2 - y^2 + 2z^2 \) at \( P(1,2,3) \) is the direction of \( \overrightarrow{PQ} \), where \( Q = (5,0,4) \) and find when it is maximum. Also find the maximum value of the directional derivative.

Solution:

\[
\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -2y, \quad \frac{\partial f}{\partial z} = 4z
\]

\[
\text{grad} f = \sum i \frac{\partial f}{\partial x} = 2x i - 2y j + 4z k
\]

At \( P(1,2,3) \), \( \text{grad} f = 2 i - 2 j + 4z k \)

\[
\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 4 i - 2 j + k
\]

Unit vector along \( \overrightarrow{PQ} \) is \( \hat{e} = \frac{4 i - 2 j + k}{\sqrt{21}} \)

\[\therefore \text{Directional derivative of } f \text{ at } P \text{ along } \overrightarrow{PQ} \text{ is } = (\text{grad } f)_P \cdot \hat{e} \]

\[
= (2 i - 4 j + 12 k) \cdot \frac{(4 i - 2 j + k)}{\sqrt{21}}
\]

\[
= \frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}}
\]

Directional derivative of \( f \) at \( P \) is maximum along \( \text{grad} f = 2 i - 4 j + 12 k \)

Maximum value of directional derivative of \( f \) at \( P \) along \( \overrightarrow{PQ} \) is \( |\text{grad } f| \)

\[
= \sqrt{4 + 16 + 144}
\]
14.7.19: Find the angle between surfaces of the spheres \( x^2 + y^2 + z^2 = 29 \) and \( x^2 + y^2 + z^2 + 4x - 6y - 82 - 47 = 0 \) at \((4, -3, 2)\).

Solution: Let \( \phi_1 = x^2 + y^2 + z^2 - 29 \)
\[ \phi_2 = x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 \]
\[ \nabla \phi_1 = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k} \]
\[ \nabla \phi_1 = (2x + 4) \mathbf{i} + (2y - 6) \mathbf{j} + (2z - 8) \mathbf{k} \]

At \( P(4, -3, 2) \), \( \nabla \phi_1 = 8 \mathbf{i} - 6 \mathbf{j} + 4 \mathbf{k} \)
\[ \nabla \phi_2 = 12 \mathbf{i} - 12 \mathbf{j} - 4 \mathbf{k} \]

If \( \theta \) is the angle between the surfaces at \( P \), then
\[
\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|} = \frac{(8 \mathbf{i} - 6 \mathbf{j} + 4 \mathbf{k}) \cdot (12 \mathbf{i} - 12 \mathbf{j} - 4 \mathbf{k})}{\sqrt{64 + 36 + 16} \sqrt{144 + 144 + 16}}
\]
\[
= \frac{152}{\sqrt{116 \times 304}} = \frac{19}{29} \Rightarrow \theta = \cos^{-1} \frac{19}{29}
\]

14.8 DIVERGENCE AND CURL

14.8.1 Definition (Divergence): If \( \overrightarrow{F} \) is a differentiable vector point function, then the divergence of \( \overrightarrow{F} \) is defined as
\[
\text{div} \overrightarrow{F} = \mathbf{i} \cdot \frac{\partial \overrightarrow{F}}{\partial x} + \mathbf{j} \cdot \frac{\partial \overrightarrow{F}}{\partial y} + \mathbf{k} \cdot \frac{\partial \overrightarrow{F}}{\partial z}
\]

14.8.2 Note: \( \text{div} \overrightarrow{F} \) is also written as \( \nabla \cdot \overrightarrow{F} = \sum \mathbf{i} \cdot \frac{\partial \overrightarrow{F}}{\partial x} \)

14.8.3 Note: If \( \overrightarrow{F} \) is a vector point function, then \( \text{div} \overrightarrow{F} \) is a scalar point function.

14.8.4 Theorem: If \( \overrightarrow{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k} \), then \( \text{div} \overrightarrow{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \)

Proof: \( \text{div} \overrightarrow{F} = \sum \mathbf{i} \cdot \frac{\partial \overrightarrow{F}}{\partial x} = \sum \mathbf{i} \left( \frac{\partial F_1}{\partial x} \mathbf{i} + \frac{\partial F_2}{\partial x} \mathbf{j} + \frac{\partial F_3}{\partial x} \mathbf{k} \right) \)
14.8.5 Definition: If $\vec{F}$ is a vector point function and $\text{div}\,\vec{F} = 0$ then $\vec{F}$ is called a solenoidal vector.

14.8.6 Note: If $\vec{F}$ is a constant vector point function, then $\vec{F}$ is solenoidal. ($\because \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is constant $\Rightarrow F_1, F_2, F_3$ are constant $\Rightarrow \frac{\partial F_1}{\partial x} = 0 = \frac{\partial F_2}{\partial y} = \frac{\partial F_3}{\partial z} \Rightarrow \text{div}\,\vec{F} = 0$).

**SOLVED PROBLEMS**

14.8.7: Show that $y^3z^2\hat{i} - 3x^2z^5\hat{j} + 5x^5y^4\hat{k}$ is solenoidal.

Solution: Let $\vec{F} = y^3z^2\hat{i} - 3x^2z^5\hat{j} + 5x^5y^4\hat{k}$

\[
\text{div}\,\vec{F} = \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial}{\partial y}(-3x^2z^5) + \frac{\partial}{\partial z}(5x^5y^4) \\
= 0 + 0 + 0 = 0 \quad \therefore \vec{F} \text{ is solenoidal.}
\]

14.8.8: Find $p$, if $\vec{F} = (x + 3y)\hat{i} + (y = 2z)\hat{j} + (x + pz)\hat{k}$ is solenoidal.

Solution: If $\vec{F}$ is solenoidal, then $\text{div}\,\vec{F} = 0$.

\[
\Rightarrow \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + pz) = 0 \\
\Rightarrow 1 + 1 + p = 0 \\
\Rightarrow p = -2
\]

14.8.9 Definition (Curl): If $\vec{F}$ is a differentiable vector point function, then curl $\vec{F}$ is defined as

\[
\text{Curl}\,\vec{F} = \hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \frac{\partial \vec{F}}{\partial z}
\]

14.8.10 Note: Curl $\vec{F}$ is a vector point function. It is also written as $\text{curl}\,\vec{F} = \sum \vec{F} \times \frac{\partial F_i}{\partial x}$.

14.8.11 Theorem: If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then curl $\vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{vmatrix}$.
Proof: \( \text{Curl} \bar{F} = \sum \mathbf{i} \times \frac{\partial F_x}{\partial x} + \mathbf{j} \times \frac{\partial F_y}{\partial y} + \mathbf{k} \times \frac{\partial F_z}{\partial z} \)
\[
= \mathbf{i} \times \left( \frac{\partial F_1}{\partial x} \mathbf{i} + \frac{\partial F_2}{\partial y} \mathbf{j} + \frac{\partial F_3}{\partial z} \mathbf{k} \right) + \mathbf{j} \times \left( \frac{\partial F_1}{\partial y} \mathbf{i} + \frac{\partial F_2}{\partial z} \mathbf{j} + \frac{\partial F_3}{\partial x} \mathbf{k} \right) + \mathbf{k} \times \left( \frac{\partial F_1}{\partial z} \mathbf{i} + \frac{\partial F_2}{\partial x} \mathbf{j} + \frac{\partial F_3}{\partial y} \mathbf{k} \right)
\]
\[
= \frac{\partial F_2}{\partial x} \mathbf{k} + \frac{\partial F_1}{\partial x} \mathbf{j} + \frac{\partial F_3}{\partial y} \mathbf{i} - \frac{\partial F_1}{\partial y} \mathbf{k} - \frac{\partial F_3}{\partial z} \mathbf{j} + \frac{\partial F_2}{\partial z} \mathbf{i} + \frac{\partial F_1}{\partial z} \mathbf{j} + \frac{\partial F_2}{\partial y} \mathbf{i} - \frac{\partial F_3}{\partial x} \mathbf{k}
\]
\[
= \mathbf{i} \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \mathbf{j} \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \mathbf{k} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)
\]
\[
= \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{vmatrix}
\]

14.8.12 Definition: If \( \bar{F} \) is a vector point function and \( \text{curl} \bar{F} = \mathbf{0} \), then \( \bar{F} \) is said to be irrotational.

14.8.13 Note: If \( \bar{F} \) is a constant vector function, then \( \bar{F} \) is irrotational \( (\therefore \bar{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k} \) is constant \( \Rightarrow F_1, F_2, F_3 \) are constant \( \Rightarrow \frac{\partial F_1}{\partial x} = 0 \).

14.8.14 SAQ: If \( \bar{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \), then show that \( \text{div} \bar{r} = 3 \) and curl \( \bar{r} = \mathbf{0} \).

14.8.15 SAQ: If \( \bar{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \), then \( \frac{\bar{r}}{r^3} \) is irrotational and \( \frac{\bar{r}}{r^3} \) is solenoidal.

SOLVED PROBLEMS

14.8.16: Show that \( \bar{r} = (y + z) \mathbf{i} + (z + x) \mathbf{j} + (x + y) \mathbf{k} \) is irrotational.

Solution: Curl \( \bar{r} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y + z & z + x & x + y
\end{vmatrix} \)
\[
= \mathbf{i} \left( \frac{\partial}{\partial y} (x + y) - \frac{\partial}{\partial z} (z + x) \right) - \mathbf{j} \left( \frac{\partial}{\partial x} (x + y) - \frac{\partial}{\partial z} (y + z) \right) + \mathbf{k} \left( \frac{\partial}{\partial x} (z + x) - \frac{\partial}{\partial y} (y + z) \right)
\]
\[
= \mathbf{i} \left( 1 - 1 \right) + \mathbf{j} \left( 1 - 1 \right) + \mathbf{k} \left( 1 - 1 \right)
\]
\[
\therefore \bar{r} \text{ is irrotational.}
14.8.17: Find $a, b, c$ such that $\mathbf{F} = (x + 2y + az) \mathbf{i} + (bx - 3y - z) \mathbf{j} + (4x + cy + 2z) \mathbf{k}$ is irrotational.

Solution: Since $\mathbf{F}$ is irrotational, we have $\nabla \times \mathbf{F} = \mathbf{0}$.

$$\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x + 2y + az & bx - 3y - z & 4x + cy + 2z
\end{vmatrix} = \mathbf{0}$$

$$\Rightarrow \mathbf{i} \left( \frac{\partial}{\partial y} (4x + cy + 2z) - \frac{\partial}{\partial z} (bx - 3y - z) \right) - \mathbf{j} \left( \frac{\partial}{\partial x} (4x + cy + 2z) - \frac{\partial}{\partial z} (x + 2y + az) \right) + \mathbf{k} \left( \frac{\partial}{\partial x} (bx - 3y - 2) - \frac{\partial}{\partial y} (x + 2y + az) \right) = \mathbf{0}$$

$$\Rightarrow \mathbf{i} (c+1) - \mathbf{j} (4-a) + \mathbf{k} (b-z) = \mathbf{0}$$

$$\Rightarrow c = -1, a = 4, b = 2$$

14.8.18: Show that $\nabla \times (\mathbf{r} \times \mathbf{a}) = -2\mathbf{a}$

Solution: $\nabla \times (\mathbf{r} \times \mathbf{a}) = \sum \mathbf{i} \times \frac{\partial}{\partial x} (\mathbf{r} \times \mathbf{a}) = \sum \mathbf{i} \times \left( \frac{\partial \mathbf{r} \times \mathbf{a}}{\partial x} \right)$

$$= \sum \mathbf{i} \times \mathbf{a} = \sum (\mathbf{a} \cdot \mathbf{i}) \mathbf{i} - \sum (\mathbf{i} \cdot \mathbf{a}) \mathbf{i}$$

$$= \sum (\mathbf{a} \cdot \mathbf{i}) \mathbf{i} - \sum \mathbf{a} = \mathbf{a} - 3\mathbf{a} = -2\mathbf{a}$$

14.8.19: Show that $\mathbf{r}^n \mathbf{F}$ is irrotational.

Solution: Let $\mathbf{F} = r^n \mathbf{F} = r^n x \mathbf{i} + r^n y \mathbf{j} + r^n z \mathbf{k}$

$$\nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
r^n x & r^n y & r^n z
\end{vmatrix}$$

$$= \mathbf{i} \left( \frac{\partial}{\partial y} (r^n z) - \frac{\partial}{\partial z} (r^n y) \right) - \mathbf{j} \left( \frac{\partial}{\partial x} (r^n z) - \frac{\partial}{\partial z} (r^n x) \right) + \mathbf{k} \left( \frac{\partial}{\partial x} (r^n y) - \frac{\partial}{\partial y} (r^n x) \right)$$
14.8.20: If $r^n \mathbf{T}$ is solenoidal, then show that $n = -3$.

Solution: Let $\mathbf{T} = r^n \mathbf{T} \Rightarrow \mathbf{T} = r^n x \mathbf{i} + r^n y \mathbf{j} + r^n z \mathbf{k}$

Since $\mathbf{T}$ is solenoidal, $\text{div} \mathbf{T} = 0$

\[
\frac{\partial}{\partial x} (r^n x) + \frac{\partial}{\partial y} (r^n y) + \frac{\partial}{\partial z} (r^n z) = 0
\]

\[
\Rightarrow \sum \left( r^n \cdot 1 + x \cdot nr^{n-1} \frac{x}{r} \right) = 0
\]

\[
\Rightarrow 3r^n + nr^{n-2} \sum x^2 = 0
\]

\[
\Rightarrow 3r^n + nr^{n-2} r^2 = 0
\]

\[
\Rightarrow (3 + n)r^4 = 0
\]

\[
\Rightarrow n = -3
\]

14.8.21 Note: $\frac{\mathbf{T}}{r^3}$ is solenoidal

14.8.22: Show that $f(r) \mathbf{T}$ is irrotational.

Solution: Let $\mathbf{F} = f(r) \mathbf{T} = f(r) x \mathbf{i} + f(r) y \mathbf{j} + f(r) z \mathbf{k}$

\[
\text{Curl} \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x f(r) & y f(r) & z f(r)
\end{vmatrix}
\]

\[
= \mathbf{i} \left( \frac{\partial}{\partial y} (z f(r)) - \frac{\partial}{\partial z} (y f(r)) \right) - \mathbf{j} \left( \frac{\partial}{\partial x} (z f(r)) - \frac{\partial}{\partial z} (x f(r)) \right) + \mathbf{k} \left( \frac{\partial}{\partial x} (y f(r)) - \frac{\partial}{\partial y} (x f(r)) \right)
\]
14.17

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\[ \frac{zf'(r)}{r} - \frac{yf'(r)}{r} = 0 \]

\( \therefore \overline{F} \) is irrotational.

14.8.23 Theorem: If \( \overline{f}, \overline{g} \) are two vector point functions, then 
\[ \text{div}(\overline{f} \pm \overline{g}) = \text{div}\overline{f} \pm \text{div}\overline{g} \].

Proof: 
\[ \text{div}(\overline{f} \pm \overline{g}) = \sum \overline{T} \cdot \frac{\partial}{\partial x} (\overline{f} \pm \overline{g}) = \sum \overline{T} \cdot \left( \frac{\partial \overline{f}}{\partial x} \pm \frac{\partial \overline{g}}{\partial x} \right) \]

\[ = \sum \overline{T} \cdot \frac{\partial \overline{f}}{\partial x} \pm \sum \overline{T} \cdot \frac{\partial \overline{g}}{\partial x} \]

\[ = \text{div}\overline{f} \pm \text{div}\overline{g} \]

14.8.24 Theorem: If \( \overline{f}, \overline{g} \) are two vector point functions, then 
\[ \text{Curl}(\overline{f} \pm \overline{g}) = \text{Curl}\overline{f} \pm \text{Curl}\overline{g} \].

Proof: 
\[ \text{Curl}(\overline{f} \pm \overline{g}) = \sum \overline{i} \times \frac{\partial}{\partial x} (\overline{f} \pm \overline{g}) = \sum \overline{i} \times \left( \frac{\partial \overline{f}}{\partial x} \pm \frac{\partial \overline{g}}{\partial x} \right) \]

\[ = \sum \overline{i} \times \frac{\partial \overline{f}}{\partial x} \pm \sum \overline{i} \times \frac{\partial \overline{g}}{\partial x} \]

\[ = \text{Curl}\overline{f} \pm \text{Curl}\overline{g} \]

14.8.25 Definitions: If \( \overline{a} \) is a vector, then the operator

\[ \overline{a} \cdot \overline{V} = (\overline{a} \cdot \overline{i}) \frac{\partial}{\partial x} + (\overline{a} \cdot \overline{j}) \frac{\partial}{\partial y} + (\overline{a} \cdot \overline{k}) \frac{\partial}{\partial z} \]

is defined as 
\[ (\overline{a} \cdot \overline{V})\phi = (\overline{a} \cdot \overline{i}) \frac{\partial \phi}{\partial x} + (\overline{a} \cdot \overline{j}) \frac{\partial \phi}{\partial y} + (\overline{a} \cdot \overline{k}) \frac{\partial \phi}{\partial z} \]

and 
\[ (\overline{a} \cdot \overline{V})\overline{F} = (\overline{a} \cdot \overline{i}) \frac{\partial \overline{F}}{\partial x} + (\overline{a} \cdot \overline{j}) \frac{\partial \overline{F}}{\partial y} + (\overline{a} \cdot \overline{k}) \frac{\partial \overline{F}}{\partial z} \]

where \( \phi \) is a scalar point function and \( \overline{F} \) is a vector point function.

If \( \overline{a} \) is a vector, then the operator

\[ \overline{a} \times \overline{V} = (\overline{a} \times \overline{i}) \frac{\partial}{\partial x} + (\overline{a} \times \overline{j}) \frac{\partial}{\partial y} + (\overline{a} \times \overline{k}) \frac{\partial}{\partial z} \]

is defined as
\[ (\vec{a} \times \nabla) \phi = (\vec{a} \times \vec{i}) \frac{\partial \phi}{\partial x} + (\vec{a} \times \vec{j}) \frac{\partial \phi}{\partial y} + (\vec{a} \times \vec{k}) \frac{\partial \phi}{\partial z} \]

where \( \phi \) is a scalar point function.

**14.8.26 Theorem:** If \( \vec{f} \) is a vector function and \( \phi \) is a scalar function which are differentiable, then

\[
\text{div} \, \phi \vec{f} = (\text{grad} \, \phi) \cdot \vec{f} \times \phi \text{div} \vec{f} \\
\text{curl} \, \vec{f} = (\text{grad} \, \phi) \times \vec{f} + \phi \text{curl} \vec{f}
\]

**Proof:**

\[
\text{div} \, \phi \vec{f} = \sum \vec{i} \cdot \frac{\partial}{\partial x} (\phi \vec{f}) = \sum \vec{i} \cdot \left( \frac{\partial \phi}{\partial x} \vec{f} + \phi \frac{\partial \vec{f}}{\partial x} \right) = \left[ \vec{i} \cdot \frac{\partial \phi}{\partial x} \vec{f} + \sum \vec{i} \cdot \phi \frac{\partial \vec{f}}{\partial x} \right] = \sum \vec{i} \cdot \frac{\partial \phi}{\partial x} \vec{f} + \phi \sum \vec{i} \cdot \frac{\partial \vec{f}}{\partial x} = (\text{grad} \, \phi) \cdot \vec{f} + \phi \text{div} \vec{f}
\]

\[
\text{curl} \, \phi \vec{f} = \sum \vec{i} \times \frac{\partial}{\partial x} (\phi \vec{f}) = \sum \vec{i} \times \left( \frac{\partial \phi}{\partial x} \vec{f} + \phi \frac{\partial \vec{f}}{\partial x} \right) = \sum \vec{i} \times \frac{\partial \phi}{\partial x} \vec{f} + \phi \sum \vec{i} \times \frac{\partial \vec{f}}{\partial x} = (\text{grad} \, \phi) \times \vec{f} + \phi \text{curl} \vec{f}
\]

**14.8.27 Theorem:** If \( \vec{f} \) and \( \vec{g} \) are differentiable vector functions then \( \text{div} (\vec{f} \times \vec{g}) = \vec{g} \cdot \text{curl} \vec{f} - \vec{f} \cdot \text{curl} \vec{g} \)

**Proof:**

\[
\text{div} (\vec{f} \times \vec{g}) = \sum \vec{i} \cdot \frac{\partial}{\partial x} (\vec{f} \times \vec{g}) = \sum \vec{i} \cdot \left( \frac{\partial \vec{f}}{\partial x} \times \vec{g} + \frac{\partial \vec{g}}{\partial x} \vec{f} \right)
\]

\[
= \sum \vec{i} \cdot \frac{\partial \vec{f}}{\partial x} \times \vec{g} + \sum \vec{i} \cdot \vec{f} \times \frac{\partial \vec{g}}{\partial x} = \sum \vec{i} \cdot \frac{\partial \vec{f}}{\partial x} \times \vec{g} - \sum \vec{i} \times \frac{\partial \vec{g}}{\partial x} \cdot \vec{f} = \text{curl} \vec{f} \cdot \vec{g} - \text{curl} \vec{g} \cdot \vec{f} = \vec{g} \cdot \text{curl} \vec{f} - \vec{f} \cdot \text{curl} \vec{g}
\]
14.8.28 Corollary: If \( \mathbf{f} \) and \( \mathbf{g} \) are irrotational, then \( \mathbf{f} \times \mathbf{g} \) is solenoidal (\( \therefore \mathbf{f}, \mathbf{g} \) are irrotational \( \Rightarrow \text{curl} \, \mathbf{f} = \mathbf{0} = \text{curl} \, \mathbf{g} \)).

14.8.29 Theorem: If \( \mathbf{f} \) and \( \mathbf{g} \) are differentiable vector point functions, then

\[
\text{grad}(\mathbf{f} \cdot \mathbf{g}) = (\mathbf{f} \cdot \nabla)\mathbf{g} + (\mathbf{g} \cdot \nabla)\mathbf{f} + \mathbf{f} \times \text{curl} \, \mathbf{g} + \mathbf{g} \times \text{curl} \, \mathbf{f}
\]

**Proof:**

\[
\mathbf{f} \times \text{curl} \, \mathbf{g} = f \times \sum \mathbf{i} \times \frac{\partial g}{\partial x} = \sum \mathbf{i} \times \left( \mathbf{i} \times \frac{\partial g}{\partial x} \right)
\]

\[
= \sum \left( \frac{\partial g}{\partial x} \right) \mathbf{i} - (\mathbf{i} \cdot \mathbf{i}) \frac{\partial g}{\partial x}
\]

\[
= \sum \left( \frac{\partial g}{\partial x} \right) \mathbf{i} - \sum (\mathbf{i} \cdot \mathbf{i}) \frac{\partial g}{\partial x}
\]

\[
\therefore \sum \left( \frac{\partial g}{\partial x} \right) \mathbf{i} = \mathbf{f} \times \text{curl} \, \mathbf{g} + (\mathbf{f} \cdot \nabla)\mathbf{g}
\]

\[
\therefore \sum \left( \frac{\partial g}{\partial x} \right) \mathbf{i} = \mathbf{f} \times \text{curl} \, \mathbf{g} + (\mathbf{f} \cdot \nabla)\mathbf{g}
\]

Similarly, \( \sum \left( \frac{\partial \mathbf{f}}{\partial x} \right) \mathbf{i} = \mathbf{g} \times \text{curl} \, \mathbf{f} + (\mathbf{g} \cdot \nabla)\mathbf{f} \)

Adding we get \( \sum \left( \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial x} \right) \mathbf{i} = \mathbf{f} \times \text{curl} \, \mathbf{g} + \mathbf{g} \times \text{curl} \, \mathbf{f} + (\mathbf{f} \cdot \nabla)\mathbf{g} + (\mathbf{g} \cdot \nabla)\mathbf{f} \)

\[
\Rightarrow \sum \frac{\partial}{\partial x} (\mathbf{f} \cdot \mathbf{g}) = \mathbf{f} \times \text{curl} \, \mathbf{g} + \mathbf{g} \times \text{curl} \, \mathbf{f} + (\mathbf{f} \cdot \nabla)\mathbf{g} + (\mathbf{g} \cdot \nabla)\mathbf{f}
\]

\[
\text{grad}(\mathbf{f} \cdot \mathbf{g}) = \mathbf{f} \times \text{curl} \, \mathbf{g} + \mathbf{g} \times \text{curl} \, \mathbf{f} + (\mathbf{f} \cdot \nabla)\mathbf{g} + (\mathbf{g} \cdot \nabla)\mathbf{f}
\]

14.8.30 Theorem: If \( \mathbf{f}, \mathbf{g} \) are differentiable vector point functions, then

\[
\text{curl}(\mathbf{f} \times \mathbf{g}) = \mathbf{f} \text{ div } \mathbf{g} - \mathbf{g} \text{ div } \mathbf{f} + (\mathbf{g} \cdot \nabla)\mathbf{f} - (\mathbf{f} \cdot \nabla)\mathbf{g}
\]

**Proof:**

\[
\text{curl}(\mathbf{f} \times \mathbf{g}) = \sum \mathbf{i} \times \frac{\partial}{\partial x} (\mathbf{f} \times \mathbf{g}) = \sum \mathbf{i} \times \left( \frac{\partial \mathbf{f}}{\partial x} \times \mathbf{g} + \mathbf{f} \times \frac{\partial \mathbf{g}}{\partial x} \right)
\]
\[\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\] and \[\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}\]

where \(\phi\) is a scalar point function and \(\vec{F}\) is a vector point function.
14.8.33 Theorem: If \( \phi \) is a scalar point function which is differentiable, then \( \text{curl} \text{grad} \phi = 0 \).

Proof: \( \text{grad} \phi = \sum \hat{r} \frac{\partial \phi}{\partial x} \).

\[ \text{curl} \text{grad} \phi = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}
\end{vmatrix} \\
= \sum \left[ \hat{i} \left( \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) \right] = 0 \\
\]

14.8.34 Note: If \( \phi \) is a scalar point function, then \( \nabla \phi \) is irrotational. The equation \( \nabla^2 \phi = 0 \) is called Laplacian equation.

14.8.35 Theorem: If \( \vec{f} \) is a differentiable vector point function, then \( \text{div} \text{curl} \vec{f} = 0 \).

Proof: Let \( \vec{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} \)

\[ \text{curl} \vec{f} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
f_1 & f_2 & f_3
\end{vmatrix} = \hat{r} \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) - \hat{j} \left( \frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) + \hat{k} \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\
\]

\[ \therefore \text{div} \text{curl} \vec{f} = \frac{\partial}{\partial x} \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) - \frac{\partial}{\partial y} \left( \frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\
= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y} \\
= 0 \\
\]

14.8.36 Note: If \( \vec{f} \) is a vector point function, then \( \text{curl} \vec{f} \) is solenoidal.

14.8.37 Theorem: If \( \vec{f} \) is differential vector point function, then \( \text{curl} \text{curl} \vec{f} = \text{grad} \text{div} \vec{f} - \nabla^2 \vec{f} \).

Proof: \( \text{curl} \text{curl} \vec{f} = \sum \hat{r} \times \frac{\partial}{\partial x} (\text{curl} \vec{f}) = \sum \hat{r} \times \frac{\partial}{\partial x} \left( \hat{r} \times \frac{\partial \vec{f}}{\partial x} + \hat{j} \times \frac{\partial \vec{f}}{\partial y} + \hat{k} \times \frac{\partial \vec{f}}{\partial z} \right) \)
\[ \nabla \cdot (\vec{\tau} \cdot \vec{\tau}) = \sum \frac{\partial}{\partial x} \left( \vec{\tau} \cdot \frac{\partial \vec{\tau}}{\partial x} + \vec{\tau} \cdot \frac{\partial \vec{\tau}}{\partial y} + \vec{\tau} \cdot \frac{\partial \vec{\tau}}{\partial z} \right) + \sum \frac{\partial}{\partial y} \left( \vec{\tau} \cdot \frac{\partial \vec{\tau}}{\partial x} + \vec{\tau} \cdot \frac{\partial \vec{\tau}}{\partial y} + \vec{\tau} \cdot \frac{\partial \vec{\tau}}{\partial z} \right) + \sum \frac{\partial}{\partial z} \left( \vec{\tau} \cdot \frac{\partial \vec{\tau}}{\partial x} + \vec{\tau} \cdot \frac{\partial \vec{\tau}}{\partial y} + \vec{\tau} \cdot \frac{\partial \vec{\tau}}{\partial z} \right) \]

\[ = \vec{i} \cdot \nabla^2 \vec{f} + \vec{j} \cdot \frac{\partial^2 \vec{f}}{\partial y \partial x} + \vec{k} \cdot \frac{\partial^2 \vec{f}}{\partial z \partial x} + \vec{i} \cdot \frac{\partial^2 \vec{f}}{\partial x \partial y} + \vec{j} \cdot \frac{\partial^2 \vec{f}}{\partial x \partial z} + \vec{k} \cdot \frac{\partial^2 \vec{f}}{\partial y \partial z} \]

SOLVED PROBLEMS

14.8.38: If \( \vec{\tau} = \nabla \left( x^3 + y^3 + z^3 - 3xyz \right) \), find \( \nabla \cdot \vec{\tau} \) and \( \nabla \times \vec{\tau} \).

Solution:

\[ \vec{\tau} = \nabla \left( x^3 + y^3 + z^3 - 3xyz \right) \]

\[ = \sum \frac{\partial}{\partial x} \left( x^3 + y^3 + z^3 - 3xyz \right) \]

\[ = \sum (3x^2 - 3yz) = (3x^2 - 3yz)\vec{i} + (3y^2 - 3xz)\vec{j} + (3z^2 - 3xy)\vec{k} \]
Differential Operators

14.23

Differential Equation, Abstract Algebra...

\[ \text{div} \mathbf{F} = \frac{\partial}{\partial x} (3x^2 - 3yz) + \frac{\partial}{\partial y} (3y^2 - 3zx) + \frac{\partial}{\partial z} (3z^2 - 3xy) \]

\[ = 6x + 6y + 6z \]

\[ \text{curl} \mathbf{F} = \frac{\partial}{\partial x} \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{array} \]

\[ = \mathbf{i}(-3x + 3x) - \mathbf{j}(-3y + 3y) + \mathbf{k}(-3z + 3z) \]

\[ = \mathbf{0} \]

14.8.39: If \( \phi = x^2yz \) find \( \text{curl} \ \text{grad} \ \phi \).

Solution: \( \text{grad} \ \phi = \sum \mathbf{i} \frac{\partial \phi}{\partial x} = 2xyz \mathbf{i} + x^2z \mathbf{j} + x^2y \mathbf{k} \)

\[ \therefore \text{curl} \ \text{grad} \ \phi = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{array} \right| \]

\[ = \mathbf{i}(x^2 - x^2) - \mathbf{j}(2xy - 2xy) + \mathbf{k}(2xz - 2xz) \]

\[ = \mathbf{0} \]

14.8.40: If \( \mathbf{F} = 2xz^2 \mathbf{i} - yz \mathbf{j} + 3xz^3 \mathbf{k} \), then find \( \text{curl} \ \text{curl} \ \mathbf{F} \) at \((1,1,1)\).

Solution: \[ \text{curl} \ \text{curl} \mathbf{F} = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & -yz & 3xz^3 \end{array} \right| = \mathbf{i}(0 + y) - \mathbf{j}(3z^3 - 4xz) + \mathbf{k}(0 - 0) \]

\[ = y \mathbf{i} + (4xz - 3z^3) \mathbf{j} \]
\[
\text{curl } \nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & 4xz - 3z^3 & 0
\end{vmatrix}
= \mathbf{i}(0 - 4x + 9z^2) - \mathbf{j}(0 - 0) + \mathbf{k}(4z - 1)
= (9z^2 - 4x)\mathbf{i} + (4z - 1)\mathbf{k}
\]

At \((1,1,1)\), \(\text{curl } \nabla \times \mathbf{F} = 5\mathbf{i} + 3\mathbf{k}

**14.8.41**: If \(\mathbf{a}\) is a constant vector, then show that

\[
\text{curl } \frac{\mathbf{a} \times \mathbf{r}}{r^3} = \frac{\mathbf{a}}{r^3} + \frac{3\mathbf{a}}{r^5}
\]

**Solution**: \[
\text{curl } \frac{\mathbf{a} \times \mathbf{r}}{r^3} = \sum \mathbf{i} \times \frac{\partial}{\partial x} \left( \frac{\mathbf{a} \times \mathbf{r}}{r^3} \right)
\]

\[
= \sum \mathbf{i} \times \left\{ -\frac{3}{r^4} \frac{\partial r}{\partial x} (\mathbf{a} \times \mathbf{r}) + \frac{1}{r^3} \frac{\partial}{\partial x} (\mathbf{a} \times \mathbf{r}) \right\}
\]

\[
= \sum \mathbf{i} \times \left\{ -\frac{3}{r^4} \frac{x}{r} (\mathbf{a} \times \mathbf{r}) + \frac{1}{r^3} \left( \mathbf{a} \times \mathbf{i} \right) \right\}
\quad \because \frac{\partial}{\partial x} (\mathbf{a} \times \mathbf{r}) = \frac{\partial \mathbf{a}}{\partial x} \times \mathbf{r} + a \times \frac{\partial \mathbf{r}}{\partial x} = 0 + a \times \mathbf{i} = \mathbf{a} \times \mathbf{i}
\]

\[
= -\frac{3}{r^5} \sum x \mathbf{i} \times (\mathbf{a} \times \mathbf{r}) + \frac{1}{r^3} \sum \mathbf{i} \times (\mathbf{a} \times \mathbf{i})
\]

\[
= -\frac{3}{r^5} \mathbf{r} \times (\mathbf{a} \times \mathbf{r}) + \frac{1}{r^3} \sum \mathbf{i} \times (\mathbf{a} \times \mathbf{i})
\]

\[
= -\frac{3}{r^5} \left[ (\mathbf{r} \cdot \mathbf{r}) \mathbf{a} - (\mathbf{r} \cdot \mathbf{a}) \mathbf{r} \right] + \frac{1}{r^3} \left( 3\mathbf{a} - \mathbf{a} \right)
\]

\[
= -\frac{3}{r^5} \left( (\mathbf{r} \cdot \mathbf{r}) \mathbf{a} - (\mathbf{r} \cdot \mathbf{a}) \mathbf{r} \right) + \frac{2\mathbf{a}}{r^3}
\]

\[
= -\frac{3}{r^5} \mathbf{r} \times (\mathbf{a} \times \mathbf{r}) + \frac{3}{r^5} (\mathbf{r} \cdot \mathbf{a}) \mathbf{r} + \frac{2\mathbf{a}}{r^3}
\]

\[
= \frac{3}{r^5} (\mathbf{r} \cdot \mathbf{a}) \mathbf{r} - \frac{\mathbf{a}}{r^3}
\]
14.8.42: Show that $\nabla^2 \frac{1}{r} = 0$

Solution:

$$\nabla^2 \frac{1}{r} = \sum \frac{\partial^2}{\partial x^2} \left( \frac{1}{r} \right) = \sum \frac{\partial}{\partial x} \frac{\partial}{\partial x} \left( \frac{1}{r} \right) = \sum \frac{\partial}{\partial x} \left( -\frac{1}{r^2} \frac{\partial r}{\partial x} \right) = \sum \frac{\partial}{\partial x} \left( -\frac{x}{r^3} \right)$$

$$= \sum \left[ -\frac{1}{r^3} - x \left( \frac{3}{r^4} \frac{dr}{dx} \right) \right]$$

$$= \sum \left( -\frac{1}{r^3} + \frac{3x^2}{r^4} \right)$$

$$= -\frac{3}{r^3} + \frac{3}{r^5} \sum x^2$$

$$= -\frac{3}{r^3} + \frac{3}{r^5} \sum x^2$$

$$= -\frac{3}{r^3} + \frac{3}{r^5} \sum x^2$$

$$= 0$$

14.8.43: Show that $\nabla^2 r^n = r(n+1)r^{n-2}$

Solution:

$$\nabla^2 r^n = \sum \frac{\partial^2}{\partial x^2} (r^n) = \sum \frac{\partial}{\partial x} \frac{\partial}{\partial x} (r^n) = \sum \frac{\partial}{\partial x} \left( nr^{n-1} \frac{\partial r}{\partial x} \right)$$

$$= \sum \frac{\partial}{\partial x} \left( nr^{n-1} \frac{x}{r} \right) = \sum \frac{\partial}{\partial x} \left( nr^{n-2} x \right)$$

$$= n \sum \left( r^{n-2} + x \cdot (n-2) x \cdot r^{n-3} \cdot \frac{x}{r} \right)$$

$$= 3nr^{n-2} + n(n-2)r^{n-4} \sum x^2$$

$$= 3nr^{n-2} + n(n-2)r^{n-4} r^2$$

$$= nr^{n-2} (3 + n - 2)$$

$$= n(n+1)r^{n-2}$$
14.8.44: If \( \overrightarrow{F} \) is a vector point function and \( \overrightarrow{a} \) is a constant vector, then show that

\[
\nabla (\overrightarrow{a} \cdot \overrightarrow{F}) = (\overrightarrow{a} \cdot \nabla) \overrightarrow{F} + \overrightarrow{a} \times \text{curl} \overrightarrow{F}.
\]

**Solution:**

\[
\overrightarrow{a} \times \text{curl} \overrightarrow{F} = \overrightarrow{a} \times \left( \overrightarrow{i} \frac{\partial \overrightarrow{F}}{\partial x} + \overrightarrow{j} \frac{\partial \overrightarrow{F}}{\partial y} + \overrightarrow{k} \frac{\partial \overrightarrow{F}}{\partial z} \right)
\]

\[
= \left( \overrightarrow{a} \cdot \frac{\partial \overrightarrow{F}}{\partial x} \right) \overrightarrow{i} - (\overrightarrow{a} \cdot \overrightarrow{j}) \frac{\partial \overrightarrow{F}}{\partial y} + \left( \overrightarrow{a} \cdot \frac{\partial \overrightarrow{F}}{\partial z} \right) \overrightarrow{k} - (\overrightarrow{a} \cdot \overrightarrow{i}) \frac{\partial \overrightarrow{F}}{\partial x} + (\overrightarrow{a} \cdot \overrightarrow{j}) \frac{\partial \overrightarrow{F}}{\partial y} - (\overrightarrow{a} \cdot \overrightarrow{k}) \frac{\partial \overrightarrow{F}}{\partial z}
\]

\[
= \sum \frac{\partial (\overrightarrow{a} \cdot \overrightarrow{F})}{\partial x} - \sum (\overrightarrow{a} \cdot \overrightarrow{i}) \frac{\partial \overrightarrow{F}}{\partial x}
\]

\[
= \text{grad} (\overrightarrow{a} \cdot \overrightarrow{F}) - (\overrightarrow{a} \cdot \nabla) \overrightarrow{F}
\]

\[
\therefore \text{grad} (\overrightarrow{a} \cdot \overrightarrow{F}) = \overrightarrow{a} \times \text{curl} \overrightarrow{F} + (\overrightarrow{a} \cdot \nabla) \overrightarrow{F}
\]

14.8.45 SAQ: Show that \( \nabla \overrightarrow{r}^3 = 3 \overrightarrow{r} \).

14.8.46 SAQ: If \( \overrightarrow{A} \) and \( \overrightarrow{B} \) are differentiable vector point functions, then show that

\[
(\overrightarrow{A} \times \nabla) \cdot \overrightarrow{B} = \overrightarrow{A} \cdot (\nabla \times \overrightarrow{B})
\]

14.8.47 SAQ: If \( \overrightarrow{F} \) is a differentiable vector point function and \( \overrightarrow{a} \) is a constant vector, then show that \( \text{div} (\overrightarrow{a} \times \overrightarrow{F}) = -\overrightarrow{a} \cdot \text{curl} \overrightarrow{F} \) and \( \text{curl} (\overrightarrow{a} \times \overrightarrow{F}) = \overrightarrow{a} \text{div} \overrightarrow{F} - (\overrightarrow{a} \cdot \nabla) \overrightarrow{F} \)

14.8.48 SAQ: Prove that \( \nabla (\overrightarrow{r} \cdot \overrightarrow{a}) = \overrightarrow{a} \), where \( \overrightarrow{a} \) is a constant vector.

14.8.49 SAQ: Prove that \( \nabla \cdot (\overrightarrow{r} \times \overrightarrow{a}) = 0 \), where \( \overrightarrow{a} \) is a constant vector.

14.9 ANSWERS TO SAQs

14.8.14 SAQ: Let \( \overrightarrow{F} = x \overrightarrow{i} + y \overrightarrow{j} + z \overrightarrow{k} \Rightarrow \text{div} \overrightarrow{F} = \sum \frac{\partial}{\partial x} (x) = \sum 1 = 3 \)

\[
\text{curl} \overrightarrow{F} = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x & y & z
\end{vmatrix}
\]

\[
= \overrightarrow{i} (0 - 0) - \overrightarrow{j} (0 - 0) + \overrightarrow{k} (0 - 0) = \overrightarrow{0}
\]
14.8.15 SAQ: Let $\vec{r} = \frac{r}{r^2}$

$$\text{curl} \vec{r} = \sum \vec{e} \times \frac{\partial}{\partial \vec{x}} \left( \frac{\vec{r}}{r^2} \right) = \sum \vec{e} \times \left( \frac{2}{r^3} \frac{\partial r}{\partial \vec{x}} \frac{\vec{r}}{r} + \frac{1}{r^2} \frac{\partial \vec{r}}{\partial \vec{x}} \right)$$

$$= \sum \vec{e} \times \left( \frac{2}{r^3} \frac{x}{r} \frac{\vec{r}}{r} + \frac{1}{r^2} \frac{\vec{r}}{r^2} \right)$$

$$= \left( \sum x \vec{e} \times \vec{r} \right) \left( \frac{-2}{r^4} \right) + \frac{1}{r^2} \sum \vec{e} \times \vec{e}$$

$$= \left( \vec{r} \times \vec{r} \right) \left( \frac{-2}{r^4} \right) + \frac{1}{r^2} \vec{0} = \vec{0}$$

$\therefore \vec{r}$ is irrotational.

Let $\vec{g} = \frac{\vec{r}}{r^3} = \frac{x}{r^3} \vec{i} + \frac{y}{r^3} \vec{j} + \frac{z}{r^3} \vec{k}$

$$\text{div} \vec{g} = \sum \frac{\partial}{\partial \vec{x}} \left( \frac{x}{r^3} \right) = \sum \left[ \frac{1}{r^3} + x \left( - \frac{3}{r^4} \frac{x}{r} \right) \right]$$

$$= \sum \frac{1}{r^3} - \frac{3}{r^5} \sum x^2$$

$$= \frac{3}{r^3} - \frac{3}{r^5} r^2 = 0$$

$\therefore \vec{g}$ is solenoidal.

14.8.45 SAQ: $\nabla r^3 = \sum \vec{e} \frac{\partial}{\partial \vec{x}} \left( r^3 \right) = \sum i 3r^2 \frac{x}{r} = 3r \sum x \vec{i} = 3r \vec{r}$

14.8.46 SAQ: $(\vec{A} \times \vec{V}) \cdot \vec{B} = \sum (\vec{A} \times \vec{r}) \cdot \frac{\partial \vec{B}}{\partial \vec{x}}$

$$= \sum \left( \vec{A} \cdot \vec{t} \times \frac{\partial \vec{B}}{\partial \vec{x}} \right),$$ since in a scalar triple product, $\cdot$ and $x$ can be interchanged.
\[
\vec{A} \cdot \sum \vec{i} \times \frac{\partial \vec{B}}{\partial x} = \vec{A} \cdot (\nabla \times \vec{B})
\]

14.8.47 SAQ: \( \text{div}(\vec{a} \times \vec{F}) = \sum \vec{i} \cdot \frac{\partial}{\partial x} (\vec{a} \times \vec{F}) = \sum \vec{i} \times \left( \vec{a} \times \frac{\partial \vec{F}}{\partial x} \right) \)

\[= -\vec{a} \cdot \sum \vec{i} \times \frac{\partial \vec{F}}{\partial x} = -\vec{a} \cdot \text{curl} \vec{F} \]

\[\text{curl}(\vec{a} \times \vec{F}) = \sum \vec{i} \times \frac{\partial}{\partial x} (\vec{a} \times \vec{F}) = \sum \vec{i} \times \left( \vec{a} \times \frac{\partial \vec{F}}{\partial x} \right) \]

\[= \sum \left( \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} \right) \vec{a} - \sum (\vec{i} \cdot \vec{a}) \frac{\partial \vec{F}}{\partial x} = \vec{a} \text{ div} \vec{F} - (\vec{a} \cdot \nabla) \vec{F} \]

14.8.48 SAQ: \( \nabla (\vec{r} \cdot \vec{a}) = \sum \vec{i} \cdot \frac{\partial}{\partial x} (\vec{r} \cdot \vec{a}) = \sum \vec{i} \left( \frac{\partial \vec{r}}{\partial x} \cdot \vec{a} \right) = \sum \vec{i} (\vec{i} \cdot \vec{a}) \)

\[= \vec{a} \left( \text{since } \vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \Rightarrow \vec{i} \cdot \vec{a} = a_1 \right) \]

14.8.49 SAQ: \( \nabla \cdot (\vec{r} \times \vec{a}) = \sum \vec{i} \cdot \frac{\partial}{\partial x} (\vec{r} \times \vec{a}) = \sum \vec{i} \cdot \left( \frac{\partial \vec{r}}{\partial x} \times \vec{a} \right) \)

\[= \sum \vec{i} \cdot (\vec{i} \times \vec{a}) = 0 \]

14.10 SUMMARY
Vector point function, equations of curl, tangents normals to the curves, level surfaces, directional derivatives, gradient, divergence and curl are discussed. Solenoidal and irrotational vectors are discussed. Some problems are discussed.

14.11 TECHNICAL TERMS
Gradient, Divergence, Curl, Solenoidal, Irrotational

14.12 EXERCISES

14.12.1: If \( a = x + y + z, b = x^2 + y^2 + z^2, c = xy + yz + zx \) prove that \( [\nabla a, \nabla b, \nabla c] = 0 \).

14.12.2: Find the directional derivative of \( f = xy + yz + zx \) in the direction of the vector \( \vec{i} + 2\vec{j} + 2\vec{k} \) at \((1, 2, 0)\).

14.12.3: Find the angle between the surfaces \( x^2 + y^2 + z^2 = 9 \) and \( x^2 + y^2 - z = 3 \) at \((1, 2, 0)\).

14.12.4: Prove that \( (\vec{i} \times \vec{v}) \cdot \vec{i} = 0 \).

14.12.5: Find \( \text{div} \vec{F} \) and \( \text{curl} \vec{F} \) if \( \vec{F} = x^2y\vec{i} - 2xz\vec{j} + 2yz\vec{k} \), at \((1, 1, 1)\).
14.12.6: Show that \( \text{div} \overrightarrow{r} = \frac{2}{r} \)

14.12.7: If \( \phi = x^2 - y^2 \), show that \( \nabla^2 \phi = 0 \)

14.12.8: If \( \overrightarrow{r} = (3x^2y - z)\overrightarrow{i} + (xz^3 + y^4)\overrightarrow{j} - 2x^3z^2\overrightarrow{k} \), then find \( \text{grad} \cdot \text{div} \overrightarrow{r} \) at \((2, -1, 0)\).

14.12.9: If \( \phi = x^2 + y^2 + z^2 \), then find \( \text{curl} \cdot \text{grad} \phi \)

14.12.10: Show that \( \text{curl} \cdot \text{grad} \overrightarrow{r}^m = \overrightarrow{0} \)

14.12.11: If \( \overrightarrow{a} \) and \( \overrightarrow{b} \) are constant vectors, then show that \( \text{div}((\overrightarrow{r} \times \overrightarrow{a}) \times \overrightarrow{b}) = -2\overrightarrow{b} \cdot \overrightarrow{a} \)

### 14.13 ANSWERS TO EXERCISES

14.12.2: \( \frac{10}{3} \)

14.12.3: \( \cos^{-1}\left(\frac{-3}{7\sqrt{6}}\right) \)

14.12.5: \( 4, 4\overrightarrow{i} - 3\overrightarrow{k} \)

14.12.8: \( -6\overrightarrow{i} + 24\overrightarrow{j} - 32\overrightarrow{k} \)

14.12.9: \( \overrightarrow{0} \)

### 14.14 MODEL EXAMINATION QUESTIONS

14.14.1: Define \( \nabla \phi, \nabla \cdot \overrightarrow{r}, \nabla \times \overrightarrow{r} \)

14.14.2: Define Solenoidal, irrotational vectors

14.14.3: Prove that \( \overrightarrow{r}^n \overrightarrow{r} \) is irrotational

14.14.4: Find \( k \), if \( (x + 3y)\overrightarrow{i} + (y - 2z)\overrightarrow{j} + (x + kz)\overrightarrow{k} \) is solenoidal

14.14.5: Show that \( \text{curl}(f(\overrightarrow{r})\overrightarrow{r}) = \overrightarrow{0} \)

### 14.5 REFERENCE


--- A. Satyanarayana Murthy
Lesson - 15

VECTOR INTEGRATION

15.1 OBJECTIVE OF THE LESSON

In this lesson, integration of vector function, line integrals, surface integrals and volume integrals are discussed and some results are proved and some problems are discussed.

15.2 STRUCTURE OF THE LESSON

This lesson has the following components.

15.3 Introduction

15.4 Integration of Vector Functions

15.5 Line Integrals

15.6 Surface Integrals

15.7 Volume Integrals

15.8 Answer to SAQ's

15.9 Summary

13.10 Technical Terms

13.11 Exercises

13.12 Answers to Exercizes

13.13 Model Examination Questions

13.14 Reference Books

15.3 INTRODUCTION

Integration of scalar functions was discussed at the intermediate level. Here we discuss the integration of vector functions, line integrals, surface integrals, volume integrals. Some results are proved and some problems are discussed.

15.4 INTEGRATION OF A VECTOR FUNCTION

15.4.1 Definition: If \( \vec{F} \) and \( \vec{F} \) are vector functions of a scalar (real) variable \( t \) such that

\[
\frac{d}{dt}(\vec{F}(t) + \vec{c}) = \vec{F}(t),
\]

where \( \vec{c} \) is any constant vector, then \( \vec{F}(t) + \vec{c} \) is called the (indefinite) integral (or primitive) of \( \vec{F}(t) \) with respect to \( t \) and we write

\[
\int \vec{F}(t) \, dt = \vec{F}(t) + \vec{c}.
\]

The constant vector \( \vec{c} \) is called the constant of integration.

15.4.2 Note: If \( \vec{F}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k} \), then

\[
\int \vec{F}(t) \, dt = \int f_1(t) \, dt \hat{i} + \int f_2(t) \, dt \hat{j} + \int f_3(t) \, dt \hat{k} + \vec{c}
\]

15.4.3 Definition: Let \( F(t) \) be a function defined on \( [a, b] \). If

\[
\int F(t) \, dt = \vec{F}(t) + \vec{c},
\]

then \( \vec{F}(b) - \vec{F}(a) \)
is called the definite integral of $\mathbf{r}(t)$ over $[a, b]$. We write $\int_a^b \mathbf{r}(t) dt = \mathbf{F}(b) - \mathbf{F}(a)$. $a$ is called the lower limit and $b$ is called the upper limit.

15.4.4 Note : $\int_a^b \mathbf{r}(t) dt = \mathbf{F}(t) + c \Rightarrow \int_a^b \mathbf{r}(t) dt = \mathbf{F}(b) - \mathbf{F}(a)$

15.4.5 Note : If $\mathbf{r}(t) = \mathbf{r}_1(t) + \mathbf{r}_2(t) + \mathbf{r}_3(t)$, then $\int_a^b \mathbf{r}(t) dt = \int_a^b \mathbf{r}_1(t) dt + \int_a^b \mathbf{r}_2(t) dt + \int_a^b \mathbf{r}_3(t) dt$

15.4.6 Note : $\int (\mathbf{r} + \mathbf{g})(t) dt = \int \mathbf{r}(t) dt + \int \mathbf{g}(t) dt$ and $\int k\mathbf{r}(t) dt = k\int \mathbf{r}(t) dt$ where $k \in \mathbb{R}$

15.4.7 Theorem : If $\mathbf{r}$ and $\mathbf{g}$ are vector functions, then

$$\int \left( \frac{d\mathbf{r}}{dt} \cdot \mathbf{g} + \mathbf{r} \cdot \frac{d\mathbf{g}}{dt} \right) dt = \mathbf{r} \cdot \mathbf{g} + c$$

Proof : $\frac{d}{dt}(\mathbf{r} \cdot \mathbf{g} + c) = \frac{d\mathbf{r}}{dt} \cdot \mathbf{g} + \mathbf{r} \cdot \frac{d\mathbf{g}}{dt} \Rightarrow \int \left( \frac{d\mathbf{r}}{dt} \cdot \mathbf{g} + \mathbf{r} \cdot \frac{d\mathbf{g}}{dt} \right) dt = \mathbf{r} \cdot \mathbf{g} + c$

15.4.8 Corollary : If $\mathbf{r}$ is a vector function, then $\int 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} dt = \mathbf{r}^2 + c$

Proof : Take $\mathbf{r}$ for $\mathbf{g}$ in Theorem 15.4.7.

15.4.9 Corollary : If $\mathbf{r}$ is a vector function, then

$$\int 2 \frac{d\mathbf{r}}{dt} \cdot \frac{d^2\mathbf{r}}{dt^2} dt = \left( \frac{d\mathbf{r}}{dt} \right)^2 + c$$

Proof : Take $\frac{d\mathbf{r}}{dt}$ for $\mathbf{r}$ in corollary 15.4.8.

15.4.10 Theorem : If $\mathbf{r}$ and $\mathbf{g}$ are vector functions, then

$$\int \left( \frac{d\mathbf{r}}{dt} \times \mathbf{g} + \mathbf{r} \times \frac{d\mathbf{g}}{dt} \right) dt = \mathbf{r} \times \mathbf{g} + c$$

Proof : $\frac{d}{dt}(\mathbf{r} \times \mathbf{g} + c) = \frac{d\mathbf{r}}{dt} \times \mathbf{g} + \mathbf{r} \times \frac{d\mathbf{g}}{dt}$
Vector Integration

15.3

Differential Equation, Abstract Algebra...

\[ \Rightarrow \int \left( \frac{df}{dt} \times \overline{g} + \frac{dg}{dt} \times \overline{f} \right) dt = \overline{f} \times \overline{g} + \overline{c} \]

15.4.11 Corollary: If \( \overline{a} \) is a constant vector and \( \overline{f} \) is a vector function, then

\[ \int \left( \overline{a} \times \frac{df}{dt} \right) dt = \overline{a} \times \overline{f} + \overline{c} \]

Proof: \( \frac{d}{dt}(\overline{a} \times \overline{f} + \overline{c}) = \overline{a} \times \frac{df}{dt} + \frac{d\overline{a}}{dt} \times \overline{f} + \overline{0} \)

\[ = \overline{a} \times \frac{df}{dt} \]

\[ \Rightarrow \int \left( \overline{a} \times \frac{df}{dt} \right) dt = \overline{a} \times \overline{f} + \overline{c} \]

15.4.12 Corollary: If \( \overline{f} \) is a vector function, then

\[ \int \overline{f} \times \frac{d^2\overline{f}}{dt^2} dt = \overline{f} \times \frac{df}{dt} + \overline{c} \]

Proof: \( \frac{d}{dt} \left( \overline{f} \times \frac{df}{dt} + \overline{c} \right) = \frac{df}{dt} \times \frac{df}{dt} + \overline{f} \times \frac{d^2\overline{f}}{dt^2} + \overline{0} \)

\[ = \overline{0} + \overline{f} \times \frac{d^2\overline{f}}{dt^2} + \overline{0} = \overline{f} \times \frac{d^2\overline{f}}{dt^2} \]

\[ \Rightarrow \int \left( \overline{f} \times \frac{d^2\overline{f}}{dt^2} \right) dt = \overline{f} \times \frac{df}{dt} + \overline{c} \]

SOLVED PROBLEMS

15.4.13: If \( \overline{f}(t) = t\overline{i} + (t^2 - 2t)\overline{j} + (3t^2 + 3t^3)\overline{k} \), then find \( \int_0^1 \overline{f}(t) dt \).

Solution: \( \int_0^1 \overline{f}(t) dt = \int_0^1 t dt + \int_0^1 (t^2 - 2t) dt + \int_0^1 (3t^2 + 3t^3) dt \)
\[= \frac{1}{2} - \frac{2}{3} + \frac{7}{4}\]

**15.4.14**: If \( \vec{r} = t \hat{i} - t^2 \hat{j} + (t-1) \hat{k} \) and \( \vec{g} = 2t^2 \hat{i} + 6t \hat{k} \), then find \( \int_0^2 (\vec{r} \cdot \vec{g}) \, dt, \int_0^2 (\vec{r} \times \vec{g}) \, dt. \)

**Solution**: \( \vec{r} \cdot \vec{g} = 2t^3 + 6t^2 - 6t \)

\[
\therefore \int_0^2 (\vec{r} \cdot \vec{g}) \, dt = \int_0^2 (2t^3 + 6t^2 - 6t) \, dt = \left( \frac{t^4}{2} + 2t^3 - 3t^2 \right)_0^2 = 8 + 16 - 12 = 12
\]

\[
\vec{r} \times \vec{g} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
t & -t^2 & t-1 \\
2t^2 & 0 & 6t
\end{vmatrix} = t(-6t^3) - j(6t^2 - 2t^3 + 2t^2) + k(2t^4) = 6t^3 \hat{i} + (2t^3 - 8t^2) \hat{j} + 2t^4 \hat{k}
\]

\[
\int_0^2 \vec{r} \times \vec{g} \, dt = -\int_0^2 6t^3 \, dt + \int_0^2 (2t^3 - 8t^2) \, dt + \int_0^2 2t^4 \, dt = -\frac{t^4}{2}_0^2 + \frac{t^4}{2} + \frac{2t^5}{5}_0^2 = -\frac{40}{3} \hat{i} + \frac{64}{5} \hat{k}
\]

**15.4.15**: If \( \frac{d^2 \vec{r}}{dt^2} = -k^2 \vec{r} \), then show that \( \left( \frac{d \vec{r}}{dt} \right)^2 = -k^2 \vec{r}^2 + c \)

**Solution**: By hypothesis, \( \frac{d^2 \vec{r}}{dt^2} = -k^2 \vec{r} \)
\[ \Rightarrow 2 \frac{d\vec{r}}{dt} \frac{d^2\vec{r}}{dt^2} = -k^2 \frac{d\vec{r}}{dt} \]

\[ \therefore \left( 2 \frac{d\vec{r}}{dt} \frac{d^2\vec{r}}{dt^2} \right) dt = -k^2 \int 2\vec{r} \cdot \frac{d\vec{r}}{dt} dt \]

\[ \Rightarrow \left( \frac{d\vec{r}}{dt} \right)^2 = -k^2 \vec{r}^2 + c, \text{ where } c \text{ is a constant of integration. Since} \]

\[ \frac{d}{dt} \left( \frac{d\vec{r}}{dt} \right)^2 = 2 \frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} \]

**15.4.16:** If \( \vec{r}(t) = 5t^2\vec{i} + t\vec{j} - t^3\vec{k} \), then find \( \int_1^2 \left( \vec{r} \times \frac{d^2\vec{r}}{dt^2} \right) dt \)

**Solution:**

\[ \frac{d\vec{r}}{dt} = 10t\vec{i} + \vec{j} - 3t^2\vec{k} \]

\[ \frac{d^2\vec{r}}{dt^2} = 10\vec{i} - 6t\vec{k} \]

\[ \vec{r} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^2 & t & -t^3 \\ 10 & 0 & -6t \end{vmatrix} = -6t^2\vec{i} + 20t^3\vec{j} - 10t\vec{k} \]

\[ \therefore \int_1^2 \left( \vec{r} \times \frac{d^2\vec{r}}{dt^2} \right) dt = \int_1^2 6t^2 dt + \int_1^2 20t^3 dt - \int_1^2 10t dt \]

\[ = -2\left( t^3 \right)_1^2 + \vec{j} \left( 5t^4 \right)_1^2 - \vec{k} \left( 5t^2 \right)_1^2 \]

\[ = -14\vec{i} + 75\vec{j} - 15\vec{k} \]

**15.4.17 SAQ:** If \( \vec{r} = \vec{a} \cos wt + \vec{b} \sin wt \), where \( \vec{a} \) and \( \vec{b} \) are constant vectors, show that \( \frac{d^2\vec{r}}{dt^2} + w^2 \vec{r} = \vec{0} \).
15.5 LINE INTEGRAL

15.5.1 Definition: Let C be a curve in space with initial point A and terminal point B. If the direction of C oriented from A to B is taken as positive, then the direction from B to A is taken as negative. If A and B coincide, then C is called a closed curve.

15.5.2 Definition: A curve \( \mathbf{r}(t) \) is called a smooth curve if

(i) \( \mathbf{r}(t) \) is differentiable at \( t \) i.e. \( \mathbf{r}'(t) \) exists.

(ii) \( \mathbf{r}'(t) \neq \mathbf{0} \)

(iii) \( \mathbf{r}'(t) \) is continuous.

15.5.3 Definition: A curve \( C \) is called piecewise smooth if it consists of a finite number of smooth curves.

15.5.4 Definition: An integral which is to be evaluated along a curve is called a line integral.

Let \( \mathbf{T} = \mathbf{T}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \) be a smooth curve \( C \) joining A and B. Then \( d\mathbf{T} \) stands for

\[
\frac{d\mathbf{T}}{dt} = \left( \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \right) dt,
\]

which is written as \( dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} \). If \( s \) denotes the arc length of a point on \( C \) from a fixed point on \( C \), then \( \frac{d\mathbf{T}}{ds} = \mathbf{T} \) is a unit vector along the target to the curve \( C \) at the point \( \mathbf{T} \). Let \( \mathbf{F}(\mathbf{T}) \) be a vector point function defined and continuous along \( C \). The component of the vector \( \mathbf{F} \) along the tangent is \( \mathbf{F} \cdot \frac{d\mathbf{T}}{ds} \). The integral of \( \mathbf{F} \cdot \frac{d\mathbf{T}}{ds} \) along \( C \) from A to B, written as

\[
\int_{A}^{B} \mathbf{F} \cdot \frac{d\mathbf{T}}{ds} \, ds = \int_{A}^{B} \mathbf{F} \cdot d\mathbf{T}
\]

is called the (tangential) line integral of \( \mathbf{F} \) along \( C \) (from A to B).

15.5.5 Definition: If \( C \) is a simple closed curve (a curve which does not intersect itself anywhere), then the tangential line integral of \( \mathbf{F} \) along \( C \) is called the calculation of \( \mathbf{F} \) along \( C \) and is denoted by \( \oint_{C} \mathbf{F} \cdot d\mathbf{T} \).
15.5.6 Cartesian Form : If $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$, then

$$\mathbf{F} \cdot d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$$

so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

If the parametric equations of the curve $C$ are $x = x(t)$, $y = y(t)$, $z = z(t)$ and if $t = t_1$ at $A$ and $t = t_2$ at $B$, then

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{t_1}^{t_2} \left( F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$

15.5.7 Work done by a force : Suppose a force $\mathbf{F}$ acts on a particle. Let the particle be displaced along a given path $C$ in space. If $\mathbf{r}$ denotes the position vector of a point on $C$, then $\frac{d\mathbf{r}}{ds}$ is a unit vector along the tangent to $C$ at the point $\mathbf{r}$ in the direction of increasing $s$. The component of force $\mathbf{F}$ along the tangent to $C$ is $\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}$. The work done by the force $\mathbf{F}$ during a small displacement $ds$ of the particle along $C$ is $\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}$ i.e. $\mathbf{F} \cdot d\mathbf{r}$. The total work done by $\mathbf{F}$ in this displacement along $C$ is $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solved Problems :

15.5.8 : Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$ and $C$ is the curve $\mathbf{r} = t \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$, $t$ varying from -1 to +1.

Solution : Given $\mathbf{r} = t \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$, $\Rightarrow x = t$, $y = t^2$, $z = t^3$

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t \mathbf{j} + 3t^2 \mathbf{k}$$

$$\mathbf{F} = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k} = t^3 \mathbf{i} + t^5 \mathbf{j} + t^4 \mathbf{k}$$

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^6 + 3t^6 = t^3 + 5t^6$$
\[ \int \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} (t^3 + 5t^6) dt = \left( \frac{t^4}{4} + \frac{5}{7} t^7 \right)_{-1}^{1} = \frac{10}{7} \]

15.5.9 : Evaluate \( \int_{C} (x^2 + y^2) dx \) where \( C \) is the arc of the parabola \( y^2 = 4ax \) from \((0,0)\) to \((a,2a)\).

**Solution :**
\[ \int_{C} (x^2 + y^2) dx = \int_{0}^{2a} (x^2 + 4ax) dx = \left( \frac{x^3}{3} + 2ax^2 \right)_{0}^{2a} = \frac{a^3}{3} + 2a^3 = \frac{7a^3}{3} \]

15.5.10 : If \( \mathbf{F} = 3xy \mathbf{i} - 5z \mathbf{j} + 10x \mathbf{k}, \) evaluate \( \int_{C} \mathbf{F} \cdot d\mathbf{r} \) along the curve \( x = t^2 + 1, y = 2t^2, z = t^3 \) from \( t = 1 \) to \( t = 2. \)

**Solution :**
\[ \mathbf{F} = 3xy \mathbf{i} - 5z \mathbf{j} + 10x \mathbf{k}, \]
\[ = 3(t^2 + 1)2t^2 \mathbf{i} - 5t^3 \mathbf{j} + 10(t^2 + 1) \mathbf{k} \]
\[ = (6t^4 + 6t^2) \mathbf{i} - 5t^3 \mathbf{j} + 10(t^2 + 1) \mathbf{k} \]
\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = (t^2 + 1) \mathbf{i} + 2t^2 \mathbf{j} + t^3 \mathbf{k} \]
\[ \frac{d\mathbf{r}}{dt} = 2t \mathbf{i} + 4t \mathbf{j} + 3t^2 \mathbf{k} \]
\[ \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^2 (t^2 + 1)2t - 5t^3 \cdot 4t + 10(t^2 + 1)3t^2 \]
\[ = 12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2 \]
\[ = 12t^4 + 12t^3 + 10t^4 + 30t^2 \]
\[ \therefore \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{1}^{2} \left( \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \right) dt = \int_{1}^{2} (12t^5 + 10t^4 + 12t^3 + 30t^2) dt \]
\[ = \left( 2t^6 + 2t^5 + 3t^4 + 10t^3 \right)_{1}^{2} = 128 + 64 + 48 + 80 - 17 = 303 \]
15.9 Vector Integration

15.5.11: If \( \mathbf{F} = \left( x^2 + y^2 \right) \mathbf{i} - 2xy \mathbf{j} \), evaluate \( \oint_{C} \mathbf{F} \cdot d\mathbf{r} \), where \( C \) is the rectangle in the XY-plane bounded by \( y = 0, x = a, y = b, x = 0 \).

**Solution:** Since \( C \) is in the XY-plane, \( \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \oint_{C} F_{1} dx + F_{2} dy \)

\[ = \oint_{C} \left( x^2 + y^2 \right) dx - 2xy dy \]

(i) Along OA: \( y = 0, \ dy = 0; \ x \) varies from 0 to a.

\[ \therefore \oint_{OA} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{a} x^2 \ dx = \frac{a^3}{3} \]

(ii) Along AB: \( x = a, \ dx = 0; \ y \) varies from 0 to b.

\[ \therefore \oint_{AB} \mathbf{F} \cdot d\mathbf{r} = -\int_{0}^{b} 2ay \ dy = -a \left( y^2 \right)_{0}^{b} = -ab^2 \]

(iii) Along BC: \( y = b, \ dy = 0; \ x \) varies from a to 0.

\[ \oint_{BC} \mathbf{F} \cdot d\mathbf{r} = \left( \int_{a}^{b} x^2 + b^2 \right) dx = \left( \frac{x^3}{3} + b^2x \right)_{a}^{b} = -\frac{a^3}{3} - ab^2 \]

(iv) Along CO: \( x = 0, \ dx = 0; \ y \) varies from b to 0.

\[ \oint_{CO} \mathbf{F} \cdot d\mathbf{r} = 0 \]

\[ \therefore \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -2ab^2 \]
15.10 : If \( \mathbf{F} = y \mathbf{i} + z \mathbf{j} + x \mathbf{k} \), find the circulation of \( \mathbf{F} \) around the curve C, where C is the circle \( x^2 + y^2 = 1, \ z = 0 \).

**Solution** : Equation of C is : \( x^2 + y^2 = 1, \ z = 0 \)

Parametric equations of C are : \( x = \cos \theta, \ y = \sin \theta, \ z = 0, \ 0 \leq \theta \leq 2\pi \)

Circulation of \( \mathbf{F} \) along C is

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C F_1 dx + F_2 dy + F_3 dz = \oint_C ydx + zdy = \int_C ydx
\]

\[
= \int_0^{2\pi} \sin \theta (\sin \theta) d\theta
\]

\[
= -\int_0^{2\pi} \sin^2 \theta d\theta = -\frac{\pi}{2} \sin^2 \theta d\theta = -4\int_0^{\pi/2} \sin^2 \theta d\theta
\]

\[
= -4 \cdot \frac{1}{2} \pi = -\pi
\]

15.6 SURFACE INTEGRAL

15.6.1 Definition : Let \( \phi : \mathbb{R}^3 \rightarrow \mathbb{R} \). The \( S = \{(x, y, z) \in \mathbb{R}^3 / \phi(x, y, z) = 0\} \) is called a (level) surface. \( \phi(x, y, z) = 0 \) is called the equation of the surface. If further \( \{(x, y, z) / \phi(x, y, z) \leq 0\} \) is bounded, then S is called a closed surface.

15.6.2 Definitions : A surface with equation \( \phi(x, y, z) = 0 \) is called a smooth surface if \( \phi \) has continuous first order partial derivatives.

If a surface S is divided into a finite number of smooth surfaces, then S is called a piecewise smooth surface.

An integral which is evaluated over a surface is called a surface integral.

Let \( \mathbf{F} \) be a continuous vector function defined over a smooth surface S.

Divide S into n sub-regions \( S_1, S_2, \ldots, S_n \) of areas \( \delta S_1, \delta S_2, \ldots, \delta S_n \). Let \( P_i \) be a point on \( S_i \) and \( \mathbf{N}_i \) be the unit normal to \( S_i \). Then \( \delta S_i \mathbf{N}_i \) is denoted by \( \delta \mathbf{A}_i \) which is the vector area of \( S_i \) i.e. a vector normal to \( S_i \) at \( P_i \) and having magnitude \( \delta S_i \). Let \( I_n = \sum_{i=1}^{n} \mathbf{F}(P_i) \cdot \mathbf{N}_i \delta S_i \). The limit of \( I_n \), as \( n \rightarrow \infty \), if it exists, is called the normal surface integral of \( \mathbf{F} \) over the surface S and is denoted by \( \int_S \mathbf{F} \cdot d\mathbf{A} \) or \( \int_S \mathbf{F} \cdot \mathbf{N} dS \).
15.6.3 Definition: Let \( S \) be a closed surface. Then the normal surface integral \( \int_{S} \vec{F} \cdot \vec{N} \, dS \) or \( \int_{S} \vec{F} \cdot d\vec{A} \) is called the flux of \( \vec{F} \) over \( S \).

15.6.4 Cartesian form: Suppose the outward drawn unit normal to the surface \( S \) at \( P \) (i.e. a unit normal to the surface \( S \) at \( P \) which is drawn outwards) makes angles \( \alpha, \beta, \gamma \) with the positive direction of the co-ordinate axes respectively. If \( \ell, m, n \) are the direction cosines of the outward drawn unit normal, then \( \vec{N} = \ell \vec{i} + m \vec{j} + n \vec{k} \). If \( \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k} \), then

\[
\vec{F} \cdot \vec{N} = F_1 \cos \alpha + F_2 \cos \beta + F_3 \cos \gamma.
\]

\( \therefore \int_{S} \vec{F} \cdot \vec{N} \, ds = \int_{S} \left( F_1 \cos \alpha + F_2 \cos \beta + F_3 \cos \gamma \right) ds \)

15.6.5 Definition (Projection of surface \( S \) in a plane): Let \( S \) be a surface and \( \pi \) be a plane. For each point \( P \) of \( S \), let \( P_1 \) be the foot of the perpendicular from \( P \) to the plane \( \pi \). Then the set of all these feet of the perpendiculars on \( \pi \) is called the projection of \( S \) on \( \pi \) i.e. \{\( P_1/P_1 \) is the the foot of the perpendicular from a point \( P \) of \( S \) on the plane \( \pi \)\} is the projection of \( S \) on \( \pi \).

15.6.6 Note: The projection of \( ds \) (in 15.6.4) on YZ plane is \( ds \cos \alpha = dy \, dz \), since \( dy \) and \( dz \) are differentiable along \( Y \), \( Z \) axes respectively and hence

\[
\int_{S} \vec{F} \cdot \vec{N} \, ds = \int_{S} \left( F_1 dy + F_2 dz + F_3 dx \right) dy
\]

15.6.7 Note: If \( R_1 \) is the projection of \( S \) in the XY-plane, then

\[
\int_{S} \vec{F} \cdot \vec{N} \, ds = \int_{R_1} \int \vec{F} \cdot \vec{N} \, dx \, dy = \int_{R_1} \int \vec{F} \cdot \frac{\vec{N} \, dx \, dy}{|\vec{N} \cdot \vec{K}|}
\]

Similarly \( \int_{S} \vec{F} \cdot \vec{N} \, ds = \int_{R_2} \int \vec{F} \cdot \vec{N} \, dy \, dx \) and \( \int_{S} \vec{F} \cdot \vec{N} \, ds = \int_{R_3} \int \vec{F} \cdot \vec{N} \, dz \, dx \)

where \( R_2, R_3 \) are the projection of \( S \) in the YZ, ZX planes respectively.

15.6.8: If \( \vec{F} = x^2 + y^2 \vec{i} - 2x \vec{j} + 2yz \vec{k} \), evaluate \( \int_{S} \vec{F} \cdot \vec{N} \, ds \), where \( S \) is the surface of the plane \( 2x + y + 2z = 6 \) in the first octant.

Solution: Let \( \phi = 2x + y + 2z - 6 \). \( \therefore \nabla \phi = 2 \vec{i} + \vec{j} + 2 \vec{k} \) is the normal to the surface.
unit normal is \( \mathbf{N} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2\mathbf{i} + \mathbf{j} + 2\mathbf{k}}{3} \)

Let \( R \) be the projection of \( S \) in the \( XY \)-plane.

\( \therefore R \) is bounded by \( X \)-axis, \( Y \)-axis, \( 2x + y = 6, z = 0 \).

\[ \mathbf{F} \cdot \mathbf{N} = (x + y^2)\frac{2}{3} - 2x \frac{1}{3} + 2yz \frac{2}{3} \]

\[ = \frac{1}{3}[2x + 2y^2 - 2x + 4yz] = \frac{2}{3}(y^2 + 2yz) \]

\[ \mathbf{N} \cdot \mathbf{k} = \frac{2}{3} \]

\[ \therefore \int \mathbf{F} \cdot \mathbf{N} \, ds = \int \int \mathbf{F} \cdot \mathbf{N} \, \frac{dx \, dy}{|\mathbf{N} \cdot \mathbf{k}|} = \int \int \frac{2}{3}(y^2 + 2yz) \, \frac{dx \, dy}{\sqrt{2/3}} \]

\[ = \int \int_R (y^2 + 2yz) \, dx \, dy = \int_{x=0}^{3} \int_{y=0}^{6-2x} [y^2+y(6-y-2x)] \, dx \, dy \]

\[ = 2 \int_{y=0}^{3} (3-x)(6-2x) \, dy = 2 \int_{x=0}^{3} (6-2x) \left( \frac{y^2}{2} \right)_{y=0}^{y=3} \, dx \]

\[ = 2 \cdot \frac{1}{2} \int_{x=0}^{3} (3-x)(6-2x)^2 \, dx = 4 \int_{0}^{3} (3-x)^3 \, dx \]

\[ = -\left[ (3-x)^4 \right]_0^3 = -[0-3^4] = 81 \]

15.6.9: Evaluate \( \int_{S} \mathbf{F} \cdot \mathbf{N} \, dS \) where \( \mathbf{F} = yz \mathbf{i} + zx \mathbf{j} + xy \mathbf{k} \) and \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 1 \) which lies in the first octant.

Solution: Let \( \phi = x^2 + y^2 + z^2 - 1 \)
Vector Integration

\[ \nabla \phi = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k} \]

\[ \mathbf{N} = \text{unit normal to } S = \frac{2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}}{2 \sqrt{x^2 + y^2 + z^2}} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \]

\[ \mathbf{F} \cdot \mathbf{N} = xyz + xyz + xyz = 3xyz \]

Let \( R \) be the projection of \( S \) in the \( YZ \)-plane.

\[ \int_S \mathbf{F} \cdot \mathbf{N} \, ds = \int_R \mathbf{F} \cdot \mathbf{N} \frac{dy \, dz}{|\mathbf{N} \cdot \mathbf{T}|} = \int_R 3xyz \frac{dy \, dz}{x} \]

\[ = 3 \int_0^1 \int_{y=0} \sqrt{1-z^2} yz \, dz \, dy = 3 \int_0^1 \left( \frac{y^2}{2} \sqrt{1-z^2} \right)_0^1 \, dz \]

\[ = \frac{3}{2} \int_{z=0}^1 \left(1-z^2\right)z \, dz = \frac{3}{2} \left[ \frac{z^2}{2} - \frac{z^4}{4} \right]_0^1 = \frac{3}{2} \left( \frac{1}{2} - \frac{1}{4} \right) \]

\[ = \frac{3}{2} \left( \frac{1}{4} \right) = \frac{3}{8} \]

15.7 VOLUME INTEGRALS

15.7.1 Definition (Volume Integral) : Suppose \( V \) is a region bounded by a surface \( S \). Suppose \( f \) is a scalar point function defined on the region \( V \). Divide the region \( V \) into subregions \( V_1, V_2, \ldots, V_n \) with volumes \( \delta V_1, \delta V_2, \ldots, \delta V_n \). Let \( P_r \) be a point in the subregion \( V_r \). If \( \lim_{n \to \infty} \sum_{r=1}^n f(P_r) \delta V_r \) exists, then the limit is called the volume integral of \( f \) over \( V \) and is denoted by \( \iiint_V f \, dv \).

If we divide the region \( V \) into small cuboids by drawing lines parallel to the coordinate axis, then \( dv = dx \, dy \, dz \) and \( \iiint_V f \, dv = \iiint_V f \, dx \, dy \, dz \).

If \( \mathbf{F} \) is a vector point function, then we can define \( \iiint_V \mathbf{F} \, dv \) on similar lines.
Solved Problems :

15.7.2 : If \( \mathbf{F} = (2x^2 - 3z)i - 2xyj - 4xk \), evaluate \( \iiint \nabla \cdot \mathbf{F} \, dv \) and \( \iiint \nabla \times \mathbf{F} \, dv \), where \( V \) is the region bounded by the places \( x = 0, y = 0, z = 0 \) and \( 2x + 2y + z = 4 \).

Solution : \( \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2x^2 - 32) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x) = 4x - 2x = 2x \)

\[
\iiint_V \nabla \cdot \mathbf{F} \, dv = \iiint_{x=0}^{2} \, dy \, dx \int_{y=0}^{2-2x-2y} 2x \, dz = \int_{x=0}^{z} z \, dx = \int_{x=0}^{2-2x-2y} 2x \, dz = \int_{y=0}^{2y} \frac{2}{0} (4x^2 - 4x) \, dx = \left[ \frac{4x^2}{2} - \frac{8x^3}{3} \right]_0^2 = 16 + 8 - \frac{64}{3} = \frac{8}{3}
\]

\( \nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix} = i \left( 0 - 0 \right) - j \left( -4 + 3 \right) + k \left( -2y \right) = j - 2yk
\]

\[
\iiint_V \nabla \times \mathbf{F} \, dv = \iiint_{x=0}^{2} \, dy \, dx \int_{y=0}^{2} \left( j - 2y \mathbf{k} \right) \, dz = \int_{x=0}^{2} \, dz \, dw \int_{y=0}^{2} \left( j - 2y \mathbf{k} \right) \, dx = \int_{x=0}^{2} \, dx \, dw \int_{y=0}^{2} \left( j - 2y \mathbf{k} \right) \, dy
\]
\[
\int_{x=0}^{2} \int_{y=0}^{2-x} (\mathbf{j} - 2y\mathbf{k})(4 - 2x - 2y) \, dy \, dx
\]
\[
= \int_{0}^{2} \left[ \left( 4y - 2xy - y^2 \right)\mathbf{j} - \left( 4y^2 - 2xy^2 - \frac{4}{3}y^3 \right)\mathbf{k} \right]_{y=0}^{2-x} \, dx
\]
\[
= \int_{0}^{2} \left[ (2-x)^2 \mathbf{j} - \frac{2}{3}(2-x)^3 \mathbf{k} \right] \, dx
\]
\[
= \left[ \frac{1}{3} (2-x)^3 \mathbf{j} - \frac{2}{3} (2-x)^4 \mathbf{k} \right]_{0}^{2}
\]
\[
= 0 - \left[ \frac{8}{3} \mathbf{j} + \frac{8}{3} \mathbf{k} \right] = \frac{8}{3} (\mathbf{j} - \mathbf{k})
\]

15.7.3 : If \( \mathbf{F} = 2xz\mathbf{i} - x\mathbf{j} + y^2\mathbf{k} \), evaluate \( \int_{V} \mathbf{F} \, dV \), where \( V \) is the region bounded by the surfaces \( x = 0, y = 0, y = 6, z = x^2, z = 4 \).

\[\text{Solution : } \int_{V} \mathbf{F} \, dV = \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=x^2}^{4} (2xz\mathbf{i} - x\mathbf{j} + y^2\mathbf{k}) \, dz \, dy \, dx\]
\[
= \int_{x=0}^{2} \int_{y=0}^{6} \left[ xz^2 \mathbf{i} - \mathbf{j} + \mathbf{k} \right]_{z=x^2}^{4} \, dy \, dx
\]
\[
= \int_{x=0}^{2} \left[ \int_{y=0}^{6} x(z)^4 \, dy - \int_{y=0}^{6} x(z)^4 \, dy \right] \, dx
\]
\[
= \int_{x=0}^{2} \int_{y=0}^{6} x(z)^4 \, dy \, dx
\]
\[ \int_{V} \vec{F} \, dV = \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=0}^{2} x(16 - x^4) \, dy \, dx + \int_{x=0}^{2} \int_{y=0}^{6} y^2(4 - x^2) \, dy \, dx + \int_{x=0}^{2} \int_{y=0}^{6} z^2(4 - x^2) \, dy \, dx \]

\[ = \frac{2}{3} \int_{0}^{2} x(16 - x^4) \, dx - \frac{2}{3} \int_{0}^{2} x(4 - x^2) \, dx + \frac{2}{3} \int_{0}^{2} (4 - x^2) \, dx \]

\[ = \frac{2}{3} [48x^2 - x^6]_0^2 - \frac{2}{3} [12x^2 - \frac{3}{2} x^4]_0^2 + \frac{2}{3} [4x - \frac{x^3}{3}]_0^2 \]

\[ = \frac{128}{3} - \frac{24}{3} + \frac{384}{3} \]

15.7.4 SAQ: Evaluate \( \int_{V} \vec{F} \, dV \) when \( \vec{F} = x\vec{i} + y\vec{j} + z\vec{k} \) and \( V \) is the region bounded by \( x = 0, y = 0, y = 6, z = 4, z = x^2 \).

15.7.5 SAQ: Evaluate \( \int_{C} \frac{dx}{x + y} \) where \( C \) is the curve \( x = at^2, y = 2at, 0 \leq t \leq 2 \).

15.7.6 SAQ: Evaluate \( \int_{S} \vec{F} \cdot \vec{N} \, dS \) where \( \vec{F} = z\vec{i} + x\vec{j} - 3y^2z\vec{k} \) and \( S \) is the surface \( x^2 + y^2 = 16 \) included in the first octant between \( z=0 \) and \( z=5 \).

15.8 ANSWERES TO SAQ’s

15.7.4 SAQ:

\[ \int_{V} \vec{F} \, dV = \int_{x=0}^{2} \int_{y=x^2}^{4} x \, dz \, dy \, dx + \int_{x=0}^{2} \int_{y=x^2}^{4} z \, dz \, dy \, dx + \int_{x=0}^{2} \int_{y=x^2}^{4} z \, dz \, dx \]
\[
\begin{align*}
&= \mathbf{i} \int_{x=0}^{2} \int_{y=0}^{6} (4-x^2) \, dy \, dx + \mathbf{j} \int_{x=0}^{2} \int_{y=0}^{6} (4-x^2) \, dy \, dx + \mathbf{k} \int_{x=0}^{2} \int_{y=0}^{6} (16-x^4) \, dy \, dx \\
&= \mathbf{i} \left( 12x^2 - \frac{3}{2}x^4 \right)_{0}^{2} + \mathbf{j} 18 \left( 4x - \frac{x^3}{3} \right)_{0}^{2} + \mathbf{k} 6 \left( 16x - \frac{x^5}{5} \right)_{0}^{2} \\
&= 24\mathbf{i} + 96\mathbf{j} + \frac{384}{5}\mathbf{k}
\end{align*}
\]

15.7.5 SAQ:
\[
\int_{C}^{x+y} \frac{dx}{2at^2 + 2at} = \int_{0}^{2} \frac{dt}{t+2} = 2 \left[ \log (t+2) \right]_{0}^{2} = 2(\log 4 - \log 2) = 2 \log \frac{4}{2} = 2 \log 2
\]

15.7.6 SAQ:
Let \( \phi = x^2 + y^2 - 16; \)
\[
\nabla \phi = 2xi + 2yj
\]
\[
\vec{N} = \frac{x\mathbf{i} + y\mathbf{j}}{4} \Rightarrow \vec{F} \cdot \vec{N} = \frac{x}{4}
\]
Let \( R \) be the projection of \( S \) in the \( YZ \)-plane.
\( y \) varies from 0 to 4
\( z \) varies from 0 to 5
\[
\int_{S} \vec{F} \cdot \vec{N} \, ds = \int_{R} \left( \frac{xz + xy}{4} \right) dy \, dz = \int_{y=0}^{4} \int_{z=0}^{5} \left( \frac{xz + xy}{4} \right) \frac{dy \, dz}{x/4}
\]
\[
= \int_{y=0}^{4} \int_{z=0}^{5} (z + y) \, dz \, dy = 90
\]
15.9 SUMMARY

In this lesson, we defined the integration of a vector function-line integrals, surface integrals and volume integrals. We proved some results and we discussed some problems.

15.10 TECHNICAL TERMS

Circulation, Flux, integral of a vector function, line, surface, volume integrals.

15.11 EXERCISES

15.11.1 : Find the value of \( \int_{0}^{1} \left( e^{t\mathbf{i}} + e^{-2t\mathbf{j}} + t\mathbf{k} \right) dt \)

15.11.2 : If \( \frac{d\mathbf{r}}{dt} = 12\cos 2t\mathbf{i} - 8\sin 2t\mathbf{j} + 16t\mathbf{k} \) and \( \mathbf{r}(0) = \mathbf{0} \) find \( \mathbf{r}(1) \).

15.11.3 : If \( \mathbf{a} \) and \( \mathbf{b} \) are constant vectors such that \( \frac{d^{2}\mathbf{r}}{dt^{2}} = \mathbf{a} + \mathbf{b} \) then find \( \mathbf{r} \)

15.11.4 : Evaluate \( \oint_{C} \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = x^{2}\mathbf{i} + y^{2}\mathbf{j} + y\mathbf{k} \) and \( C \) is the curve \( y^{2} = 4x \) in the XY-plane from \( (0,0) \) to \( (4,4) \).

15.11.5 : If \( \mathbf{F} = (3x^{2} + 6y)\mathbf{i} - 14yz\mathbf{j} + 20xz^{2}\mathbf{k} \), evaluate \( \oint_{C} \mathbf{F} \cdot d\mathbf{r} \) along the straight line from \( (0,0,0) \) to \( (1,0,0) \), then to \( (1,1,0) \) and then to \( (1,1,1) \).

15.11.6 : Evaluate \( \iint_{S} \mathbf{F} \cdot \mathbf{N} dS \) where \( \mathbf{F} = y\mathbf{i} + 2x\mathbf{j} - z\mathbf{k} \) and \( S \) is the surface of the plane \( 2x + y + 6 \) in the first octant cut off by the plane \( z = 4 \).

15.11.7 : If \( \mathbf{F} = 2xy\mathbf{i} + yz^{2}\mathbf{j} + xz\mathbf{k} \), find \( \iint_{S} \mathbf{F} \cdot \mathbf{N} dS \), where \( S \) is the surface of the parallelloiped \( x = 0, y = 0, z = 0, x = 2, y = 1 \) and \( z = 3 \).

15.11.8 : Evaluate \( \iiint_{V} \mathbf{F} \cdot \mathbf{d}V \), where \( \mathbf{F} = (2x^{2} - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k} \) and \( V \) is the volume of the region enclosed by \( x = 0, y = 0, z = 0, x + y + z = 1 \).

15.12 ANSWERS TO EXERCISES

15.11.1 : \( (e-1)\mathbf{i} + \frac{1}{2}(1-e^{-2})\mathbf{j} + \frac{1}{2}\mathbf{k} \)
15.11.2 : \[ 6 \sin 2t + (4 \cos 2t - 4) \hat{j} + t^2 \hat{k} \]

15.11.3 : \[ \hat{r} = \frac{1}{6} \hat{m}^3 + \frac{1}{2} \hat{b} t^2 + \hat{c} t + \hat{d} \]

15.11.4 : 264

15.11.5 : \[ \frac{23}{3} \]

15.11.6 : 108

15.11.7 : 30

15.11.8 : \[ \frac{1}{12} \]

15.13 MODEL EXAMINATION QUESTIONS

15.13.1 : If \[ \hat{F} = \hat{m} \cos wt + \hat{b} \sin wt \], where \( \hat{m}, \hat{b} \) are constant vectors, prove that \[ \frac{d^2 \hat{r}}{dt^2} + \omega^2 \hat{r} = \hat{0} \]

15.13.2 : If \[ \hat{F} = yz \hat{i} + xz \hat{j} + xy \hat{k} \], evaluate \[ \int_S \hat{F} \cdot \hat{n} \, dS \], where \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 1 \) which lies in the first constant.

15.13.3 : Evaluate \[ \oint_C \hat{F} \cdot d\hat{r} \], where \( \hat{F} = x^2 \hat{i} + y^2 \hat{j} \) and the curve \( C \) is \( y^2 = 4x \) in the XY-plane from \((0, 0)\) to \((4, 4)\).

15.13.4 : If \( \hat{F} = (2x^2 - 3z) \hat{i} - 2xy \hat{j} - 4x \hat{k} \), then evaluate \[ \int_V \int \nabla \cdot \hat{F} \, dv \] where \( V \) is the closed region bounded by the planes \( x = 0, y = 0, z = 0 \) and \( 2x + 2y + z = 4 \).

15.14 REFERENCES


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Lesson - 16

GAUSS'S DIVERGENCE THEOREM, GREEN'S THEOREM AND STOKE'S THEOREM

16.1 OBJECTIVE OF THE LESSON
In this lesson, we prove the Gauss's Divergence theorem, Green's Theorem and Stoke's Theorem. Some consequences and some problems are discussed.

16.2 STRUCTURE OF THE LESSON
This lesson has the following components.
16.3 Introduction
16.4 Gauss's Divergence Theorem
16.5 Green's Theorem
16.6 Stoke's Theorem
16.7 Answers to SAQ's
16.8 Summary
16.9 Technical Terms
16.10 Exercises
16.11 Answers to Exercises
16.12 Model Examination Questions

16.3 INTRODUCTION
In this lesson, Gauss's divergence theorem, Green's Theorem and Stoke's Theorem are proved. Some results are proved and some problems are solved.

16.4 GAUSS'S DIVERGENCE THEOREM
16.4.1 Theorem (Gauss's Divergence Theorem) : If \( \mathbf{F} \) is a vector function, continuous and having first order partial derivatives in some domain containing a closed surface \( S \) and \( V \) is the region enclosed by \( S \), then

\[
\iiint_V \text{div} \mathbf{F} \, dv = \iint_S \mathbf{F} \cdot \mathbf{N} \, dS
\]
where \( \mathbf{N} \) is the outward drawn unit normal vector at any points of \( S \).

**Proof:** Let \( \mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k} \), where \( F_1, F_2, F_3 \) are scalar point functions.

\[
\mathbf{F} = \frac{\partial F_1}{\partial x} \mathbf{i} + \frac{\partial F_2}{\partial y} \mathbf{j} + \frac{\partial F_3}{\partial z} \mathbf{k}
\]

\[
\therefore \text{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}
\]

\[
\therefore \int \int \text{div} \mathbf{F} \, dv = \int \int \left( \int \frac{\partial F_1}{\partial x} \, dv + \int \frac{\partial F_2}{\partial y} \, dv + \int \frac{\partial F_3}{\partial z} \, dv \right)
\]

**Case 1:** Let \( S \) be a closed surface. First let us choose the coordinate axes so that any line parallel to the axes meets the surface \( S \) at most two points.

Let \( R \) be the projection of \( S \) in the XY-plane. Let \( z = f(x, y), z = g(x, y) \) be the equations of \( S_1 \) and \( S_2 \) respectively, so that \( f(x, y) \leq z \leq g(x, y) \).

Now, \( \int \int \frac{\partial F_3}{\partial z} \, dv = \int \int \left( \int \frac{\partial F_3}{\partial z} \, dv \right) \, dx \, dy \, dz \)

\[
= \left[ \begin{array}{c} g(x, y) \\ f(x, y) \\ z = f(x, y) \\ z = g(x, y) \\ \end{array} \right] \, dx \, dy
\]
For the upper part $S_2$, $dx\,dy = dS\cos\theta = \vec{N}\cdot\vec{k}\,dS$

since normal to $S_2$ makes an acute angle $\gamma$ with $\vec{k}$.

\[
\therefore \int \int \int_{S_2} F_3(x,y,g)\,dx\,dy = \int \int \int_{S_2} F_3\vec{N}\cdot\vec{k}\,dS
\]

For the lower part $S_1, dx\,dy = -\cos\gamma\,ds = -\vec{N}\cdot\vec{k}\,dS$, since the normal to $S_1$, makes an obtuse angle $\gamma$ with $\vec{k}$.

\[
\therefore \int \int \int_{S_1} F_3(x,y,f)\,dx\,dy = -\int \int \int_{S_1} F_3\vec{N}\cdot\vec{k}\,dS
\]

\[
\therefore \int \int \int_{V} \frac{\partial F_3}{\partial z}\,dv = \int \int \int_{S_2} F_3\vec{N}\cdot\vec{k}\,ds + \int \int \int_{S_1} F_3\vec{N}\cdot\vec{x}\,dS
\]

\[
= \int \int \int_{S} F_3\vec{K}\cdot\vec{N}\,dS
\]

Similarly \[
\int \int \int_{V} \frac{\partial F_2}{\partial y}\,dv = \int \int \int_{S} F_2\vec{j}\cdot\vec{N}\,dS \quad \text{and} \quad \int \int \int_{V} \frac{\partial F_1}{\partial x}\,dv = \int \int \int_{S} F_1\vec{i}\cdot\vec{N}\,dS
\]

\[
\therefore \int \int \int_{V} \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right)\,dv = \int \int \int_{S} F_1\vec{i}\cdot\vec{N}\,dS + \int \int \int_{S} F_2\vec{j}\cdot\vec{N}\,dS + \int \int \int_{S} F_3\vec{k}\cdot\vec{N}\,dS
\]

\[
= \int \int \int_{S} \left( F_1\vec{i} + F_2\vec{j} + F_3\vec{k} \right)\cdot\vec{N}\,dS
\]
Case 2: If lines parallel to coordinate axes meet the surface in more than two points, sub-divide the region bounded by $S$ into subregions which satisfy the condition in case (1). From case (1), it follows that the theorem is true for each subregion and adding all these, we get the theorem for the given region.

Hence the Theorem.

16.4.2 Cartesian form of the Gauss’s Divergence Theorem: Let $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$

If $\alpha, \beta, \gamma$ are the angles made by $\mathbf{N}$ (outward drawn unit normal) with the positive directions of the axes, then $\cos \alpha, \cos \beta, \cos \gamma$ are the direction cosines of $\mathbf{N}$ so that $\mathbf{N} = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}$.

\[ \mathbf{F} \cdot \mathbf{N} = F_1 \cos \alpha + F_2 \cos \beta + F_3 \cos \gamma \]

\[ \therefore \text{ Gauss’s divergence theorem is} \]

\[ \iint_V \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) \ dx \ dy \ dz = \int_S \left( F_1 \cos \alpha + F_2 \cos \beta + F_3 \cos \gamma \right) \ ds \]

\[ = \int_S F_1 \ dy \ dz + F_2 \ dz \ dx + F_3 \ dx \ dy \]

16.4.3 Theorem: If $\mathbf{F}$ is a differentiable vector function having continuous first order partial derivatives and $S$ is a closed surface enclosing a region $V$, then

\[ \int_S \mathbf{N} \times \mathbf{F} \ ds = \int_V \nabla \times \mathbf{F} \ dv \]

Proof: Let $\mathbf{F} = \mathbf{\pi} \times \mathbf{F}$, where $\mathbf{\pi}$ in any constant vector. By Gauss’s divergence theorem, we have

\[ \int_S \mathbf{F} \cdot \mathbf{N} \ ds = \int_V \nabla \cdot \mathbf{F} \ dv \]

\[ \Rightarrow \int_S \mathbf{\pi} \times \mathbf{F} \cdot \mathbf{N} \ ds = \int_V \nabla \cdot (\mathbf{\pi} \times \mathbf{F}) \ dv \]

\[ \Rightarrow \int_S \mathbf{\pi} \cdot \mathbf{F} \times \mathbf{N} \ ds = - \int_V \mathbf{\pi} \cdot \nabla \times \mathbf{F} \ dv \]
\[ \Rightarrow \vec{a} \cdot \int_{S} \vec{F} \times \vec{N} \, ds = -\vec{a} \cdot \int_{V} \nabla \times \vec{F} \, dv \]

\[ \Rightarrow \vec{a} \cdot \int_{S} \vec{F} \times \vec{N} \, ds + \vec{a} \int_{V} \nabla \times \vec{F} \, dv = 0 \]

\[ \Rightarrow \vec{a} \left( \int_{S} \vec{F} \times \vec{N} \, ds + \int_{V} \nabla \times \vec{F} \, dv \right) = 0 \]

Since \( \vec{a} \) is arbitrary, we may take it as non-zero not perpendicular to

\[ \int_{S} \vec{F} \times \vec{N} \, ds + \int_{V} \nabla \times \vec{F} \, dv = 0 \]

\[ \int_{S} \vec{F} \times \vec{N} \, ds = \int_{V} \nabla \times \vec{F} \, dv \]

16.4.4 Theorem: If \( \phi \) is a continuously differentiable scalar point function and \( S \) is a closed surface enclosing a region with volume \( V \), then

\[ \int_{S} \vec{N} \phi \, ds = \int_{V} \nabla \phi \, dv \]

Proof: Let \( \vec{F} = \vec{a} \phi \) where \( \vec{a} \) is any constant vector.

\[ \Rightarrow \int_{S} \phi \vec{N} \, ds = \int_{S} \nabla \phi \, dV \Rightarrow \int_{S} \phi \vec{N} \, ds = \int_{S} \nabla \phi \, dV \]

\[ \Rightarrow \vec{a} \cdot \int_{S} \phi \vec{N} \, ds - \vec{a} \cdot \int_{V} \nabla \phi \, dv = 0 \]

\[ \Rightarrow \vec{a} \left( \int_{S} \phi \vec{N} \, ds - \int_{V} \nabla \phi \, dv \right) = 0 \]

Since \( \vec{a} \) is arbitrary, we get

\[ \int_{S} \phi \vec{N} \, ds = \int_{V} \nabla \phi \, dv . \]
SOLVED PROBLEMS

16.4.5 : Verify Gauss's divergence theorem for \( \int_S \left( (x^3 - yz) \hat{i} - 2x^2y \hat{j} + z \hat{k} \right) \cdot \vec{N} \, dS \) over the surface of the cube bounded by the co-ordinate planes and \( x = y = z = a \).

Solution : \( \vec{F} = (x^3 - yz) \hat{i} - 2x^2y \hat{j} + z \hat{k} \)

\[ \text{div} \vec{F} = \frac{\partial}{\partial x} (x^3 - yz) + \frac{\partial}{\partial y} (-2x^2y) + \frac{\partial}{\partial z} (z) \]

\[ = 3x^2 - 2x^2 + 1 = x^2 + 1 \]

To verify \( \int_V \text{div} \vec{F} \, dv = \int_S \vec{F} \cdot \vec{N} \, dS \)

\[ \text{LHS} = \int_V \text{div} \vec{F} \, dv = \int_0^a \int_0^a \left( x^2 + 1 \right) dx \, dy \, dz = \int_0^a \left( x^3 + x \right)_0^a \, dy \, dz \]

\[ = \int_0^a \left( \frac{a^3}{3} + a \right) dy \, dz = \left( \frac{a^3}{3} + a \right) \int_0^a \left( \frac{x^3}{3} + x \right)_0^a \, dy \, dz \]
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\[ \int_{0}^{a} \left( \frac{a^3}{3} + a \right) dy dz = \left( \frac{a^3}{3} + a \right) \int_{0}^{a} y^0 dx = \left( \frac{a^3}{3} + a \right) (x)^0 = \frac{a^5}{3} + a^3 \]

RHS:

(1) For \( S_1 \) : the face PMAN, \( \mathbb{N} = \mathbb{T}, x = a, ds = dy dz; F \cdot \mathbb{N} = x^3 - yz = a^3 - yz \)

\[
\int_{S_1} F \cdot \mathbb{N} ds = \int_{y=0}^{a} \int_{z=0}^{a} (a^3 - yz) dy dz = \int_{y=0}^{a} \left( a^3 y - \frac{y^2 z}{2} \right)_{y=0}^{a} dz = \int_{z=0}^{a} \left( a^4 - \frac{a^2 z^2}{2} \right) dz
\]

\[
= \left( a^4 z - \frac{a^2 z^2}{2} \right)_{z=0}^{a} = a^5 - \frac{a^4}{4}
\]

(2) For \( S_2 \) : the face OBLC : \( \mathbb{N} = -\mathbb{T}, x = 0, ds = dy dz \)

\[ F \cdot \mathbb{N} = -\left( x^3 - yz \right) = yz \]

\[ \therefore \int_{S_2} F \cdot \mathbb{N} ds = \int_{0}^{a} \int_{0}^{a} yz dy dz = \int_{0}^{a} \left( \frac{y^2}{2} \right) z dz = \frac{a^2}{2} \left( \frac{z^2}{2} \right)_{0}^{a} = \frac{a^4}{4} \]

(3) For \( S_3 \) : Face PNBL; \( \mathbb{N} = \mathbb{T}, y = a, F \cdot \mathbb{N} = -2x^2 y = -2ax^2; ds = dx dz \)

\[
\int_{S_3} F \cdot \mathbb{N} ds = \int_{0}^{a} \int_{0}^{a} -2ax^2 dx dz = -2a \int_{0}^{a} \left( \frac{x^3}{3} \right) dz = -2a \cdot \frac{a^3}{3} (z)^a
\]

\[ = -\frac{2a^4}{3} \cdot a = -\frac{2a^5}{3} \]

(4) For \( S_4 \) : Face OAMC : \( \mathbb{N} = -\mathbb{T}, y = 0, ds = dx dz; F \cdot \mathbb{N} = 2x^2 y = 0 \)

\[ \therefore \int_{S_4} F \cdot \mathbb{N} ds = 0 \]
(5) For \( S_5 \): Face PLCM: \( \mathbf{N} = \mathbf{k}, z = a, ds = dx
dy, \mathbf{F} \cdot \mathbf{N} = z = a. \)

\[
\int_{S_5} \mathbf{F} \cdot \mathbf{N} ds = \int_{0}^{a} a dx \int_{0}^{a} (x) dy = a \int_{0}^{a} dy = a^{2} \left( y \right)_{0}^{a} = a^{3}
\]

(6) For \( S_6 \): Face OANB; \( \mathbf{N} = -\mathbf{k}, ds = dx
dy, z = 0, \mathbf{F} \cdot \mathbf{N} = -z = 0 \). \( \therefore \int_{S_6} \mathbf{F} \cdot \mathbf{N} dS = 0 \)

\[
\therefore \int \mathbf{F} \cdot \mathbf{N} ds = \int \mathbf{F} \cdot \mathbf{N} ds + \int \mathbf{F} \cdot \mathbf{N} ds + \int \mathbf{F} \cdot \mathbf{N} ds + \int \mathbf{F} \cdot \mathbf{N} ds + \int \mathbf{F} \cdot \mathbf{N} ds + \int \mathbf{F} \cdot \mathbf{N} ds
\]

\[
= a^{5} - a^{4} + a^{4} \cdot \frac{2a^{5}}{3} + 0 + a^{3} + 0
\]

\[
= \frac{a^{5}}{3} + a^{3}
\]

\( \therefore \text{LHS} = \text{RHS} \). \( \therefore \) Gauss's divergence theorem is verified.

16.4.6: Evaluate \( \int_{S} (ax^{2} + by^{2} + cz^{2}) ds \) over the surface \( x^{2} + y^{2} + z^{2} = 1 \).

**Solution**: By Gauss's divergence theorem \( \int_{S} \mathbf{F} \cdot \mathbf{N} ds = \int \nabla \cdot \mathbf{F} dv \).

Here \( \mathbf{F} \cdot \mathbf{N} = ax^{2} + by^{2} + cz^{2} \)

Let \( \phi : x^{2} + y^{2} + z^{2} - 1 \)

\[ \nabla \phi = \sum \frac{\partial \phi}{\partial x} \mathbf{i} = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k} \]

\( \mathbf{N} = \text{outward drawn unit normal} = \frac{2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}}{\sqrt{4x^{2} + 4y^{2} + 4z^{2}}} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \)

Since \( \mathbf{F} \cdot \mathbf{N} = ax^{2} + by^{2} + cz^{2} \), we have \( \mathbf{F} = ax \mathbf{i} + by \mathbf{j} + cz \mathbf{k} \)

\( \nabla \cdot \mathbf{F} = a + b + c \)

\( \therefore \) By Gauss's Divergence Theorem, \( \int_{S} \mathbf{F} \cdot \mathbf{N} ds = \int \nabla \cdot \mathbf{F} dv = \int (a + b + c) dv \)
- \(\frac{4\pi}{3}(a + b + c)\), when \(V\) is the volume of the sphere \(\frac{4\pi}{3}\).

**16.4.7 SAQ**: If \(\vec{N}\) is the outward drawn unit normal to any closed surface \(S\), show that \(\int_{V} \text{div} \vec{N} \, dv = S'\), when \(S'\) is the surface area of the surface \(S\).

**16.4.8 SAQ**: Using Gauss's divergence theorem, prove that \(\int_{S} \vec{r} \cdot \vec{N} \, ds = 3V\), where \(V\) is the volume of the region enclosed by \(S\).

**16.4.9 SAQ**: Show that \(\int_{S} (ax\hat{i} + by\hat{j} + cz\hat{k}) \cdot \vec{N} \, ds = \frac{4\pi}{3}(a + b + c)\), where \(S\) is the surface of the sphere \(x^2 + y^2 + z^2 = 1\).

**16.4.10**: Prove that
\[
\int_{S} \vec{r} \cdot \text{curl} \vec{g} \, dv = \int_{S} \vec{g} \times \vec{r} \cdot \vec{N} \, ds + \int_{V} \vec{g} \cdot \text{curl} \vec{r} \, dv
\]

**Solution**: Applying the Gauss's divergence theorem to \(\vec{g} \times \vec{r}\), we get
\[
\int_{S} \vec{g} \times \vec{r} \cdot \vec{N} \, ds = \int_{V} \text{div} \vec{g} \times \vec{r} \, dv
\]
\[
= \int_{V} \big( \vec{r} \cdot \text{curl} \vec{g} - \vec{g} \cdot \text{curl} \vec{r} \big) \, dv
\]
\[
= \int_{V} \vec{r} \cdot \text{curl} \vec{g} \, dv - \int_{V} \vec{g} \cdot \text{curl} \vec{r} \, dv
\]
\[
\therefore \int_{V} \vec{r} \cdot \text{curl} \vec{g} \, dv = \int_{S} \vec{g} \times \vec{r} \cdot \vec{N} \, ds + \int_{V} \vec{g} \cdot \text{curl} \vec{r} \, dv
\]

**16.4.11 SAQ**: For any closed surface \(S\), prove that \(\int_{S} \vec{N} \, ds = 0\)

**16.5 GREEN'S THEOREM**

**16.5.1 Theorem (Green's Theorem in Plane)**: Let \(S\) be a closed region in the XY-Plane enclosed by a curve \(C\). If \(P\) and \(Q\) are differentiable real valued functions of \(x\) and \(y\) in \(S\), then
\[
\oint_{C} P \, dx + Q \, dy = \iint_{S} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy
\]
**Proof**: The line integral being taken along the entire boundary $C$ of $S$ such that $S$ is on the left as one moves along $C$.

**Case 1**: Let any line parallel to either co-ordinate axis cut in atmost two points.

Let $S$ be lie between the lines $x = a, x = b, y = c,$ and $y = d$.

Let $y = f(x)$ be the curve $C_1(AEB)$

Let $y = g(x)$ be the curve $C_2(ADB)$

where $f(x) \leq g(x)$

\[
\therefore \int \int_{S} \frac{\partial P}{\partial y} \, dx \, dy = \int_{a}^{b} \int_{y = f(x)}^{g(x)} \frac{\partial P}{\partial y} \, dy \, dx
\]

\[
= \int_{a}^{b} \left[ P(x, y) \right]_{y = f(x)}^{g(x)} \, dx = \left[ P(x, g) - P(x, f) \right]_{a}^{b}
\]
\[ \int_{a}^{b} P(x,g) \, dx - \int_{a}^{b} P(x,f) \, dx \]

\[ = -\int_{b}^{a} P(x,y) \, dx - \int_{b}^{a} P(x,f) \, dx \]

\[ = -\int_{C_2}^{C_1} P(x,y) \, dx - \int_{C} P(x,y) \, dx = -\int_{C_2}^{C_1} P(x,y) \, dx \]

\[ \therefore \int_{C} P \, dx = -\int_{S} \frac{\partial P}{\partial y} \, dy \quad \text{--------- (1)} \]

Similarly we can prove that \[ \int_{C} Q \, dy = \int_{S} \frac{\partial Q}{\partial x} \, dx \quad \text{--------- (2)} \]

Adding (1) & (2), we get \[ \int_{C} P \, dx + Q \, dy = \int_{S} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \]

**Case 2** : If a line parallel to either axis cuts \( S \) in more than two points, then divide \( S \) into subregions which satisfy the condition in case (1). From case (1), it follows that the Theorem is true for each subregion and adding all these, we get the theorem for the given region.

Hence the theorem.

**16.5.2 Theorem (Green's Identities)** : If \( f \) and \( g \) are differentiable scalar point functions over the region \( V \) enclosed by the surface \( S \), then

(a) \[ \int_{V} \left( f \nabla^2 g + \nabla f \cdot \nabla g \right) \, dv = \int_{S} \left( f \nabla g \right) \cdot \vec{N} \, ds \]

(b) \[ \int_{V} \left( f \nabla^2 g - g \nabla^2 f \right) \, dv = \int_{S} \left( f \nabla g - g \nabla f \right) \cdot \vec{N} \, ds \]

**Proof** : (a) Let \( \vec{F} = f \nabla g \). Then

\[ \nabla \cdot \vec{F} = \nabla \cdot (f \nabla g) \]

\[ = f \left( \nabla \cdot \nabla g \right) + \nabla f \cdot \nabla g = f \nabla^2 g + \nabla f \cdot \nabla g \]
By Gauss divergence Theorem. Then we have
\[ \nabla \cdot F = \int_{S} F \cdot N \, ds \]

\[ \int_{V} \left( \nabla \cdot (\nabla^2 f + \nabla f \cdot \nabla g) \right) \, dv = \int_{S} (\nabla f \cdot \widetilde{N}) \, ds \quad \text{(1)} \]

(b) Interchanging \( f \) and \( g \), we have
\[ \int_{V} \left( g \nabla^2 f + \nabla f \cdot \nabla g \right) \, dv = \int_{S} (g \nabla f \cdot \widetilde{N}) \, ds \quad \text{(2)} \]

Subtracting (2) from (1), we get
\[ \int_{V} \left( f \nabla^2 g - g \nabla^2 f \right) \, dv = \int_{S} (f \nabla g - g \nabla f) \cdot \widetilde{N} \, ds \]

Hence the Theorem.

16.5.3 **Note**: In Theorem 16.5.2, (a) is called Green's first identity and (b) is called Green's second identity or Green's Theorem in symmetrical form.

16.5.4 **Definition**: A scalar point function \( \phi \) is called harmonic function, if \( \nabla^2 \phi = 0 \). The equation \( \nabla^2 \phi = 0 \) is called the harmonic equation of \( \phi \).

16.5.5 **Note**: If \( f \) and \( g \) are harmonic functions, then
\[ \int_{S} (f \nabla g - g \nabla f) \cdot \widetilde{N} \, dS = 0 \]

This follows from the Green's second identity.

**SOLVED PROBLEMS**

16.5.6 : Show that the area enclosed by a closed curve \( C \) is given by
\[ \frac{1}{2} \int_{C} x \, dy - y \, dx \]
and hence find the area bounded by the ellipse. \( x = a \cos \theta, y = b \sin \theta, 0 \leq \theta \leq 2\pi \).

**Solution**: By Green's Theorem, we have
\[ \frac{1}{2} \int_{C} x \, dy - y \, dx = \frac{1}{2} \int (-y) \, dx + x \, dy \]
\[ = \frac{1}{2} \int_{S} \left( \frac{\partial}{\partial x} (x) - \frac{\partial}{\partial y} (-y) \right) \, dx \, dy \]
\[ = \frac{1}{2} \int_S (1 + 1) \, dx \, dy = \int_S 1 \, dx \, dy \]

\[ = A, \text{ where } A \text{ is the area of the region } S. \]

Area of the ellipse \[ = \frac{1}{2} \int_C x \, dy - y \, dx \]

\[ = \frac{1}{2} \int_0^{2\pi} \left[ a \cos \theta (b \cos \theta) - b \sin \theta (-a \sin \theta) \right] \, d\theta \]

\[ = \frac{1}{2} \int_0^{2\pi} ab \left( \cos^2 \theta + \sin^2 \theta \right) \, d\theta \]

\[ = \frac{1}{2} \cdot 2\pi \cdot ab \cdot (0 \cdot 2\pi) = \frac{1}{2} \cdot ab \cdot 2\pi = \pi ab \text{ square units.} \]

16.5.7: Verify Green's Theorem in plane for \( \oint_C (x^2 + y^2) \, dx + x^2 \, dy \), where \( C \) is the closed curve of the region bounded by \( y = x \) and \( y = x^2 \).

Solution: We have to verify \( \oint_C P \, dx + Q \, dy = \iint_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \)

where \( P = xy + y^2, \ Q = x^2 \)

\[ \therefore \frac{\partial P}{\partial y} = x + 2y; \frac{\partial Q}{\partial x} = 2x \]
LHS = \[ \int_{y=x^2}^{x^2} (Pdx + Qdy) + \int_{y=x}^{x} (Pdx + Qdy) \]

= \[ \int_{0}^{1} [x \cdot x^2 + x^4 + x^2 \cdot 2x]dx + \int_{0}^{1} [(x \cdot x + x^2) + x^2]dx \]

= \[ \int_{0}^{1} (3x^3 + x^4)dx + \int_{0}^{1} 3x^2dx \]

= \[ \left( \frac{3}{4} x^4 + \frac{x^5}{5} \right)_{0}^{1} + (x^3)_{0}^{1} = \frac{3}{4} + \frac{1}{5} - 1 = \frac{15 + 4 - 20}{20} = -\frac{1}{20} \]

RHS = \[ \int_{S} [2x - (x + 2y)]dx + \int_{0}^{y=x^2} (x - 2y)dx dy \]

= \[ \int_{0}^{5} (x - 2y)dy = \int_{0}^{1} (xy - y^2)_{y=x^2}x dx \]

= \[ \int_{0}^{1} ((x^2 - x^2) - (x^3 - x^4))dx = -\left( \frac{x^4}{4} - \frac{x^5}{5} \right)_{0}^{1} = -\left( \frac{1}{4} - \frac{1}{5} \right) = -\frac{1}{20} \]

\therefore \text{LHS} = \text{RHS}

\therefore \text{Green's Theorem is verified.}

16.5.8: Verify Green's Theorem in the plane for \( \oint_{C} (3x^2 - 8y^2)dx + (4y - 6xy)dy \), where C is the region bounded by \( y = \sqrt{x} \) and \( y = x^2 \).

Solution: We have to verify \( \oint_{C} Pdx + Qdy = \int_{S} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)dx dy \),
where \( P = 3x^2 - 8y^2 \), \( Q = 4y - 6xy \)

\[
\frac{\partial P}{\partial y} = -16y, \quad \frac{\partial Q}{\partial x} = -6y
\]

\[
\oint_{C_1} P\,dx + Q\,dy = \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^2)\,2x\,dx
\]

\[
= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx = (x^3 + 2x^4 - 4x^5) \bigg|_0^1 = 1 + 2 - 4 = -1
\]

\[
\oint_{C_2} P\,dx + Q\,dy = \int_1^0 (3x^2 - 8x)\,dx + \left(4\sqrt{x} - 6x\sqrt{x}\right)\frac{1}{2\sqrt{x}}\,dx
\]

\[
= \int_1^0 (3x^2 - 8x + 2 - 3x)\,dx = \int_1^0 (3x^2 - 11x + 2)\,dx = \left(\frac{x^3 - 11x^2}{2} + 2x\right) \bigg|_1^0
\]

\[
= -\left(1 - \frac{11}{2} + 2\right) = \frac{5}{2}
\]

\[
\therefore \oint_{C} P\,dx + Q\,dy = -1 + \frac{5}{2} = \frac{3}{2}
\]
\[ \text{RHS} = \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \int_0^{\sqrt{x}} (-6y + 16y) \, dx \, dy \]

\[ = \int_0^{\sqrt{x}} 10y \, dx = \int_0^1 (5y^2) \frac{\sqrt{x}}{x} \, dx \]

\[ = \int_0^1 5(x - x^4) \, dx = 5 \left( \frac{x^2}{2} - \frac{x^5}{5} \right)_0^1 = 5 \left( \frac{1}{2} - \frac{1}{5} \right) = \frac{3}{2} \]

\[ \therefore \text{LHS} = \text{RHS} \]

\[ \therefore \text{Green's Theorem is verified.} \]

**16.5.9** Using Green's Theorem, evaluate \[ \int_C (x^2 + y^2) \, dx + 3xy^2 \, dy \] where \( C \) is the circle \( x^2 + y^2 = 4 \).

**Solution** : We have \( \int_C P \, dx + Q \, dy = \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \)

where \( P = x^2 + y^2, \ Q = 3xy^2 \)

\[ \therefore \frac{\partial P}{\partial y} = 2y, \ \frac{\partial Q}{\partial x} = 3y^2 \]

\[ \therefore \int_C P \, dx + Q \, dy = \int_0^{\sqrt{4-x^2}} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3y^2 - 2y) \, dy \, dx \]

\[ = \int_{x=-2}^2 \left( y^3 - y^2 \right) \sqrt{4-x^2} \, dx = 2 \int_{-2}^2 \left( 4 - x^2 \right)^{3/2} \, dx \]
Gauss’s Divergence Theorem, Green’s Theorem and Stoke’s...

\[ \frac{2}{3} \int (4-x^2)^{3/2} \, dx \quad \text{Put } x = 2 \sin \theta \]
\[ \frac{dx}{d\theta} = 2 \cos \theta \theta \text{ varies from } 0 \text{ to } \frac{\pi}{2} \]

\[ \frac{\pi}{2} \]
\[ 4 \int_0^{\pi/2} (4 \cos^2 \theta)^{3/2} 2 \cos \theta d\theta \]

\[ = 64 \int_0^{\pi/2} \cos^4 \theta d\theta \]

\[ = 64 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \]

\[ = 12\pi \]

16.6 STOKE’S THEOREM

16.6.1 Theorem (Stoke’s Theorem) : Let \( S \) be a surface bounded by a closed, non-intersecting curve \( C \). If \( \vec{F} \) is any differentiable vector point function, then \( \oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot \vec{N} \, ds \), where \( \vec{N} \) is the outward drawn unit normal vector to \( S \) and \( C \) is the boundary of \( S \) which is traversed in the positive direction (in the sense that if a person walking on the boundary of \( S \) in this direction, with his head pointing in the direction of outward drawn normal \( \vec{N} \) to \( S \), has the surface on his left.

Proof : Let \( S \) be a surface which is such that its projections on \( XY \), \( YZ \), \( ZX \) planes are regions bounded by simple closed curves. Let \( S \) have equations \( z = f(x, y) \) or \( x = g(y, z) \) or \( y = h(z, x) \) where \( f, g, h \) are differentiable functions.

Let \( \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k} \Rightarrow \text{curl} \vec{F} = \nabla \times \vec{F} = \nabla \times (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}) \)

\[ \vec{F}_i = \text{curl} F_1 \vec{\tau} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix} \]

\[ = \frac{\partial F_1}{\partial z} \vec{j} - \frac{\partial F_1}{\partial y} \vec{k} \]
Let $z = f(x, y)$ be the equation of $S$. For any point of $S$

$$\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = x \mathbf{i} + y \mathbf{j} + f(x, y) \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \frac{\partial z}{\partial y} \mathbf{k}$$

Since $\frac{\partial \mathbf{r}}{\partial y}$ is the tangent vector to $S$, $\mathbf{N} \cdot \frac{\partial \mathbf{r}}{\partial y} = 0$

$$\mathbf{N} \cdot \mathbf{j} + \mathbf{N} \cdot \mathbf{k} \frac{\partial z}{\partial y} = 0 \Rightarrow \mathbf{N} \cdot \mathbf{j} - \mathbf{N} \cdot \mathbf{k} \frac{\partial z}{\partial y}$$

$$\therefore (\nabla \times \mathbf{F}) \cdot \mathbf{N} \, ds = -\left( \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} \frac{\partial z}{\partial y} \right) \mathbf{N} \cdot \mathbf{k} \, ds$$

$$= -\frac{\partial}{\partial y} F_1(x, y, z) \cos \gamma \, ds$$

$$= -\frac{\partial F_1}{\partial y} dx \, dy$$

Let $R$ be the projection of $S$ on $XY$ Plane and $\sigma$ be the boundary of $R$. 
Gauss’ Divergence Theorem, Green’s Theorem and Stoke’s...  

\[ \mathbf{\nabla} \times \mathbf{F} \cdot \mathbf{N} = \int_{S} \left( 0 - \frac{\partial F_1}{\partial y} \right) \, dx \, dy = \int_{\sigma} F_1 \, dx + \sigma \, dy, \text{ by Green’s Theorem} \]

Since \( F_1(x, y, z) \) of \( C \) is the same as \( F_1(x, y, f(x, y)) \), we have

\[
\int_{C} F_1 \, dx = \int_{C} F_2 \, dy, \quad \therefore \int_{S} \mathbf{\nabla} \times \mathbf{F} \cdot \mathbf{N} \, ds = \int_{C} F_1 \, dx
\]

Similarly by taking projections on \( YZ, ZX \) planes, we have

\[
\int_{S} \mathbf{\nabla} \times \mathbf{F} \cdot \mathbf{N} \, ds = \int_{C} F_2 \, dy \quad \text{and} \quad \int_{S} \mathbf{\nabla} \times \mathbf{F} \cdot \mathbf{N} \, ds = \int_{C} F_3 \, dz
\]

Adding we get \( \int_{S} \mathbf{\nabla} \times \mathbf{F} \cdot \mathbf{N} \, ds = \int_{C} \mathbf{\nabla} \cdot \mathbf{F} \cdot \mathbf{dr} \)

16.6.2 **Stoke’s Theorem in Plane**: Let the surface \( S \) lie in the \( XY \)-plane. Then \( z = 0 \), \( z - \) axis is normal \( \mathbf{N} \) so that \( \mathbf{N} = \mathbf{k} \).

Let \( \mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j}; \quad \mathbf{F} = x \mathbf{i} + y \mathbf{j}; \quad z = 0, \, dz = 0 \)

\[
\Rightarrow \oint_{C} \mathbf{F} \cdot \mathbf{dr} = \oint_{C} F_1 \, dx + F_2 \, dy; \quad \text{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix}
\]

\[
= -\frac{\partial F_2}{\partial z} \mathbf{i} + \frac{\partial F_1}{\partial z} \mathbf{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}
\]

\[ \text{Curl} \mathbf{F} \cdot \mathbf{N} = \text{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \]

Stoke’s theorem is \( \int_{C} \mathbf{F} \cdot \mathbf{dr} = \oint_{S} \text{curl} \mathbf{F} \cdot \mathbf{N} \, ds = \int_{S} \text{curl} \mathbf{F} \cdot \mathbf{N} \, ds \)
\[ \Rightarrow \int_C (F_1 \, dx + F_2 \, dy) = \int_S \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy \]

which is the Green's Theorem.

16.6.3 Note: Stoke's Theorem is plane is Green's Theorem.

**SOLVED PROBLEMS**

16.6.4: Verify Stoke's Theorem for \( \mathbf{A} = (2x - y) \mathbf{i} - yz^2 \mathbf{j} - y^2 z \mathbf{k} \), where \( S \) is the upper half surface of the sphere \( x^2 + y^2 + z^2 = 1 \) and \( C \) is its boundary.

**Solution**: The boundary \( C \) of \( S \) in the circle \( x^2 + y^2 = 1, z = 0 \) in the \( XY \)-plane with parametric equations: \( x = \cos t, y = \sin t, z = 0, 0 \leq t \leq 2\pi \).

\[ \therefore \int_C \mathbf{A} \cdot d\mathbf{r} = \int_C (2x - y) \, dx + 0 \, dy + 0 \, dz = \int_C (2 \cos t - \sin t)(-\sin t) \, dt \]

\[ = \int_0^{2\pi} -\sin 2t \, dt + \int_0^{2\pi} \sin^2 t \, dt \]

\[ = \left[ \frac{\cos 2t}{2} \right]_0^{2\pi} + 4 \cdot \frac{1}{2} \frac{\pi}{2} = \pi \]

\[ \text{curl} \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \mathbf{i}(-2yz + 2yz) - \mathbf{j}(0 - 0) + \mathbf{k}(0 + 1) = \mathbf{k} \]

\[ \int_S \text{curl} \mathbf{A} \cdot \mathbf{N} \, ds = \int_S \mathbf{N} \cdot \mathbf{A} \, ds = \int_S \mathbf{N} \cdot \mathbf{k} \, ds = \int_S \int_R \, dx \, dy, \text{ where } R \text{ is the projection of } S \text{ in the } XY \text{ plane.} \]

\[ = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx = 4 \int_{0}^{1} \int_{0}^{1} \, dy \, dx \]
16.21

Gauss’s Divergence Theorem, Green’s Theorem and Stoke’s...

\[
\frac{1}{2} \left( \frac{2 \pi}{22} + \frac{1}{2} \sin^{-1} x \right) \bigg|_0^1 = 4 \left( \frac{1}{22} \right) = \pi
\]

\[
\mathbf{A} \cdot d\mathbf{r} = \int \mathbf{curl} \mathbf{A} \cdot \mathbf{N} ds
\]

\[
\therefore \mathbf{A} \cdot d\mathbf{r} = \int \mathbf{curl} \mathbf{A} \cdot \mathbf{N} ds
\]

\[
\therefore \text{Stoke’s Theorem is verified.}
\]

16.6.5: Verify Stoke’s Theorem for \( \mathbf{F} = -y^3 \mathbf{i} + x^3 \mathbf{j} \) where \( S \) in the circular disc \( x^2 + y^2 \leq 1, z = 0 \)

**Solution:** Given \( \mathbf{F} = -y^3 \mathbf{i} + x^3 \mathbf{j} \)

Boundary of \( S \) in the circle \( x^2 + y^2 = 1, z = 0 \) in the XY-Plane, with parametric equations:

\[
x = \cos t, \ y = \sin t, \ z = 0, \ 0 \leq t \leq 2\pi.
\]

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y^3 \, dx + x^3 \, dy = \int_0^{2\pi} \left( -\sin^3 t (\sin t) + \cos^3 t \cos t \right) dt
\]

\[
= 2\pi \int_0^{\pi/2} \left( \cos^4 t + \sin^4 t \right) dt = 4 \int_0^{\pi/2} \left( \cos^4 t \, dt + \sin^4 t \, dt \right)
\]

\[
= 4 \left( \frac{3}{2} \frac{\pi}{22} + \frac{3}{4} \frac{\pi}{22} \right) = \frac{3\pi}{2}
\]

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-3y^3 & x^3 & 0
\end{vmatrix} = \mathbf{k} \left( 3x^2 + 3y^2 \right)
\]

\[
\int_S \nabla \times \mathbf{F} \cdot \mathbf{N} ds = 3 \int_R \left( x^2 + y^2 \right) \mathbf{K} \cdot \mathbf{N} ds = 3 \int_R \left( x^2 + y^2 \right) dx \, dy, \quad \text{since} \quad \mathbf{N} = \mathbf{K} \quad \text{and} \quad R \quad \text{is the region in the XY-Plane.}
\]

Let \( x = r \cos \theta, \ y = r \sin \theta, \) so that \( dx \, dy = r \, dr \, d\theta \)

r varies from 0 to 1 and \( \theta \) varies from 0 to \( 2\pi \).

\[
\therefore \int_S \nabla \times \mathbf{F} \cdot \mathbf{N} ds = 3 \int_0^{2\pi} \left( 1 \right) \int_0^1 r^2 r \, dr \, d\theta = 3 \int_0^{2\pi} \left( \frac{r^4}{4} \right) \bigg|_0^1 \, d\theta
\]

\[
= \frac{3\pi}{4}
\]
\[
\frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}
\]

\[
\therefore \int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl} \vec{F} \cdot \vec{N} \, dS.
\]

\[
\therefore \text{Stoke's Theorem is verified.}
\]

16.7 ANSWERS TO SAQ's

16.4.7 SAQ: \(\int \text{div} \vec{N} \, dv = \int_S \vec{N} \cdot \vec{N} \, ds\) by Gauss's divergence theorem.

\[
= \int_S ds = S', \text{ where } S' \text{ is the surface area of the surface } S.
\]

16.4.8 SAQ: \(\int_S \vec{r} \cdot \vec{N} \, ds = \int_V \text{div} \vec{r} \, dv = \int_V 3dv = 3V\)

where \(V\) is the volume of the volume of the region bounded by \(S\).

16.4.9 SAQ: \(\int_S \left(ax \mathbf{i} + by \mathbf{j} + cz \mathbf{k}\right) \cdot \vec{N} \, ds = \int_V \text{div} \left(ax \mathbf{i} + by \mathbf{j} + cz \mathbf{k}\right)dv\)

\[
= \int_V (a + b + c)dv = (a + b + c) \left(\text{volume of the sphere}\right)
\]

\[
= \frac{4\pi}{3} (a + b + c)
\]

16.4.11 SAQ: \(\int_S \vec{N} \, ds = \int_S \vec{N} \, ds = \int_V \nabla \phi \, dv = 0\)

by taking \(\phi = 1\) in 16.4.4 Theorem.

16.8 SUMMARY

In this lesson, the stated and proved the three important theorems namely Gauss's divergence Theorem, Green's Theorem and Stoke's Theorem. We discussed some consequences such as Green's identities etc. and some problems.

16.9 TECHNICAL TERMS

Gauss's divergence Theorem - Green's Theorem - Stoke's Theorem - Green's Identities.
16.10 EXERCISES

16.10.1 : Verify Gauss’s divergence theorem for the function \( \mathbf{F} = (x^2 - yz) \mathbf{i} + (y^2 - zx) \mathbf{j} + (z^2 - xy) \mathbf{k} \) taken over the rectangular parallelepiped \( 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c \).

16.10.2 : Evaluate \( \int_{S} \mathbf{F} \cdot \mathbf{N} \, dS \) using Gauss's divergence Theorem where \( \mathbf{F} = 4x \mathbf{i} - 2y \mathbf{j} + z^2 \mathbf{k} \) and \( S \) is the region bounded by \( x^2 + y^2 = 4, z = 0, z = 3 \).

16.10.3 : Verify Green's theorem (in plane) for \( \oint_{C} (2xy - x^2) \, dx + (x^2 + y^2) \, dy \), where \( C \) is the boundary of the region enclosed by \( y = x^2 \) and \( y^2 = x \) described in the positive sense.

16.10.4 : Using Green's Theorem, evaluate \( \oint_{C} (3x + 4y) \, dx + (2x - 3y) \, dy \) where \( C \) is the circle \( x^2 + y^2 = 4 \).

16.10.5 : Using Stoke's Theorem, prove that \( \text{curl} \, \text{grad} \, \phi = 0 \).

16.10.6 : Verify Stoke's Theroem for \( \mathbf{F} = 2y \mathbf{i} + 3x \mathbf{j} - z^2 \mathbf{k} \) where \( S \) is the upper half surface of the sphere \( x^2 + y^2 + z^2 = 9 \) and \( C \) is its boundary.

16.11 ANSWERS TO EXERCISE

16.10.2 : \( 84\pi \)
16.10.3 : \( 0 \)
16.10.4 : \( -8\pi \)

16.12 MODEL EXAMINATION QUESTIONS

16.12.3 : State and prove Stoke's theorem.
16.12.4 : Verify Gauss divergence Theorem for \( \mathbf{F} = (x^3 - yz) \mathbf{i} - 2x^2 \mathbf{y} + z \mathbf{k} \), where \( S \) is the surface bounded by the Co-ordinate planes and \( x = y = z = a \).
16.12.5: Verify Green’s Theorem for \[ \oint_C \left( x^2 + y^2 \right) dx + x^2 dy. \]

16.12.6: Verify Stoke’s Theorem for \( \mathbf{F} = -y^3 \mathbf{i} + x^3 \mathbf{j} \) where \( S \) is the circular disc \( x^2 + y^2 \leq 1, z = 0. \)

16.13 REFERENCES


Lesson Writer,

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Experiment - 1

EXACT DIFFERENTIAL EQUATIONS

Aim: To solve a differential equation which is either exact or can be reduced to exact form.

Definitions: A differential equation \( M \, dx + N \, dy = 0 \)

1. where \( M \) and \( N \) are functions of \( x, y \) is said to be exact if there exists a function \( u \) of \( x, y \) such that \( M \, dx + N \, dy = du \), where \( du = \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy \).

2. If the equation is not exact and becomes exact when it is multiplied by a suitable function \( f(x, y) \), then the equation is said to be reducible to exact form, \( f(x, y) \) is called an Integrating factor (IF).

Results used:

1. An equation of the form \( M \, dx + N \, dy = 0 \) is exact iff \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \).

2. If the equation of the form \( M \, dx + N \, dy = 0 \) is not exact, \( M \, dx + N \, dy = 0 \) is a homogeneous differential equation, \( Mx + Ny \neq 0 \), then the integrating factor is \( \frac{1}{Mx + Ny} \).

3. If the equation of the form \( yf(xy) \, dx + xg(xy) \, dy = 0 \) is not exact, \( Mx - Ny \neq 0 \) then the integrating factor is \( \frac{1}{Mx - Ny} \).

4. If the equation of the form \( M \, dx + N \, dy = 0 \) is not exact and \( f(x) = \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \) is a function of \( x \) only. Then the integrating factor is \( e^{\int f(x) \, dx} \).

5. If the equation of the form \( M \, dx + N \, dy = 0 \) is not exact and \( g(y) = \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \) is a function of \( y \) only. Then the integrating factor is \( e^{\int g(y) \, dy} \).

Procedure: Take the given differential equation of the form \( M \, dx + N \, dy = 0 \) and verify \( \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x} \) are equal or not.
Step 1 :

(1) If \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \), then compute the general solution by the rule, \( \int M \, dx + N \, dy \), where \( v = \int M \, dx \).

(2) If \( \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \), then find a suitable integrating factor \( f(x, y) \).

(3) Multiply given equation with \( f(x, y) \), then given equation takes the form \( M_1 \, dx + N_1 \, dy = 0 \), where \( M_1 = Mf(x, y) \), \( N_1 = Nf(x, y) \).

(4) Verify whether \( \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \).

(5) Write general solution using Step 1 with \( M_1 \) in place of \( M \) and \( N_1 \) in place of \( N \).

Example 1 : Solve \( (x^2y - 2xy^2) \, dx - (x^3 - 3x^2y) \, dy = 0 \)

Solution :

Here \( M = x^2y - 2xy^2 \) \hspace{1cm} \( N = -x^3 + 3x^2y \)

\[ \Rightarrow \frac{\partial M}{\partial y} = x^2 - 4xy, \Rightarrow \frac{\partial N}{\partial x} = -3x^2 + 6xy \]

\[ \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \]

Since given equation is homogeneous integrating factor is \( \frac{1}{Mx + Ny} \).

\[ \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)x + (-x^3 + 3x^2y)y} = \frac{1}{x^2y^2} \]

Multiplying given equation with \( \frac{1}{x^2y^2} \) we get

\[ \left( \frac{x^2y - 2xy^2}{x^2y^2} \right) \, dx + \left( \frac{-x^3 + 3x^2y}{x^2y^2} \right) \, dy = 0 \]
Exact Differential Equations

1.3

Differential Equations

Here \( M_1 = \frac{x^2 y - 2xy^2}{x^2 y^2} \)

\[ \frac{1}{y} - \frac{2}{x} \]

\( N_1 = \frac{-x^3 + 3x^2 y}{x^2 y^2} \)

\[ \frac{-x}{y^2} + \frac{3}{y} \]

\( \therefore \frac{\partial M_1}{\partial y} = \frac{1}{y^2} \)

\( \frac{\partial N_1}{\partial x} = \frac{1}{y^2} \)

\( \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \)

Now given equation is reduced to exact form.

Now \( V = \int M_1 \, dx = \int \left( \frac{1}{y} - \frac{2}{x} \right) \, dx = \frac{x}{y} - 2 \log x \)

\( y \) constant

\( \frac{\partial V}{\partial y} = \frac{-x}{y^2} \)

\( N_1 - \frac{\partial V}{\partial y} = -\frac{x}{y^2} + \frac{3}{y} + \frac{x}{y^2} = \frac{3}{y} \)

Now \( \int \left( N_1 - \frac{\partial V}{\partial y} \right) \, dy = \int \frac{3}{y} \, dy = 3 \log |y| \)

\( \therefore \) General solution is \( \frac{x}{y} - 2 \log x + 3 \log |y| = C \)

Example 2 : Solve \( y (xy + 2x^2 y^2) \, dx + x (xy - x^2 y^2) \, dy = 0 \)

Solution : Here \( M = y (xy + 2x^2 y^2) \)

\( = xy^2 + 2x^2 y^3 \)

\( N = x (xy - x^2 y^2) \)

\( = x^2 y - x^3 y^2 \)

\( \therefore \frac{\partial M}{\partial y} = 2xy + 6x^2 y^2 \)

\( \frac{\partial N}{\partial x} = 2xy - 3x^2 y^2 \)
\[
\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}
\]

Integrating factor = \( \frac{1}{Mx - Ny} \)

I.F. = \( \frac{1}{Mx - Ny} = \frac{1}{(xy^2 + 2x^2y^3)x - (x^2y - x^3y^2)y} \)

= \( \frac{1}{x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3} = \frac{1}{3x^3y^3} \)

Multiplying given equation with \( \frac{1}{3x^3y^3} \), we get

\[
\left( \frac{xy^2 + 2x^2y^3}{3x^3y^3} \right)dx + \left( \frac{x^2y - x^3y^2}{3x^3y^3} \right)dy = 0
\]

Here \( M_1 = \frac{1}{3x^2y} + \frac{2}{3x} \)

\( N_1 = \frac{1}{3xy^2} - \frac{1}{3y} \)

\[\Rightarrow \frac{\partial M_1}{\partial y} = -\frac{1}{3x^2y^2} \quad \Rightarrow \frac{\partial N_1}{\partial x} = -\frac{1}{3x^2y^2}\]

\[\Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \]

Now the equation is reduced to exact form.

Now \( V = \int M_1 \, dx \) = \( \int \left( \frac{1}{3x^2y} + \frac{2}{3x} \right) \, dx = -\frac{1}{3xy} + \frac{2}{3} \log|x| \)

\[\therefore \frac{\partial V}{\partial y} = \frac{1}{3xy^2}\]

\[N_1 - \frac{\partial V}{\partial y} = \frac{1}{3xy^2} - \frac{1}{3y} - \frac{1}{3xy^2} = -\frac{1}{3y}\]
Now \[ \int \left( N_1 - \frac{\partial V}{\partial y} \right) dy = \int \frac{-1}{3y} dy = -\frac{1}{3} \log |y| \]

General solution is \[ \frac{-1}{3xy} + \frac{2}{3} \log |x| - \frac{1}{3} \log |y| = C \]

**Example 3**: Solve \((3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0\)

**Solution**: Here \(M = 3xy - 2ay^2\) \(N = x^2 - 2axy\)

\[ \Rightarrow \frac{\partial M}{\partial y} = 3x - 4ay \quad \Rightarrow \frac{\partial N}{\partial x} = 2x - 2ay \]

\[ \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \]

\[ \text{Now } \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{x^2 - 2axy} 
(3x - 4ay - 2x + 2ay) = \frac{x - 2ay}{x(x - 2ay)} = \frac{1}{x} = f(x) \]

By result 4 Integrating factor \(e^{\int f(x)dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x\)

Multiplying given equation with \(x\) we get

\((3x^2y - 2axy^2)dx + (x^3 - 2ax^2y)dy = 0\)

Here \(M_1 = 3x^2y - 2axy^2\) \(N_1 = x^3 - 2ax^2y\)

\[ \frac{\partial M_1}{\partial y} = 3x^2 - 4axy \quad \frac{\partial N_1}{\partial x} = 3x^2 - 4axy \]

\[ \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \]

Now the given equation is reduced to exact form.

Now, \(V = \int M_1 dx = \int \left(3x^2y - 2axy^2\right) dx \)

\[ = x^3y - ax^2y^2 \]
\[ \frac{\partial V}{\partial y} = x^3 - 2ax^2y \]

\[ \therefore N_1 - \frac{\partial V}{\partial y} = (x^3 - 2ax^2y) - (x^3 - 2ax^2y) = 0 \]

Now, \[ \left( N_1 - \frac{\partial V}{\partial y} \right) dy = f(0) dy = 0 \]

General solution is \( x^3y - ax^2y^2 = C \).

**Example 4:** Solve \( (xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0 \)

**Solution:** Here \( M = xy^3 + y \) \( N = 2x^2y^2 + 2x + 2y^4 \)

\[ \Rightarrow \frac{\partial M}{\partial y} = 3xy^2 + 1 \quad \frac{\partial N}{\partial x} = 4xy^2 + 2 \]

\[ \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \text{ given equation is not exact.} \]

Now, \[ \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{xy^3 + y} \left( 4xy^2 + 2 - 3xy^2 - 1 \right) \]

\[ = \frac{1}{y(x^2 + 1)} \left( y^2 + 1 \right) = \frac{1}{y} = g(y) \]

By result (5) Integrating factor \( = e^{\int g(y)dy} \)

\[ = e^{\int \frac{1}{y} dy} = e^{\log y} = y \]

Multiplying given equation with \( y \) we get \( (xy^4 + y^2)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0 \)

Here \( M_1 = xy^4 + y^2 \) \( N_1 = 2x^2y^3 + 2xy + 2y^5 \)

\[ \Rightarrow \frac{\partial M_1}{\partial y} = 4xy^3 + 2y, \quad \frac{\partial N_1}{\partial x} = 4xy^3 + 2y \]

\[ \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \]

\[ \therefore \text{Now the equation is exact form.} \]
Now \( V = \int M_1 \, dx \) keeping \( y \) constant

\[
\begin{align*}
\Rightarrow \frac{\partial V}{\partial y} &= 2x^2y^3 + 2xy \\
\therefore N_1 - \frac{\partial V}{\partial y} &= 2x^2y^3 + 2xy + 2y^5 - 2x^2y^3 - 2xy = 2y^5
\end{align*}
\]

Now, \( \int \left( N_1 - \frac{\partial V}{\partial y} \right) \, dy = \int 2y^5 \, dy = \frac{y^6}{3} \)

General solution is \( \frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = C \)

**Suggested Problems:**

1. Solve \( y^2 \, dx + \left( x^2 - xy - y^2 \right) \, dy = 0 \)

2. Solve \( x^2 \, y \, dx - \left( x^3 + y^3 \right) \, dy = 0 \)

3. Solve \( \left( x^2y^2 + xy + 1 \right) \, y \, dx + \left( x^2y^2 - xy + 1 \right) \, x \, dy = 0 \)

4. Solve \( (xy \sin xy + \cos xy) \, y \, dx + (xy \sin xy - \cos xy) \, x \, dy = 0 \)

5. Solve \( 2xy \, dy - \left( x^2 + y^2 + 1 \right) \, dx = 0 \)

6. Solve \( \left( x^2 + y^2 + x \right) \, dx + xy \, dy = 0 \)

7. Solve \( \left( y^4 + 2y \right) \, dx + \left( xy^3 + 2y^4 - 4x \right) \, dy = 0 \)

8. Solve \( \left( x + y^2 \right) \, dx + xy \, dy = 0 \)
Experiment - II

LINEAR DIFFERENTIAL EQUATIONS

AIM: To solve a differential equation which is either in Linear form or can be reduced to linear form.

DEFINITIONS:

(1) A differential equation of the form \(\frac{dy}{dx} + P(x)y = Q(x)\), where \(P(x)\) and \(Q(x)\) are functions of \(x\), is called a linear differential equation of first order in \(y\).

(2) A differential equation of the form \(\frac{dx}{dy} + P(y)x = Q(y)\), where \(P(x)\) and \(Q(y)\) are functions of \(y\) is called a linear differential equation of first order in \(x\).

(3) An equation of the form \(n\frac{dy}{dx} + Py = Qy^n\) where \(P\) and \(Q\) are functions of \(x\) only is called Bernoulli's equation.

RESULTS USED:

(1) \(\frac{dy}{dx} + P \cdot y = Q\) is a linear differential equation. Its general solution is \(y e^{\int P \, dx} = \int Q e^{\int P \, dx} \, dx + c\)

(2) \(\frac{dx}{dy} + P \cdot x = Q\) where \(P\) and \(Q\) are functions of \(y\) is a linear differential equation it's general solution \(x \cdot e^{\int P \, dy} = \int Q \cdot e^{\int P \, dy} \, dy + c\)

PROCEDURE:

Step I: Reduce the given equation to standard form

\(\frac{dy}{dx} + P(x)y = Q(x)\) or \(\frac{dx}{dy} + P(y)x = Q(y)\)

Step II: Find the Integrating factor \(e^{\int P \, dx}\) or \(e^{\int P \, dy}\) as the case may be.

Step III: General solution is \(y \cdot e^{\int P \, dx} = \int Q \cdot e^{\int P \, dx} \, dx + c\) or \(x \cdot e^{\int P \, dy} = \int Q e^{\int P \, dy} \, dy + c\) as the case may be.
Example 1: Solve \( \frac{3}{x^3} \frac{dy}{dx} + (2 - 3x^2) y = x^3 \)

Solution: Given differential equation is \( \frac{3}{x^3} \frac{dy}{dx} + (2 - 3x^2) y = x^3 \), Dividing by \( x^3 \) we get

\[
\frac{dy}{dx} + \left( -\frac{2 - 3x^2}{x^3} \right) y = 1
\]

Here \( P = -\frac{2 - 3x^2}{x^3} \), \( Q = 1 \)

\[
P = \frac{2}{x^3} - \frac{3}{x} \]

Now, \( e^{\int P \, dx} = e^{\int \frac{2}{x^3} \, dx} \)

\[
e^{\frac{2}{x^3} \, \frac{-1}{2x^2} \, \log x}
\]

\[
e^{-\frac{1}{x}} \, \log x
\]

\[
e^{-\frac{1}{x}} \, \log x^3
\]

\[
e^{-\frac{1}{x}} \, \log x^{-3}
\]

\[
e^{-\frac{1}{x^2}} \cdot e^{\log x^{-3}}
\]

\[
e^{-\frac{1}{x^2}} \cdot \frac{1}{x^3}
\]

General solution is \( y \cdot e^{\int P \, dx} = \int Q e^{\int P \, dx} \, dx \)

\[
= \frac{1}{2} \int e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} \, dx
\]
Linear Differential Equations

2.3

Differential Equations

\[ \frac{1}{2} e^{\frac{x}{3}} + C \left[ \text{Put} \quad \frac{1}{x^2} = t \Rightarrow \frac{2}{x^3} \frac{dx}{dt} = dt \right] \]

**Example 2:** Solve \( (x + 2y^3) \frac{dy}{dx} = y \)

Given differential equation is \( (x + 2y^3) \frac{dy}{dx} = y \)

\[ \Rightarrow x + 2y^3 = y \frac{dx}{dy} \]

\[ \Rightarrow y \frac{dx}{dy} - x = 2y^3 \]

\[ \Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2 \]

This is a linear differential equation in \( x \).

Here \( P = -\frac{1}{y} \) \quad \( Q = 2y^2 \)

\[ \therefore e^{\int P \, dy} = e^{\int \frac{1}{y} \, dy} = e^{-\log y} = e^{\log \frac{1}{y}} = \frac{1}{y} \]

General solution of the given differential equation is

\[ x \cdot e^{\int P \, dy} = \int Q \cdot e^{\int P \, dy} \, dy \]

\[ \Rightarrow x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y} \, dy \]

\[ = y^2 + c \]

General solution is \( \frac{x}{y} = y^2 + c \) or \( x = y^3 + cy \)
Example 3 : Solve \( \frac{dy}{dx} + \frac{y}{x-1} = xy^3 \)

Solution :

Given equation is
\[
\frac{dy}{dx} + \frac{y}{x-1} = xy^3
\]

\[
\Rightarrow \frac{1}{y^3} \frac{dy}{dx} + \frac{2}{x-1} = x \quad \text{-------- (1)}
\]

Put \( \frac{2}{3} y^3 = u \quad \text{------- (2)} \)

Differentiating with respect to \( x \).  

\[
\frac{2}{3} y^3 \frac{dy}{dx} = \frac{du}{dx}
\]

\[
\Rightarrow \frac{1}{y^3} \frac{dy}{dx} = \frac{3}{2} \frac{du}{dx} \quad \text{------- (3)}
\]

Substituting (2) and (3) in (1) we get

\[
\frac{3}{2} \frac{du}{dx} + \frac{u}{x-1} = x
\]

\[
\Rightarrow \frac{du}{dx} + \frac{2}{3} \frac{u}{x-1} = \frac{2}{3} x
\]

This is a linear differential equation in \( u \)

Here \( P = \frac{2}{3(x-1)} \) \quad \text{and} \quad \text{Q} = \frac{2}{3} x 

\[
e^{\int Pdx} = e^{\int \frac{2}{3(x-1)} dx} = e^{\frac{2}{3} \log|x-1|} = e^{\log(x-1)^{\frac{2}{3}}}
\]

\[
= (x-1)^{\frac{2}{3}}
\]
General solution is \( u \cdot (x - 1)^{\frac{2}{3}} = \int \frac{2}{3}x(x - 1)^{\frac{2}{3}} \, dx \)

\[
= \frac{2}{3} x \int (x - 1)^{\frac{2}{3}} \, dx - \int \left( \frac{2}{3} \int (x - 1)^{\frac{2}{3}} \, dx \right) \, dx + c
\]

\[
= \frac{2}{3} x \left( \frac{5}{3} \right) - \frac{2}{3} \left( \frac{5}{3} \right)^{\frac{5}{3}} - c
\]

\[
= \frac{2}{5} x(x - 1)^{\frac{5}{3}} - \frac{2}{5} \left( \frac{8}{3} \right)^{\frac{8}{3}} + C
\]

\[
= \frac{2}{5} x(x - 1)^{\frac{5}{3}} - \frac{3}{20} (x - 1)^{\frac{8}{3}} + C
\]

\[\therefore \text{ General solution is} \]

\[
y^{\frac{2}{3}} (x - 1)^{\frac{2}{3}} = \frac{2}{5} x(x - 1)^{\frac{5}{3}} - \frac{3}{20} (x - 1)^{\frac{8}{3}} + C
\]

**Example 4 :** Solve \( \frac{dy}{dx} - y \tan x = \frac{\sin x \cos^2 x}{y^2} \)

**Solution :** Given differential equation is \( \frac{dy}{dx} - y \tan x = \frac{\sin x \cos^2 x}{y^2} \) \( \text{--------- (1)} \)

This is in Bernoulli's form

Multiplying with \( y^2 \), we get

\[
y^2 \frac{dy}{dx} - y^3 \tan x = \sin x \cos^2 x \text{ \( \text{--------- (1)} \)}
\]

Put \( y^3 = u \text{ \( \text{--------- (2)} \) }

Differentiating with respect to \( x \).
\[ 3y^2 \frac{dy}{dx} = \frac{du}{dx} \]

\[ \Rightarrow y^2 \frac{dy}{dx} = \frac{1}{3} \frac{du}{dx} \quad \text{(3)} \]

Substitute (3) and (4) in (1) we get

\[ \frac{1}{3} \frac{du}{dx} - u \tan x = \sin x \cos^2 x \]

\[ \Rightarrow \frac{du}{dx} - 3u \tan x = 3 \sin x \cos^2 x \]

This is a linear differential equation in \( u \)

Here \( P = -3 \tan x \quad Q = 3 \sin x \cos^2 x \)

\[ e^{\int P \, dx} = e^{\int -3 \tan x \, dx} = e^{-3 \log \cos x} = e^{\log \frac{1}{\sec^3 x}} = \frac{1}{\sec^3 x} = \cos^3 x \]

General solution \( u \cdot \cos^3 x = \int 3 \sin x \cos^2 x \cdot \cos^3 x \, dx \)

\[ \Rightarrow y^3 \cos^3 x = \int 3 \sin x \cos^5 x \, dx \]

\[ = -\int 3 \cos^5 x \sin x \, dx \]

\[ = -\frac{3 \cos^6 x}{6} + c \quad [\text{Put } \cos x = t \Rightarrow \sin x \, dx = dt] \]

\[ = \frac{\cos^6 x}{2} + c \]

General solution is \( y^3 \cos^3 x = \frac{\cos^6 x}{2} + c \)

**SUGGESTED PROBLEMS**:

(1) Solve \( x(x-1) \frac{dy}{dx} - (x-2)y = x^3(2x-1) \)
(2) Solve \( x(1-x^2) \frac{dy}{dx} + (2x^2 - 1)y = ax^2 \)

(3) Solve \( \left( y - e^{\sin^{-1}x} \right) \frac{dx}{dy} + \sqrt{1-x^2} = 0, \ |x| < 1 \)

(4) Solve \( (1+y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0 \)

(5) Solve \( \frac{dy}{dx} + \frac{2}{x} y = 3x^2 y^{-\frac{4}{3}} \)

(6) Solve \( \frac{1}{y} \frac{dy}{dx} + \frac{x}{1-x^2} = xy^{-\frac{1}{2}} \)

(7) Solve \( \frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^y \sec y \)

(8) Solve \( \frac{dy}{dx} = e^{x-y} \left( e^x - e^y \right) \)
Experiment - III

TOTAL DIFFERENTIAL EQUATIONS

Aim: To solve a system of total differential equations.

Definitions:

(1) An equation of the form \( Pdx + Qdy + Rdz = 0 \), where \( P, Q, R \) are functions of \( x, y, z \) is called a total differential equation.

(2) System of equations of the form \( \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \), where \( P, Q, R \) are functions of \( x, y, z \) are called simultaneous total differential equations.

Procedure - A: When the given system of equations is of the form \( \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \) -------- (1)

Step (1): Consider the three sets of equations

\[
\begin{align*}
\frac{dx}{P} &= \frac{dy}{Q} = \frac{dz}{R} \\
\end{align*}
\] ------------ (2)

Step (2): If any two equations of the set (2) are integrable, find their solutions by separating the variables. The pair of such solutions gives the general solution of the given system of equations (1)

Step (3): If one equation only of the set (2) is integrable, find its general solution by separating the variables use their solution to find the solution of another set. The pair of these solutions gives the complete solution (general solution) of the given system of equation (1).

Procedure - B: Methods of Multipliers

If no equation of set (2) is integrable, write

\[
\begin{align*}
\frac{dx}{P} &= \frac{dy}{Q} = \frac{dz}{R} = \frac{\ell_1 dx + m_1 dy + n_1 dz}{\ell_1 P + m_1 Q + n_1 R} = \frac{\ell_2 dx + m_2 dy + n_2 dz}{\ell_2 P + m_2 Q + n_2 R}
\end{align*}
\]

(\( \ell_1, m_1, n_1 \) and \( \ell_2, m_2, n_2 \) are called multipliers).

Step (4): Choose \( \ell_1, m_1, n_1 \) and \( \ell_2, m_2, n_2 \) such that \( \ell_1 P + m_1 Q + n_1 R = 0 \) and \( \ell_2 P + m_2 Q + n_2 R = 0 \) then we have \( \ell_1 dx + m_1 dy + n_1 dz = 0 \) and \( \ell_2 dx + m_2 dy + n_2 dz = 0 \) which on integration gives two equations.

These equations together give the complete solution of the system (1).
Step (5) : If we choose $\ell_1, m_1, n_1$ and $\ell_2, m_2, n_2$ such that

$$\ell_1P + m_1Q + n_1R \neq 0, \quad \frac{\ell_1dx + m_1dy + n_1dz}{\ell_1P + m_1Q + n_1R} = d\phi$$

and

$$\ell_2P + m_2Q + n_2R \neq 0, \quad \frac{\ell_2dx + m_2dy + n_2dz}{\ell_1P + m_1Q + n_1R} = d\psi$$

then $\phi(x, y, z) = C_1, \psi(x, y, z) = C_2$ will give the complete solution of the system of equations (1).

Example 1 : Solve the system of equations

$$\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{y^2}$$

Taking

$$\frac{xdx}{y^2z} = \frac{dy}{x^2z}$$

$$\Rightarrow x^3dx = y^3dy$$

Integrating

$$\int x^3dx = \int y^3dy \Rightarrow \frac{x^4}{4} = \frac{y^3}{3} + C_1 \Rightarrow 3x^4 - 4y^3 = C_1 \quad -------- (1)$$

Also taking

$$\frac{xdx}{y^2z} = \frac{dz}{y^2}$$

$$\Rightarrow xdx = zdz$$

Integrating

$$\int xdx = \int zdz \Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + C_2 \Rightarrow x^2 - z^2 = C_2 \quad -------- (2)$$

\therefore General solution of the given system of equations is given by (1) & (2).

i.e. $3x^4 - 4y^3 = C_1; \quad x^2 - z^2 = C_2$, $C_1, C_2$ are arbitrary constants.

Example 2 : Solve the system of equations

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y + 2x)}$$

Taking

$$\frac{dx}{1} = \frac{dy}{-2}$$
Integrating
\[-2\int \, dx = \int \, dy \Rightarrow y + 2x = C_1 \quad \text{-------- (1)}\]

Also taking
\[\frac{dx}{1} = \frac{dz}{3x^2 \sin(y + 2x)}\]

\[\Rightarrow dx = \frac{dz}{3x^2 \sin C_1} \quad \text{(using (1))} \Rightarrow 3x^2 \sin C_1 \, dx = dz\]

Integrating
\[3\sin C_1 \int x^2 \, dx = \int dz \Rightarrow 3\sin C_1 \frac{x^3}{3} = z + C_2\]

\[\Rightarrow x^3 \sin(y + 2x) - z = C_2 \quad \text{-------- (2)}\]

\[\therefore \text{complete solution of the given system is given by (1) \& (2)}\]

i.e. \[y + 2x = C_1, \quad x^3 \sin(y + 2x) - z = C_2\]

**Example 3** : Solve the system of equations \(\frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)}\) using \((1, 1, 1)\) as multipliers

Each fraction = \(\frac{dx + dy + dz}{x(y - z) + y(z - x) + z(x - y)}\)

\[= \frac{dx + dy + dz}{0}\]

\[\Rightarrow dx + dy + dz = 0\]

Integrating we get
\[x + y + z = C_1 \quad \text{-------- (1)}\]

Again using \(\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)\) as multipliers, then
each fraction = \( \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{\frac{1}{x} (y-z) + \frac{1}{y} (z-x) + \frac{1}{z} (x-y)} \)

\( \Rightarrow \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0 \)

Integrating

\[ \int \frac{1}{x} dx + \int \frac{1}{y} dy + \int \frac{1}{z} dz = 0 \]

\( \Rightarrow \log x + \log y + \log z = \log C_2 \quad \Rightarrow \log xyz = \log C_2 \)

\( \Rightarrow xyz = C_2 \quad \text{(2)} \)

\( \therefore \) Complete solution of the given system is \( x + y + z = C_1, \ xyz = C_2 \)

Suggested Problems:

Solve the system of equations

1) \( \frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2} \)

2) \( \frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z} \)

3) \( \frac{dx}{-y^2 - z^2} = \frac{dy}{xy} = \frac{dz}{zx} \)

4) \( \frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{2x - 3y} \)

5) \( \frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \)

6) \( \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x + y)z} \)
Experiment - IV

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

Aim : To find general solution of differential equation of first order and higher degree.

Definitions :

(1) An equation of the form \( f(x, y, p) = 0 \) where \( p \) is not first degree \( \left( p = \frac{dy}{dx} \right) \) is called a differential equation of first order and of higher degree.

(2) The differential equation of the form \( y = px + \phi(p) \) is called Clairaut's equation.

Procedure : Consider the given differential equation \( f(x, y, p) = 0 \) ---------- (1)

Step (1) : If (1) can be expressed as a product of \( n \) linear factors in \( p \), then solve each factor by separating variables to get \( n \) solutions say \( F_1(x, y, c_1) = 0, F_2(x, y, c_2) = 0, \ldots, F_n(x, y, c_n) = 0 \). Then general solution of (1) is

\[
F_1(x, y, c_1) \cdot F_2(x, y, c_2) \cdot \ldots \cdot F_n(x, y, c_n) = 0.
\]

Where \( c_1 = c_2 = \ldots = c_n = c \) (say) is arbitrary constant.

Step (2) : If (1) cannot be resolved into linear factors in \( p \) and (1) is of the first degree in \( y \). i.e. (1) can be expressed in the form of \( y = F(x, p) \) then differentiate this with respect to 'x' to get an equation of the form \( P = \phi \left( x, P, \frac{dP}{dx} \right) \) which can be solved. We get a solution and finally eliminate \( P \) in this solution by using (1) we get general solution of (1).

Step (3) : If (1) cannot be resolved into linear factors in \( P \) and (1) is of the form \( x = F(y, P) \) then differentiate this with respect to 'y' to get an equation of the form \( \frac{1}{P} = \phi \left( y, P, \frac{dp}{dy} \right) \) which can be solved. We get a solution and finally eliminate \( \phi \) in this solution by using (1) to get general solution of (1).

Step (4) : If (1) can be written in the form \( y = Px + \phi(P) \) (Clairaut's form) then the general solution of (1) is obtained by replacing \( P \) by 'C' in (1) (C is arbitrary constant).

Note : The substitution \( x^2 = X, y^2 = Y \) may some times reduce the given equation to Clairaut's form.
Example 1: Solve \( x^2p^2 + 3xyp + 2y^2 = 0 \)

Solution: Given differential equation \( x^2p^2 + 3xyp + 2y^2 = 0 \) ------- (1)

\[
\Rightarrow x^2p^2 + 2xyp + xyp + 2y^2 = 0
\]

\[
\Rightarrow xp(xp + 2y) + y(xp + 2y) = 0
\]

\[
\Rightarrow (xp + y)(xp + 2y) = 0
\]

\[
\Rightarrow xp + y = 0 \quad \text{or} \quad xp + 2y = 0
\]

\[
\Rightarrow x \frac{dy}{dx} = -y \quad \Rightarrow x \frac{dy}{dx} + 2y = 0
\]

\[
\Rightarrow \frac{dy}{y} = - \frac{dx}{x}
\]

Integrating

\[
\int \frac{dy}{y} = - \int \frac{dx}{x}
\]

\[
\Rightarrow \log y + \log x = \log c
\]

\[
\Rightarrow xy = c
\]

or \( xy - c = 0 \)

\[
\Rightarrow \log y = -2 \log x + \log c
\]

\[
\Rightarrow yx^2 = c \quad \text{or} \quad x^2y - c = 0
\]

\[
\therefore \text{General solution of (1) is } (xy - c)(x^2y - c) = 0
\]

Example 2: Solve \( x^3p^2 + x^2yp + 4 = 0 \)

Solution: Given differential equation is \( x^3p^2 + x^2yp + 4 = 0 \) ------- (1)

\[
\Rightarrow y = \frac{-4 - x^3p^2}{x^2p} = \frac{-4}{x^2p} - xp
\]
Solving for $y$: Differentiating with respect to 'x'

\[
p = \frac{4}{(x^2 p)^2} \left[ 2xp + x^2 \frac{dp}{dx} \right] - \left[ p + x \frac{dp}{dx} \right]
\]

\[
\Rightarrow 2p + x \frac{dp}{dx} = \frac{4 \left( 2p + x \frac{dp}{dx} \right) x}{x^4 p^2}
\]

\[
\Rightarrow \left( 2p + x \frac{dp}{dx} \right) \left( 1 - \frac{4}{x^3 p^2} \right) = 0
\]

\[
\Rightarrow 2p + x \frac{dp}{dx} = 0
\]

\[
\Rightarrow \frac{dp}{p} + 2 \frac{dx}{x} = 0
\]

Integrating

\[
\int \left( \frac{dp}{p} + 2 \right) \frac{dx}{x} = \log c
\]

\[
\Rightarrow \log p + 2 \log x = \log c
\]

\[
\Rightarrow px^2 = c
\]

\[
\Rightarrow p = \frac{c}{x^2} \quad \text{-------- (2)}
\]

Eliminating $p$ from (1) and (2)

we get $x^3 \left( \frac{c^2}{x^4} \right) + x^2 y \left( \frac{c}{x^2} \right) + 4 = 0$

i.e. $c^2 + cxy + 4x = 0$ which is general solution of (1)

**Example (3):** Solve $y^2 \log y = xpy + p^2$

**Solution:** Given differential equation is $y^2 \log y = xpy + p^2 \quad \text{-------- (1)}$
Solving for $x$: Differentiating with respect to '$y' 
\[
\frac{1}{p} = \left[ \log y + \frac{1}{y} \right] p - y \log y \frac{dp}{dy} - \left[ y \frac{dp}{dy} - p \right] \frac{1}{y^2}
\]

Multiply with $p^2 y^2$
\[
p y^2 = (1 + \log y) p y^2 - y^3 \log y \frac{dp}{dy} - p^2 \left( y \frac{dp}{dy} - p \right)
\]
\[
\Rightarrow y \frac{dp}{dy} \left( y^2 \log y + p^2 \right) = p \left( y^2 \log y + p^2 \right)
\]
\[
\Rightarrow y \frac{dp}{dy} = p \quad [\text{discarded the factor } \left( y^2 \log y + p^2 \right)]
\]
\[
\Rightarrow \frac{dp}{dy} = \frac{dy}{y}
\]

Integrating
\[
\log p = \log y + \log c
\]
\[
\Rightarrow p = cy \quad -------- (2)
\]

Eliminating $p$ from (1) and (2)

We get $y^2 \log y = x (cy) y + (cy)^2$

i.e. $\log y - cx - c^2 = 0$ which is general solution of (1)

**Example 4**: Solve $x^2 (y - px) = yp^2$ by reducing into Clairaut's form

**Solution**: Given differential equation is $x^2 (y - px) = yp^2 \quad -------- (1)$

\[
\Rightarrow y - px = \frac{yp^2}{x^2}
\]
\[ y^2 - pxy = \left( \frac{yp}{x} \right)^2 \] 
\text{--------- (2) (multiply with 'y')} 

Put \( x^2 = X \) and \( y^2 = Y \) \( \implies 2x \, dx = dX \) \( \text{and} \) \( 2y \, dy = dY \)

\[
\frac{dy}{dx} = p \ \text{Let} \quad \frac{dY}{dX} = P \\
\therefore \frac{dY}{dX} = \frac{2y \, dy}{2x \, dx} \implies P = \frac{yp}{x} \\
\implies Px^2 = pxy \implies PX = pxy
\]

Now substituting these transformations in (2)

we get \( Y - PX = P^2 \)

\[ \implies Y = PX + P^2 \]

This is in the Clairaut's form

\[ \therefore \text{General solution of (1) is} \quad Y = CX + C^2 \]

\[ \implies y^2 = cx^2 + c^2 \]

c is arbitrary constant.

\textbf{Suggested Problems :}

\textbf{Solve the following}

1) \( xy(p^2 + 1) = (x^2 + y^2)p \)
2) \( p^3 + (2x - y^2)p^2 = 2xy^2p \)
3) \( px + y - p^2x^4 = 0 \)
4) \( xp^3 - 2yp^2 + 4x^2 = 0 \)
5) \( p^3 - 4xp + 8y^2 = 0 \)
6) \( 4xp^2 + 4yp - y^4 = 0 \)
7) \((px - y)(py + x) = 2p\) by reducing into Clairauts form.
8) \( y = 2px + p^2y\) by reducing into Clairauts form.
Experiment - V

NON-HOMOTENEOUS EQUATIONS - I

Aim: To solve a non-homogeneous linear differential equation \( f(D)y = Q(x) \) where \( Q(x) \) is a non-zero polynomial in \( x \).

Formulae:

1. \( \frac{1}{D} x^m = \frac{x^{m+1}}{m+1} \)
2. \( (1 + D)^{-1} = 1 - D + D^2 - D^3 + D^4 - \ldots \)
3. \( (1 - D)^{-1} = 1 + D + D^2 + D^3 + D^4 + \ldots \)
4. \( (1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \ldots \)
5. \( (1 - D)^{-2} = 1 + 2D + 3D^2 - 4D^3 + \ldots \)

Procedure:

Step (1): Write the auxilary equation \( f(m) = 0 \)
Step (2): Find the roots of \( f(m) = 0 \)
Step (3): Write the complementary function (C.F.)
Step (4): Express \( f(D) \) as \( f(D) = CD^m (1 + \psi(D))^n \)
Step (5): Expand \( (1 + \psi(D))^{-n} \) using the relevant formula.
Step (6): Apply the operator \( CD^{-m}(1 + \psi(D))^{-n} \) on each term of \( Q(x) \)
Step (7): Write the particular integral P.I. as the sum of the term obtained in step (6).
Step (8): Write general solution as \( y = C.F. + P.I. \)

Example (1): \( (D-1)y = x^3 \)

A.E. is \( m - 1 = 0 \Rightarrow m = 1 \)

\[ \therefore \text{C.F.} = y_c = c_1 e^x \]

\[ \text{P.I.} = y_p = \frac{1}{D-1} x^3 = \frac{-1}{1-D} x^3 = -(1-D)^{-1} x^3 \]
\[-(1 + D + D^2 + D^3 + D^4 + \cdots) x^3\]
\[-(x^3 + 3x^2 + 6x + 6)\]
\[\therefore \text{ General solution is } y = y_c + y_p\]
\[y = c_1 e^x - (x^3 + 3x^2 + 6x + 6)\]

**Example 2**: \((D^2 + D + 1)y = x^2\)

A.E. is \(m^2 + m + 1 = 0 \Rightarrow m = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\)

C.F. is \(y_c = e^{-\frac{x}{2}} \left[ c_1 \cos \left(\frac{\sqrt{3}}{2} x\right) + c_2 \sin \left(\frac{\sqrt{3}}{2} x\right) \right]\)

P.I. = \(y_p = \frac{1}{D^2 + D + 1} x^2\)
\[= \frac{1}{1 + (D^2 + D)} x^2\]
\[= \left[1 + (D^2 + D) \right]^{-1} x^2\]
\[= \left[1 - (D^2 + D) + (D^2 + D)^2 + \cdots \right] x^2\]
\[= x^2 - (2x + 2) + (D^4 + 2D^3 + D^2)(x^2)\]
\[= x^2 - (2x + 2) + 2 = x^2 - 2x\]

General solutions is \(y = y_c + y_p\)
\[y = e^{-\frac{x}{2}} \left[ c_1 \cos \left(\frac{\sqrt{3}}{2} x\right) + c_2 \sin \left(\frac{\sqrt{3}}{2} x\right) \right] + x^2 - 2x\]

**Example 3**: \((D^4 - 2D^3 + D^2)y = 3x\)

A.E. is \(m^4 - 2m^3 + m^2 = 0\)
\[m^2 (m^3 - 2m + 1) = 0\]
\[m^2 (m - 1)^2 = 0 \Rightarrow m = 0, 0, 1, 1\]
C.F. \( y_c = c_1 + c_2 x + (c_3 + c_4 x) e^x \)

P.I. \( y_p = \frac{1}{D^4 - 2D^3 + D^2} 3x \)

\[ = 3 \cdot \frac{1}{D^2} (1 - D)^{-2} (x) \]

\[ = 3 \cdot \frac{1}{D^2} [1 + 2D + 3D^2 + \cdots] (x) \]

\[ = 3 \cdot \frac{1}{D^2} (x + 2) \]

\[ = 3 \cdot \frac{1}{D} \left( \frac{x^2}{2} + \frac{2x^2}{2} \right) \]

\[ = \frac{x^3}{2} + 3x^2 \]

\[ \therefore \text{ General solution is } y = y_c + y_p \]

\[ y = c_1 + c_2 x + (c_3 + c_4 x) e^x + \frac{x^3}{2} + 3x^2 \]

Conclusion: The general solution of the given problems are

(i) \( y = c_1 e^x \left( x^3 + 3x^2 + 6x \right) \)

(ii) \( y = e^{-x} \left( c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) + x^2 - 2x \)

(iii) \( y = c_1 + c_2 x + (c_3 + c_4 x) e^x + \frac{x^2}{2} + 3x^2 \)

Suggested Problems for the Students:

(i) Solve \( \left( D^3 + 2D^2 + D \right) y = x^2 + x \)

(ii) Solve \( (D - 1)^2 y = x \)

(iii) Solve \( \left( D^3 - 2D + 4 \right) y = x^4 + 3x^2 - 5x + 2 \)
Experiment - VI

NON-HOMOGENEOUS EQUATIONS - II

Aim : To solve the non-homogeneous linear differential equation with constant coefficients

\[ f(D)y = Q_1(x) + Q_2(x) + \cdots + Q_k(x). \]

Where \( Q_j(x) = b_j e^{a_j x} \) by means of polynomial operators.

Formula : 

\[ \frac{1}{f(D)} b e^{ax} = \frac{b e^{ax}}{f(a)} \text{ if } f(a) \neq 0 \]

\[ \frac{1}{(D-a)^r} b e^{ax} = \frac{b x^r e^{ax}}{r!} \]

Procedure :

Step (1) : Write A.E. \( f(m) = 0 \) and find the roots of \( f(m) = 0 \)

Step (2) : Write the C.F.

Step (3) : For each \( j, 1 \leq j \leq k \) we find the particular integral of \( f(D)y = Q_j(x) \) as follows.

Step (4) : If \( f(a_j) \neq 0 \) then write the P.I. as

\[ P \cdot I_j = \frac{b e^{a_j x}}{f(a_j)} \text{ for } j = 1, 2, \ldots, k \]

Step (5) : If \( f(a_j) = 0 \) then express \( f(D) \) as \( f(D) = (D-a_j)^r \psi(D) \) where \( \psi(a_j) \neq 0 \).

Then \( P \cdot I_j = b \frac{x^r e^{a_j x}}{r! \psi(a_j)} \) for \( j = 1, 2, \ldots, k \).

Step (6) : Write P.I. of \( f(D) = Q(x) \) as

\[ P \cdot I = P \cdot I_1 + P \cdot I_2 + \cdots + P \cdot I_k \]

Step (7) : Write general solution as

\[ y = \text{C.F.} + P \cdot I \]
Example 1: Solve \( \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = e^x \)

Solution: The given equation in operator form is \( (D^2 - 2D + 1)y = e^x \)

A.E. is \( m^2 - 2m + 1 = 0 \Rightarrow (m - 1)^2 = 0 \Rightarrow m = 1, 1 \)

C.F. is \( y_c = (c_1 + c_2x)e^x \)

\( P.I. = y_p = \frac{1}{D^2 - 2D + 1} e^x = \frac{1}{(D-1)^2} e^x = \frac{x^2}{2!} e^x \)

\( \therefore \) General solution is

\( y = y_c + y_p \)

\( y = (c_1 + c_2x)e^x + \frac{x^2}{2!} e^x \)

Example 2: Solve \( (D^3 - 5D^2 + 7D - 3)y = e^{2x} \cosh x \)

Solution: A.E. is \( m^3 - 5m^2 + 7m - 3 = 0 \)

\( \Rightarrow (m-1)(m^2 - 4m + 3) = 0 \)

\( \Rightarrow (m-1)(m-1)(m-3) = 0 \)

\( \Rightarrow m = 1, 1, 3 \)

C.F is \( y_c = (c_1 + c_2x)e^x + c_3e^{3x} \)

\( P.I. = y_p = \frac{1}{D^3 - 5D^2 + 7D - 3} e^{2x} \cosh x \)

\( = \frac{1}{(D-1)^2(D-1)} e^{2x} \left( \frac{e^x + e^{-x}}{2} \right) \)

\( = \frac{1}{(D-1)^2(D-3)} \left( \frac{1}{2} e^{3x} + \frac{1}{(D-1)^2(D-3)} \frac{1}{2} e^x \right) \)
Non-Homogeneous Equations - II

6.3 Differential Equations

\[
\begin{align*}
\frac{1}{2} \left\{ \frac{1}{D - 3} e^{3x} \right\} + \frac{1}{2} \left\{ \frac{1}{(D - 1)^2} e^x \right\}
&= \frac{1}{8!} e^{3x} - \frac{1}{4!} e^x \\
&= \frac{xe^{3x}}{8} - \frac{x^2 e^x}{8}
\end{align*}
\]

\[
\Rightarrow \text{ General solution is}
\]

\[
y = y_c + y_p
\]

\[
y = \left( c_1 + c_2 x \right) e^x + c_3 e^{3x} + \frac{xe^{3x}}{8} - \frac{x^2 e^x}{8}
\]

**Example 3** : Solve \( (4D^2 + 16D + 15) y = 4e^{\frac{3}{2}x} \) and \( Dy = -\frac{11}{2}, y = 3 \), where \( x = 0 \).

**Solution** : A.E. is \( 4m^2 + 16m + 15 = 0 \)

\[
\Rightarrow (2m + 3)(2m + 5) = 0
\]

\[
\Rightarrow m = -\frac{3}{2}, -\frac{5}{2}
\]

C.F. is \( y_c = c_1 e^{\frac{-3x}{2}} + c_2 e^{\frac{-5x}{2}} \)

P.I. \[
\frac{1}{4D^2 + 16D + 15} 4 e^{\frac{3x}{2}}
\]

\[
= \frac{4}{4(2D + 3)(2D + 5)} e^{\frac{3x}{2}}
\]

\[
= \frac{e^{-\frac{x}{2}}}{\frac{3}{2} + \frac{5}{2}} \left\{ \frac{1}{D + \frac{3}{2}} - \frac{1}{D + \frac{5}{2}} \right\}
\]
Given when \( y = 3 \) when \( x = 0 \)

\[
\therefore 3 = c_1 + c_2 \quad \text{(2)}
\]

Dy = \(-\frac{3}{2} c_1 e^{-\frac{3}{2} x} - \frac{5}{2} c_2 e^{-\frac{5}{2} x} + e^{-\frac{3}{2} x} - \frac{3}{2} x e^{-\frac{5}{2} x}\)

Given \( Dy = -\frac{11}{2} \) when \( x = 0 \)

\[
\therefore -\frac{11}{2} = -\frac{3}{2} c_1 - \frac{5}{2} c_2 + 1
\]

i.e. \( 3c_1 + 5c_2 = 13 \quad \text{(3)} \)

Solving (2) & (3) we get

\[
c_1 = 1, \quad c_2 = 2
\]

\[
\therefore \text{ solution is}
\]

\[
y = e^{-\frac{3}{2} x} + 2e^{-\frac{5}{2} x} + x e^{-\frac{3}{2} x}
\]

\[
= (1 + x) e^{-\frac{3}{2} x} + 2e^{-\frac{5}{2} x}
\]

**Conclusion**: The general solution of the given problems are

1. \((c_1 + c_2 x)e^x + \frac{x^2}{2!}e^x\)

2. \((c_1 + c_2 x)e^x + c_3 e^{3x} + \frac{xe^{3x}}{2} - \frac{x^2 e^x}{8}\)

3. \((1 + x)e^{-\frac{3}{2} x} + 2e^{-\frac{5}{2} x}\)

**Suggested Problems for the Students**:

1. Solve \((D^3 - D)y = \cosh x\)

2. Solve \(D^2 y - 2Dy + y = 7e^x\)

3. Solve \((D^2 - 3D + 2)x = e^{2t}\), given that \( x = 0, \) when \( t = 0 \) and \( x = 2(1 + 2t) \) when \( t = \log_2 e^2 \).
Experiment - VII

NON-HOMOGENEOUS EQUATIONS - III

Aim : (A) To solve non-homogeneous linear differential equation of the form \( f(D)y = b\sin ax \) or \( b\cos ax \).

(B) To solve non-homogeneous linear differential equation of the form \( f(D)y = e^{ax} \cdot V \),
where \( V \) is a continuous function of \( x \).

(C) To solve non-homogeneous linear differential equation, of the form \( f(D)y = x \cdot V \)
where \( V \) is a continuous function of \( x \).

Formulae : For any linear differential operator \( \phi(D) \)

(i) \[
\frac{1}{\phi(D^2)} b\sin ax = \frac{b}{\phi(-a^2)} \text{ if } \phi(-a^2) \neq 0
\]

(ii) \[
\frac{1}{\phi(D^2)} b\cos ax = \frac{b}{\phi(-a^2)} \text{ if } \phi(-a^2) \neq 0
\]

(iii) \[
\frac{1}{D^2 + a^2} \sin ax = \frac{-x}{2a} \cos ax
\]

(iv) \[
\frac{1}{D^2 + a^2} \cos ax = \frac{-x}{2a} \sin ax
\]

(v) \[
\frac{1}{\phi(D)} (e^{ax} V) = e^{ax} \cdot \frac{1}{\phi(D + a)} V
\]

(vi) \[
\frac{1}{\phi(D)} (x \cdot V) = x \cdot \frac{1}{\phi(D)} V - \frac{\phi'(D)}{[\phi(D)]^2} V
\]

Procedure :

Step (1) : Find the C.F. of the given differential equation.

Step (2) : Evaluate the P.I's of (A), (B) and (C) as follows.

Procedure for (A) : Let \( Q(x) = b\sin ax \) or \( b\cos ax \).

(i) If \( f(D) \) is an even function of \( D \) then express \( f(D) \) as a function \( \phi(D^2) \) of \( D^2 \).
If \( (D^2 + a^2) \) is a repeated factor of \( \phi(D^2) \) which is repeated \( r \) times then express \( \phi(D^2) \) as

\[ \phi(D^2) = (D^2 + a^2)^r F(D^2) \quad \text{where} \quad F(-a^2) \neq 0 \]

Then evaluate particular integral using

\[ \text{P.I.} = \frac{1}{f(D)} Q(x) = \frac{1}{F(-a^2)} \frac{1}{(D^2 + a^2)^r} Q(x) \]

To evaluate \( \frac{1}{(D^2 + a^2)^r} Q(x) \) use formula (iii) or (iv) which ever is applicable repeatedly.

(ii) If \( f(D) \) contains odd powers of \( D \) then express \( f(D) \) as \( p + qD \) where \( p = \phi_1(D^2) \) and \( q = \phi_2(D^2) \) are even functions of \( D \). Then evaluate.

\[ \frac{1}{f(D)} Q(x) = \frac{(p - qD)}{p^2 + a^2q^2} Q(x) \]

**Procedure for B:**

(i) Let the given equation be \( f(D)y = e^{ax} \cdot V \)

(ii) To find \( \text{P.I.} = \frac{1}{f(D)} \left( e^{ax} \cdot V \right) \), shift \( e^{ax} \) outside and after replacing \( D \) by \( (D+a) \), operate

\[ V \text{ by } \frac{1}{f(D + a)} \]

(iii) \[ \text{P.I.} = \frac{1}{f(D)} \left( e^{ax} \cdot V \right) = e^{ax} \cdot \frac{1}{f(D + a)} V \]

**Procedure for C:**

(i) Let the given equation be \( f(D)y = x \cdot V \).

(ii) Differentiate \( f(D) \) with respect to \( D \) to obtain the derivative \( f'(D) \) of \( f(D) \).
(iii) Evaluate P.I. as

$$\text{P.I.} = \frac{1}{f(D)} (x \cdot V) = x \cdot \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V$$

**Step (3)**: Write the general solution as

$$y = C \cdot F + P \cdot I.$$ 

**Example 1**: Solve \((D^2 + 9)y = \cos^3 x\)

**Solution**: A.E. is \(m^2 + 9 = 0 \Rightarrow (m + 3i)(m - 3i) = 0\)

\[\Rightarrow m = \pm 3i\]

\[\therefore \text{C.F. } y_c = (c_1 \cos 3x + c_2 \sin 3x)\]

\[y_p = \frac{1}{D^2 + 9} \cos^3 x = \frac{1}{D^2 + 9} \left(\frac{\cos 3x + 3\cos x}{4}\right)\]

\[= \frac{1}{4} \cdot \begin{align*} x & \quad \sin 3x + \frac{3}{4} \cdot \frac{1}{-1 + 9} \cos x \\ & = \frac{x}{24} \cos 3x + \frac{3}{32} \cos x \end{align*}\]

\[\therefore \text{General solution of given Differential equation is}\]

\[y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{24} \cos 3x + \frac{3}{32} \cos x\]

**Example 2**: Solve \((D^4 + 3D^2 - 4)y = \cos^2 x - \cosh x \quad (1)\)

**Solution**: A.E. is \(m^4 + 3m^2 - 4 = 0 \Rightarrow (m^2 + 4)(m^2 - 1) = 0\)

\[\Rightarrow m^2 = -4 \quad \text{or} \quad m^2 = 1\]

\[\Rightarrow m = \pm 2i, \pm 1\]

\[\therefore \text{C.F. } y_c = c_1 \cos 2x + c_2 \sin 2x + c_3 e^x + c_4 e^{-x}\]
P.L.\(=y_p = \frac{1}{D^4+3D^2-4}\cos^2x - \cosh x = \frac{1}{(D^2+4)(D^2-1)}\left(\frac{1+\cos 2x}{2} - e^x + e^{-x}\right)\)

\[= \frac{1}{2}\left[\frac{1}{(D^2+4)(D^2-1)}e^{0x} + \frac{1}{(D^2+4)(D^2-1)}\cos 2x - \frac{1}{(D^2+4)(D^2-1)}e^x - \frac{1}{(D^2+4)(D^2-1)}e^{-x}\right]\]

Now, \(\frac{1}{(D^2+4)(D^2-1)}e^{0x} = \frac{1}{(0+4)(0-1)} = -\frac{1}{4}\) \(\ldots\ldots (2)\)

\(\frac{1}{(D^2+4)(D^2-1)}\cos 2x = \frac{1}{-4-1}\left(\frac{1}{D^2+2^2}\cos 2x\right) = -\frac{1}{5} \cdot \frac{x}{2} \cdot 2\cdot 2\)

\[= -\frac{x}{20} \sin 2x \ldots (3)\]

\(\frac{1}{(D^2+4)(D^2-1)}e^x = \frac{1}{(D^2+4)(D+1)(D-1)}e^x = \frac{1}{(1+9)(1+1)}\left(\frac{1}{D-1}e^x\right) = \frac{x}{10}e^x \ldots (4)\)

\(\frac{1}{(D^2+4)(D^2-1)}e^{-x} = \frac{1}{(D^2+4)(D-1)(D+1)}e^{-x} = \frac{1}{((-1)^2+4)(-1-1)} \frac{1}{D+1}e^{-x} = -\frac{x}{10}e^{-x} \ldots (5)\)

substituting (2), (3), (4) & (5) in \(y_p\) we get

\[y_p = \frac{1}{2}\left[\frac{1}{-4} \cdot \frac{x}{20} \sin 2x + \frac{x}{10}e^x - \frac{x}{10}e^{-x}\right]\]

\[= \frac{1}{40}\left[-5 - x \sin 2x + 2x (e^x - e^{-x})\right]\]

\[\because\ \text{General solution of (1) is}\]

\[y = y_c + y_p\]

\[= c_1 \cos 2x + c_2 \sin 2x + c_3e^x + c_4e^{-x} + \frac{1}{40}\left[-5 - x \sin 2x + 2x (e^x - e^{-x})\right]\]

**Example 3** : Solve \((D^2 - 2D + 1)y = x^2 e^{3x}\)

**Solution** : A.E. is \(m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1\)

\[\because\ \text{C.F.} = y_c = (c_1 + c_2 x)e^x\]
\[ \text{PI} = y_p = \frac{1}{(D-1)^2} x^2 e^{3x} = e^{3x} \cdot \frac{1}{(D+3-1)^2} x^2 = e^{3x} \cdot \frac{1}{(D+2)^2} x^2 \]

\[= e^{3x} \cdot \frac{1}{4 \left( \frac{1+D}{2} \right)} x^2 = e^{3x} \left( \frac{1+D}{4} \right)^{-1} x^2 \]

\[= e^{3x} \left( 1 - \frac{D}{2} + \frac{D^2}{4} + \ldots \right) x^2 \]

\[= e^{3x} \left( x^2 - Dx^2 + \frac{3}{4} D^2 x^2 \right) \]

\[= e^{3x} \left( x^2 - 2x + \frac{3}{4} \right) \]

\[= e^{3x} \left( x^2 - 2x + \frac{3}{2} \right) \]

\[= e^{3x} \left( 2x^2 - 4x + 3 \right) \]

\[\therefore \text{ General solution is }\]

\[y = y_c + y_p\]

\[y = (c_1 + c_2 x) e^x + e^{3x} \left( \frac{2x^2 - 4x + 3}{8} \right) \]

**Example 4:** \( (D^2 + 3D + 2) y = xe^x \sin x \)

A.E. is \( m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0 \)

\[\Rightarrow m = -1, -2 \]

\[\therefore \text{ C.F. } = y_c = c_1 e^{-x} + c_2 e^{-2x} \]

P.I. \[= y_p = \frac{1}{D^2 + 3D + 2} xe^x \sin x = e^x \cdot \frac{1}{(D+1)^2 + 3(D+1) + 2} (x \sin x) \]

\[= e^x \cdot \frac{1}{D^2 + 5D + 6} (x \sin x) = e^x \left[ \frac{x}{D^2 + 5D + 6} \sin x - \frac{2D+5}{(D^2 + 5D + 6)^2} \sin x \right] \]

\[= e^x \cdot x \left[ \frac{1}{-l^2 + 5D + 6} - \frac{2D+5}{(-l^2 + 5D + 6)^2} \sin x \right] \]
\[ y = e^x \left[ \frac{x}{25} \cdot \frac{D - 1}{D^2 - 1} \sin x - \frac{2D + 5}{2D + 1} \sin x \right] \]

\[ y = e^x \left[ \frac{x}{25} \cdot \frac{D - 1}{D^2 - 1} \sin x - \frac{2D + 5}{2D + 1} \sin x \right] \]

\[ y = e^x \left[ -\frac{x}{10} \left( \cos x - \sin x \right) - \frac{1}{25} \frac{D}{2} \sin x \right] \]

\[ y = e^x \left[ -\frac{x}{10} \left( \cos x - \sin x \right) - \frac{1}{25} \sin x + \frac{1}{10} \cos x \right] \]

\[ \therefore \text{ General solution is } y = y_c + y_p \]

\[ y = c_1 e^{-x} + c_2 e^{-2x} + e^x \left[ -\frac{x}{10} \left( \cos x - \sin x \right) - \frac{1}{25} \sin x + \frac{1}{10} \cos x \right] \]

**Conclusion :** The solution of the given problems are

(i) \[ c_1 \cos 3x + c_2 \sin 3x + \frac{x}{24} \cos 3x + \frac{3}{32} \cos x \]

(ii) \[ c_1 \cos 2x + c_2 \sin 2x + c_3 e^x + c_4 e^{-x} + \frac{1}{40} \left[ -5 - x \sin 2x + 2x \left( e^x - e^{-x} \right) \right] \]

(iii) \[ y = (c_1 + c_2 x)e^x + \frac{e^{3x}}{8} \left[ 2x^2 - 4x + 3 \right] \]

(iv) \[ y = c_1 e^{-x} + c_2 e^{-2x} + e^x \left[ -\frac{x}{10} \left( \cos x - \sin x \right) - \frac{1}{25} \sin x + \frac{1}{10} \cos x \right] \]

**Suggested Problems :**

(1) \( (D^4 + 2D^2 + 1) y = x^2 \cos x \)

(2) \( (D^2 - 1) y = x^2 \sin 3x \)

(3) \( (D^2 - 4) y = x \sinh x \)

(4) \( (D^2 - 2D + 4) y = e^x \sin \frac{x}{2} \)
Experiment - VIII

NON-HOMOGENEOUS EQUATIONS - IV

Aim: To solve a non-homogeneous L.D.E. \( f(D)y = P(x)e^{ax} \), where \( P(x) \) is a polynomial of degree \( k \).

Formule: If \( a \) is a repeated root of the A.E. \( f(m) = 0 \), which is repeated \( r \) times then a particular integral is given by

\[
P.I. = x^r \left( C_0 + C_1 x + C_2 x^2 + \cdots + C_k x^k \right) e^{ax}
\]

where the coefficients \( C_0, C_1, \cdots, C_k \) are determined by substituting P.I. for \( y \) in the given D.E.

Procedure:

Step (1): Write C.F. for the given D.E.

Step (2): Determine \( r \) the number of times \( a \) is repeated as a root of the A.E.

\[
r = 0, \text{ if } a \text{ is not a root of A.E.}
\]

Step (3): Write the P.I. as

\[
P.I. = \left( C_0 x^r + C_1 x^{r+1} + \cdots + C_k x^{r+k} \right) e^{ax}
\]

Step (4): Substitute P.I. for \( y \) in the given D.E. and determine the coefficients \( C_0, C_1, \cdots, C_k \).

Step (5): Write the general solution

\[
y = C.F + P.I
\]

Note:

1. \( \sin ax \) can be expressed as \( \frac{e^{iax} - e^{-iax}}{2i} \)
2. \( \cos ax \) can be expressed as \( \frac{e^{iax} + e^{-iax}}{2} \)
3. If the equation is of the form

\[
f(D) y = Q(x) = P_1(x)e^{a_1 x} + P_2(x)e^{a_2 x} + \cdots + P_k(x)e^{a_k x}
\]

Where \( P_1(x), P_2(x), \cdots, P_k(x) \) are polynomials.

Evaluate \( f(D) y = P_j(x)e^{a_j x} \) for \( j = 1, 2, \cdots, k \) and add these P.I.'s to get the P.I. of \( f(D) y = Q(x) \).
Example 1: \((D^2 - 2D + 3)y = x^3 + \sin x \quad \cdots \cdots (1)\)

A.E. is \(m^2 - 2m + 3 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4 - 12}}{2} = 1 \pm i\sqrt{2}\)

\(\therefore y_c = e^x \left[ c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x \right]\)

Here, \(Q(x) = x^3 + \sin x = x^3 + \frac{e^{ix}}{2} - \frac{e^{-ix}}{2i}\)

The corresponding P.I. is

\[y_p = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 \frac{e^{ix}}{2i} + c_5 \frac{e^{-ix}}{2}\]

where \(c_0, c_1, c_2, c_3, c_4, c_5\) are to be determined.

\[y_p' = c_1 + 2c_2 x + 3c_3 x^2 + c_4 \frac{e^{ix}}{2} - c_5 \frac{e^{-ix}}{2}\]

\[y_p'' = 2c_2 + 6c_3 x + ic_4 \frac{e^{ix}}{2} + ic_5 \frac{e^{-ix}}{2}\]

Substituting the values of \(y_p, y_p', y_p''\) in \((1)\), we get

\[
\left(2c_2 + 6c_3 x + ic_4 \frac{e^{ix}}{2} + ic_5 \frac{e^{-ix}}{2}\right) - 2 \left( c_1 + 2c_2 x + 3c_3 x^2 + c_4 \frac{e^{ix}}{2} - c_5 \frac{e^{-ix}}{2}\right) + 3 \left( c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 \frac{e^{ix}}{2} + c_5 \frac{e^{-ix}}{2}\right) = x^3 + \frac{e^{ix}}{2i} - \frac{e^{-ix}}{2i}
\]

\[
\Rightarrow (2c_2 - 2c_1 + 3c_0) + (6c_3 - 4c_2 + 3c_1) x + (-6c_3 + 3c_2) x^2 + (3c_3 x^3)
\]

\[+ \frac{e^{ix}}{2} \left( ic_4 - 2c_4 + \frac{3c_4}{i}\right) + \frac{e^{-ix}}{2} \left( ic_5 + 2c_5 + \frac{3c_5}{i}\right) = x^3 + \frac{e^{ix}}{2i} - \frac{e^{-ix}}{2i}
\]

Comparing like terms on both sides we get

\[3c_3 = 1 \Rightarrow c_3 = \frac{1}{3}\]
Non-Homogeneous Equations - IV

8.3  

Differential Equations

\[-6c_3 + 3c_2 = 0 \Rightarrow c_2 = \frac{2}{3}\]

\[6c_3 - 4c_2 + 3c_1 = 0 \Rightarrow c_1 = \frac{2}{9}\]

\[2c_2 - 2c_1 + 3c_0 = 0 \Rightarrow c_0 = -\frac{8}{27}\]

\[c_4 \left(i - 2 + \frac{3}{i}\right) = \frac{1}{i} \Rightarrow c_4 (-2 - 2i) = -i \quad \left(\therefore \frac{1}{i} = -i\right)\]

\[\Rightarrow c_4 = \frac{i(1-i)}{4}\]

\[c_5 (i + 2 - 3i) = \frac{-1}{i} = i \Rightarrow c_5 = \frac{i(1+i)}{4(1-i)} = \frac{i(1+i)}{4}\]

\[\therefore y_p = x^3 + 2x^2 + 2x - \frac{8}{27} + \frac{i(1-i)}{4} e^{ix} + \frac{i(1+i)}{4} e^{-ix}\frac{e^{ix} + e^{-ix}}{2i}\]

\[= x^3 + 2x^3 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4} \left(\frac{e^{ix} - e^{-ix}}{2i}\right) + \frac{1}{4} \left(\frac{e^{ix} + e^{-ix}}{2}\right)\]

\[= x^3 + 2x^3 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4} \left(\sin x + \cos x\right)\]

\[\therefore \text{The general solution of (1) is}\]

\[y = y_c + y_p\]

\[y = e^x \left(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x\right) + x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4} \left(\sin x + \cos x\right)\]

Example 2: Solve \(\left(D^2 - 3D + 2\right)y = 2x^2 + 3e^{2x} \quad \text{--------- (1)}\)

Solution: A.E. is \(m^2 - 3m + 2 = 0 \Rightarrow (m - 1)(m - 2) = 0 \Rightarrow m = 1, 2\)

\[\therefore y_c = c_1 e^x + c_2 e^{2x}\]

Let \(y_p = (c_0 + c_1x + c_2x^2) + c_3xe^{2x}\)

where \(c_0, c_1, c_2, c_3\) are to be determined.
\[ y_0' = c_1 + 2c_2 x + 2c_3 x e^{2x} + c_3 e^{2x} \]

\[ y_p'' = 2c_2 + 4c_3 x e^{2x} + 4c_3 e^{2x} \]

Substituting the values of \( y_p, y_p', y_p'' \) in (1) we get

\[
\left(2c_2 + 4c_3 x e^{2x} + 4c_3 e^{2x}\right) - 3\left(c_1 + 2c_2 x + 2c_3 x e^{2x} + c_3 e^{2x}\right)
+ 2\left(c_0 + c_1 x + c_2 x^2 + c_3 x e^{2x}\right) = 2x^2 + 3e^{2x}
\]

\[ \Rightarrow (2c_2 - 3c_0 + 2c_0) + (-6c_2 + 2c_1)x + 2c_2 x^2 + (-6c_3 + 2c_3)x e^{2x} + c_3 e^{2x} = 2x^2 + 3e^{2x} \]

Equating the coefficients of like terms on both sides, we get

\[ 2c_2 = 2 \Rightarrow c_2 = 1 \]
\[ -6c_2 + 2c_1 = 0 \Rightarrow c_1 = 3 \]
\[ 2c_2 - 3c_1 + 2c_0 = 0 \Rightarrow c_0 = \frac{2}{7} \]
\[ c_3 = 3 \]

\[ \therefore y_p = x^2 + 3x + \frac{7}{2} + 3xe^{2x} \]

\[ \therefore \text{The general solution of (1) is } \; y = y_c + y_p \]

\[ y = c_1 e^x + c_2 e^{2x} + x^2 + 3x + \frac{7}{2} + 3xe^{2x} \]

**Example (3) :** Solve \( (D^2 + 4D + 4)y = 3xe^{-2x} \) \——— (1)

**Solution :** A.E. is \( m^2 + 4m + 4 = 0 \Rightarrow (m + 2)^2 = 0 \Rightarrow m = -2, -2 \)

\[ \therefore y_c = (c_1 + c_2 x)e^{-2x} \]

Here \( m = -2 \) repeated twice

\[ \therefore \text{P.I. is of the form} \]

\[ y_p = x^2(Ax + B)e^{-2x} \]

\[ y_p = Ax^3 e^{-2x} + Bx^2 e^{-2x} \]

\[ \Rightarrow y_p' = 3Ax^2 e^{-2x} - 2Ax^3 e^{-2x} + 2Be^{-2x} - 2Bx^2 e^{-2x} \]
\[ y''_p = (3A - 2B)(2xe^{-2x} - 2x^2e^{-2x}) - 6Ax^2e^{-2x} + 4Ax^3e^{-2x} + 2Be^{-2x} - 4Be^{-2x} \]

substituting the values of \( y_p, y'_p, y''_p \) in (1) we get

\[ 6Ax e^{-2x} + 2Be^{-2x} = 3x e^{-2x} \]

Comparing coefficients of like terms, we have

\[ 6A = 3 \Rightarrow A = \frac{1}{2}, \quad B = 0 \]

\[ \therefore y_p = \frac{1}{2}x^3e^{-2x} \]

The general solution of (1) is \( y = y_c + y_p \)

\[ y = (c_1 + c_2x)e^{-2x} + \frac{1}{2}x^3e^{-2x} \]

**Conclusion**: The general solution of the given problems are.

(i) \( y = e^x \left( c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x \right) + \frac{x^3}{3} + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x) \)

(ii) \( y = c_1e^x + c_2e^{2x} + x^2 + 3x + \frac{7}{2} + 3xe^{2x} \)

(iii) \( y = (c_1 + c_2x)e^{-2x} + \frac{1}{2}x^3e^{-2x} \)

**Suggested Problems**:

1. Solve \( (D^2 - 2D + 1)y = xe^x \)
2. Solve \( (D^2 - 3D + 2)y = xe^{2x} + \sin x \)
3. Solve \( (D^2 - 2D)y = e^x \sin x \)
Experiment - IX

CONGRUENCES AND LINEAR CONGRUENCE EQUATIONS

Aim: To solve the linear congruence \( ax \equiv b \pmod{m} \) and apply results of congruences to find the remainder of \( m^n \) when divided by \( k \).

Results:

(a) \( a \equiv b \pmod{m} \Rightarrow a^k \equiv b^k \pmod{m} \), \( k \geq 1 \), \( k \in \mathbb{Z} \)

(b) \( a \equiv b \pmod{m} \), \( b \equiv c \pmod{m} \Rightarrow a \equiv c \pmod{m} \)

(c) \( a \equiv b \pmod{m} \), \( c \equiv d \pmod{m} \Rightarrow a + c \equiv b + d \pmod{m} \) and \( ac \equiv bd \pmod{m} \)

(d) \( ac \equiv bc \pmod{m} \), \((c, m) = 1 \Rightarrow a \equiv b \pmod{m} \)

(e) \( ac \equiv bc \pmod{m} \), \( c \mid m \Rightarrow a \equiv b \left( \frac{m}{c} \right) \mod{m} \)

(f) The linear congruence \( ax \equiv b \pmod{m} \) has a solution if \( \text{g.c.d. } d \) of \( a \) and \( m \) i.e. \( d = (a, m) \) divides \( b \). In this case the congruence \( ax \equiv b \pmod{m} \) has exactly \( d \) incongruent solutions \( \mod{m} \).

Procedure to Solve linear congruence \( ax \equiv b \pmod{m} \):

Step (1): Find \( d = \text{g.c.d. } a, m = (a, m) \).

Step (2): Verify whether \( d \mid b \).

Step (3): If \( d \nmid b \) then the congruence \( ax \equiv b \pmod{m} \) has no solution.

Step (4): If \( d = 1 \), then the congruence \( ax \equiv b \pmod{m} \) has only one solution. By using suitable results listed; find solution.

Step (5): If \( d \neq 1 \) and \( d \mid b \) then the congruence \( ax \equiv b \pmod{m} \) has exactly \( d \) incongruent solutions.

Step (6): Find one solution \( x_0 \), by using suitable results.

Step (7): Find the other incongruent solutions \( x_1, x_2, \ldots, x_{d-1} \) by using the formula

\[ x_k = x_0 + k \left( \frac{m}{d} \right) \text{ for } k = 1, 2, \ldots, d - 1 \]
9.1 Example: Solve $15x \equiv 6 \pmod{21}$

Solution: The congruence is $15x \equiv 6 \pmod{21}$  \hspace{1cm} (1)

Comparing this equation with standard linear equation $ax \equiv b \pmod{m}$, we get

$a = 15, \ b = 6, \ m = 21$

$d = (a, m) = (15, 21) = 3$ which divides 6.

Therefore the given congruence has $3 (= d)$ solutions incongruent $\pmod{21}$.

The given congruence is $3 \times 5x \equiv 3 \times 2 \pmod{3 \times 71}$

$\therefore \ 5x \equiv 2 \pmod{7}$

Also $0 \equiv 28 \pmod{7}$

Adding $5x \equiv 30 \pmod{7}$

$\Rightarrow x \equiv 6 \pmod{7}$

$\therefore x_0 = 6 \ \text{is a solution of} \ 5x \equiv 2 \pmod{7} \ \text{and hence, is also a solution of congruence} \ (1)$.

$\therefore$ All the three incongruent $\pmod{21}$ solutions of

$15x \equiv 6 \pmod{21}$ are given by

$x_0 + k \left( \frac{m}{d} \right)$ \hspace{1cm} where $k = 0, 1, 2, \ldots, (d - 1)$

Here $x_0 = 6, \ d = 3, \ \frac{m}{d} = 21 \div 3 = 7$

$\therefore$ 3 solutions of the given congruence are $x = x_0 + k \left( \frac{m}{d} \right)$

i.e. $x = 6 + 7k, \ k = 0, 1, 2$

$x = 6, \ 13, \ 20 \pmod{21}$

9.2 Example: Solve $15x \equiv 12 \pmod{36}$

Solution: $15x \equiv 12 \pmod{36}$  \hspace{1cm} (1)

Comparing this with $ax \equiv b \pmod{m}$ we get

$a = 15, \ b = 12, \ m = 36$

$d = (a, m) = (15, 36) = 3$ which divides $12 (= b)$
The congruence (1) has 3 (= d) incongruent solutions mod 36.

From (1) \( 3 \times 5(x) \equiv 3 \times 4 \pmod{3 \times 12} \)

\[ \Rightarrow 5x \equiv 4 \pmod{12} \] \hspace{1cm} (2) \hspace{1cm} (\because 3 \mid 36)

Also \( 0 \equiv 36 \pmod{12} \)

Adding \( 5x \equiv 40 \pmod{12} \)

\[ \Rightarrow x \equiv 8 \pmod{12} \]

\[ \therefore x_0 = 8 \text{ is a solution of (2) and hence a solution of (1)} \]

\[ \therefore \text{All the three solutions of (1) are given by} \]

\[ x_k = x_0 + k \frac{m}{d} \quad \text{where} \quad x_0 = 8, \quad \frac{m}{d} = \frac{36}{3} = 12, \quad d = 3, \quad k = 0, 1, 2, \ldots, \quad d - 1 \]

\[ x = 8 + 12k, \quad k = 0, 1, 2 \]

\[ \therefore x \equiv 8, 20, 32 \pmod{36} \]

**9.3 Example:** Solve \( 259x \equiv 5 \pmod{11} \)

**Solution:** Given congruence is

\[ 259x \equiv 5 \pmod{11} \] \hspace{1cm} (1)

\[ 259 \equiv 6 \pmod{11} \] \hspace{1cm} (\because 259 = 23 \times 11 + 6)

\[ \Rightarrow 259x \equiv 6x \pmod{11} \] \hspace{1cm} (2)

from (1) & (2) \( 6x \equiv 5 \pmod{11} \) \hspace{1cm} (3)

Comparing this with \( ax \equiv b \pmod{m} \) we get \( a = 6, \ b = 5, \ m = 11 \)

\[ d = (a, m) = (6, 11) = 1 \text{ which divides } 5(= b) \]

\[ \therefore \text{The congruence (3) and hence, congruence (1) has exactly one solution.} \]

Congruenc (3) is

\[ 6x \equiv 5 \pmod{11} \]

Also \( 0 \equiv 55 \pmod{11} \)

Adding \( 6x \equiv 60 \pmod{11} \)

\[ \Rightarrow x \equiv 10 \pmod{11} \]

\[ x = 10 \text{ is a solution of (3) and hence, solution of (1)}. \]
9.4 Example: Find the remainder of $3^{40}$ when divided by 23.

Solution: We know that $3^1 \equiv 3 \pmod{23}$

$$3^2 \equiv 9 \pmod{23}$$

$$\Rightarrow 3^4 \equiv 81 \pmod{23} \Rightarrow 3^4 \equiv 12 \pmod{23} \quad \text{---------- (1)}$$

$$3^3 \equiv 4 \pmod{23} \Rightarrow (3^3)^3 \equiv 4^3 \pmod{23}$$

$$\Rightarrow 3^9 \equiv 64 \pmod{23}$$

$$\Rightarrow 3^9 \equiv -5 \pmod{23} \quad (\because 64 \equiv -5 \pmod{23})$$

$$\Rightarrow (3^9)^4 \equiv (-5)^4 \pmod{23}$$

$$\Rightarrow 3^{36} \equiv 625 \pmod{23}$$

$$\Rightarrow 3^{36} \equiv 4 \pmod{23} \quad \text{---------- (2)} \quad (\because 625 \equiv 4 \pmod{23})$$

From (1) & (2) $3^{36} \times 3^4 \equiv 12 \times 4 \pmod{23}$

$$\Rightarrow 3^{40} \equiv 48 \pmod{23}$$

$$\Rightarrow 3^{40} \equiv 2 \pmod{23} \quad (\because 48 \equiv 2 \pmod{23})$$

The remainder when $3^{40}$ is divided by 23 is 2.

Note: For examination under this practical 3 problems on solution of linear congruences and one problem on finding remainder are to be given.

Suggested Problems:

1. Solve the following congruences.
   (i) $36x \equiv 27 \pmod{45}$
   (ii) $342x \equiv 5 \pmod{13}$
   (iii) $13x \equiv 9 \pmod{25}$

2. Find the remainder in the division of $2^{20}$ by 7.

Writer

Smt. N. Rajani
THE ALTERNATING GROUP $A_4$

Aim: To write the multiplication table for $A_4$, the alternating group on 4 symbols.

Definitions:
(a) A permutation $\sigma$ of a set $A$ is called a cycle of length $n$ if $\exists a_1, a_2, \ldots, a_n \in A$ such that for any $a \in A$

$$a \sigma = \begin{cases} a_{i+1} & \text{if } a = a_i, \ i < n \\ a_1 & \text{if } a = a_n \\ a & \text{if } a \notin \{a_1, a_2, \ldots, a_n\} \end{cases}$$

(b) A cycle of length 2 is called a transposition.

(c) A permutation $\sigma$ of a finite set $A = \{a_1, a_2, \ldots, a_n\}$ is called an even (odd) permutation if $\sigma$ can be written as a product of an even(odd) number of transpositions.

(d) The group of permutations on $\{1, 2, \ldots, n\}$ is denoted by $S_n$. $S_n$ is called the symmetric group on $n$ symbols or of degree $n$. The set $A_n$ of all even permutations on $n$ symbols is a normal subgroup of $S_n$ and is called the alternating group on $n$ symbols or of degree $n$.

Results used:
1. Every permutation $\sigma$ of a finite set $A$ is a product of disjoint cycles.
2. Every cycle and hence every permutation of a finite set is a product of transpositions.

Procedure:
Step 1: List all the 24 permutations in $S_4$.
Step 2: Decompose each $\sigma \in S_4$ into disjoint cycles by finding the orbits $\theta_\sigma(a)$ under $\sigma$ and finding the cycle corresponding to each orbit with at least two elements. Then $\sigma$ is the product of the cycles corresponding to the orbits with at least 2 elements.
Step 3: Decompose $\sigma$ into a product of transpositions by decomposing each cycle of $\sigma$ into a product of transpositions using the rule

$$(a_1, a_2, \ldots, a_n) = (a_1, a_2)(a_1, a_3)\cdots(a_1, a_n)$$

Step 4: Find all the even permutations in $S_4$. The set of all these permutations is $A_4$. 
Step 5: Write the multiplication table for $A_4$. For $\sigma_1, \sigma_2 \in A_n$, $\sigma_1 \sigma_2$ is computed by the rule

$$a(\sigma_1 \sigma_2) = (a\sigma_1)\sigma_2 \forall a \in \{1, 2, 3, 4\}$$

**Example:**

(a) $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $\theta_{\sigma_1}(1) = \{1, 2, 3, 4\}$

$\therefore \sigma_1 = (1,2,3,4) = (1,2)(1,3)(1,4)$

$\sigma_1$ is an odd permutation.

$\sigma_1 \not\in A_4$

(b) $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$, $\theta_{\sigma_2}(1) = \{1\}$

$\therefore \sigma_2 = (2,4,3) = (2,4)(2,3)$

$\sigma_2$ is an even permutation. $\sigma \in A_4$

(c) $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$, $\theta_{\sigma_3}(1) = \{1,3\}$

$\theta_{\sigma_3}(2) = \{2,4\}$

$\sigma_3 = (1,3)(2,4)$

$\sigma_3$ is an even permutation.

$\sigma_3 \in A_4$

(d) $\sigma_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$

$\theta_{\sigma_4}(1) = \{1\}, \theta_{\sigma_4}(2) = \{2\}$

$\theta_{\sigma_4}(3) = \{3,4\}$

$\therefore \sigma_4 = (3, 4)$ is odd.

$\sigma_4 \not\in A_4$

List all the elements in $S_4$ and for each element examine if it is even. Write the multiplication table of $A_4$.

**Writer:**

*Smt. N. Rajani*
Experiment - XI

PERMUTATIONS

Aim:
(a) To find the order of a non identity permutation \( \sigma \) in \( S_n \), the symmetric group on \( n \) symbols.

(b) To verify that for a cycle \( \sigma \) of length \( n \), with \( \sigma^r \) is a cycle if and only if \( \gcd(r,n) = 1 \) with \( n = 5,6 \).

Definitions:
1. The order of a permutation \( \sigma \) in \( S_n \) is the least positive integer \( r \) such that \( \sigma^r \) is the identity permutation \( I \) in \( S_n \).

2. A permutation \( \sigma \) of a set \( A \) is called a cycle of length \( n \) if \( \exists \ a_1, \ldots, a_n \in A \) such that

\[
\begin{align*}
    a_{i+1} & \text{ if } a = a_i, \ i < n \ \\
    a_1 & \text{ if } a = a_n \ \\
    a & \text{ if } a \not\in \{a_1, \ldots, a_n\}
\end{align*}
\]

for any \( a \in A \), \( a \sigma = \circ \).

Results Used:
1. Every \( \sigma \in S_n \) can be written as a product of disjoint cycles.
2. The order of each cycle is its length.
3. Disjoint cycles commute.
4. The order of \( \sigma \in S_n \) is the l.c.m. of the orders of all the disjoint cycles in the decomposition of \( \sigma \) into disjoint cycles.

Procedure:
(a) Take the given non-identity permutation \( \sigma \) in \( S_n \). Decompose \( \sigma \) into \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_k \) where \( \sigma_1, \sigma_2, \ldots, \sigma_k \) are disjoint cycles of lengths \( r_1, r_2, \ldots, r_k \) \( (r_i > 1, i = 1, \ldots, k) \) respectively. Then order of \( \sigma = \text{l.c.m.} \{r_1, r_2, \ldots, r_k\} \).

Examples:
1. \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix} \)

\( \sigma = (1, 3, 5)(2, 4, 6) \)
Let \( \sigma_1 = (1,3,5), \sigma_2 = (2,4,6) \)
Then \( \sigma = \sigma_1 \sigma_2 \)
\[ r_1 = \text{order of } \sigma_1 = 3 \]
\[ r_2 = \text{order of } \sigma_2 = 3. \]
\[ \text{l.c.m.} \{r_1, r_2\} = 3. \quad \therefore \text{order of } \sigma = 3. \]

**Verification:**
\[ \sigma^2 = (\sigma_1 \sigma_2)^2 = \sigma_1^2 \sigma_2^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix} \]
\[ \sigma^3 = (\sigma_1 \sigma_2)^3 = \sigma_1^3 \sigma_2^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \]
\[ o(\sigma) = 3 = \text{l.c.m.} \{r_1, r_2\} \]

2. \[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 3 & 6 \end{pmatrix} \]
\[ \sigma = (1,2) (3,4,5) = \sigma_1 \sigma_2 \]

where \[ \sigma_1 = (1,2) \]
\[ \sigma_2 = (3,4,5) \]
\[ r_1 = o(\sigma_1) = 2, r_2 = o(\sigma_2) = 3 \]
\[ \therefore o(\sigma) = \text{l.c.m.} \{r_1, r_2\} = \text{l.c.m.} \{2,3\} = 6 \]

**Verification:**
\[ \sigma_1^2 = I, \sigma_2^3 = I \]
\[ \sigma^2 = \sigma_1^2 \sigma_2^2 = (3,5,4) \]
\[ \sigma^3 = (\sigma_1 \sigma_2)^3 = \sigma_1^3 \sigma_2^3 = (1,2) \]
\[ \sigma^4 = (\sigma_1 \sigma_2)^4 = \sigma_1^4 \sigma_2^4 = (\sigma_1^2)^2 (\sigma_2^3) \sigma_2 = \sigma_2 \]
\[ \sigma^5 = (\sigma_1 \sigma_2)^5 = \sigma_1^5 \sigma_2^5 = \sigma_1^4 \sigma_1^3 \sigma_2^2 = \sigma_1 \sigma_2^2 \]
\[ = (1,2) (3,5,4) \]
\[ = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix} \]
\[ \sigma^6 = (\sigma_1 \sigma_2)^6 = \sigma_1^6 \sigma_2^6 = (\sigma_1^2)^3 (\sigma_2^3)^2 = I^3 I^2 = I \]
\[ o(\sigma) = 6 = \text{l.c.m.} \{r_1, r_2\} \]
(b):

Example 1: \( n = 5 \), \( \sigma = (1,3,4,2,5) \in S_6 \)

\( n = \text{length of } \sigma = 5 \).

g.c.d.\( \{2,5\} = 1 \), \( \sigma^2 = (1,4,5,3,2) \) is a cycle.

g.c.d.\( \{3,5\} = 1 \), \( \sigma^3 = (1,2,3,5,4) \) is a cycle.

g.c.d.\( \{4,5\} = 1 \), \( \sigma^4 = (1,5,2,4,3) \) is a cycle.

For any \( 0 < t < 5 \), \( \sigma^t \) is a cycle \( \Leftrightarrow \) g.c.d.\( \{t,5\} = 1 \)

Suppose \( \sigma^t \neq I \).

\( \exists q, t \in \mathbb{Z} \, | \, r = 5q + t, 0 \leq t < 5 \), \( \sigma^t \neq I \Rightarrow 0 < t < 5 \).

Now, g.c.d.\( \{r,5\} = \text{g.c.d.}\{t,5\} \)

\( \therefore \sigma^t = \sigma^t \) is a cycle, iff g.c.d.\( \{t,5\} = \text{g.c.d.}\{r,5\} = 1 \)

Example 2: \( n = 6 \)

\( \sigma = (1,3,5,2,4,6) \)

g.c.d.\( \{2,6\} = 2 \), \( \sigma^2 = (1,5,4) \) (3,2,6) is not a cycle.

g.c.d.\( \{3,6\} = 3 \), \( \sigma^3 = (1,2) \) (3,4)(5,6) is not a cycle.

g.c.d.\( \{4,6\} = 2 \), \( \sigma^4 = (1,4,5) \) (3,6,2) is not a cycle.

g.c.d.\( \{5,6\} = 1 \), \( \sigma^5 = (1,6,4,2,5,3) \) is a cycle.

\( \sigma^6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = I \)

For \( 0 < t < 6 \), \( \sigma^t \) is a cycle iff \( t,6 = 1 \). Suppose \( \sigma^t \neq I \). \( \exists q, t \in \mathbb{Z} \, | \, r = 6q + t \), \( \text{g.c.d.}\{r,6\} = \text{g.c.d.}\{t,6\}, 0 < t < 6 \).

\( \sigma^t = \sigma^t \) is a cycle, iff \( t,6 = 1 \Leftrightarrow (r,6) = 1 \)

Note: For practical examination the examiner shall set two question for (a) and 2 questions for (b)

Writer:

Smt. N. Rajani
Experiment - XII

FINITE CYCLIC GROUPS

Aim:  
(a) To find the order of a subgroup generated by any element $x$ in a given finite cyclic group.
(b) To find the generators of a given finite cyclic group.
(c) To find the number of generators of a finite cyclic group.

Definitions:

1. A group $G$ is called a cyclic group if $\exists$ an element $a \in G \ni \{a^n \mid n \in \mathbb{Z}\}$ $a$ is called a generator of $G$. If the group is written additively we write $na$ for $a^n$.
2. A cyclic group $G$ is called a finite cyclic group if $G$ is a finite set.
3. The order of a subgroup $H$ of a finite group is the number of elements in $H$.

Results Used:

1. Let $G$ be a cyclic group of order $n$, generated by an element $a$. Then the order of the subgroup of $G$ generated by $a^s$ is $\frac{n}{\gcd(n, s)}$.
2. Let $G$ be a finite cyclic group of order $n$. Let $x = a^s$. Then $x$ is a generator of $G$, if and only if $\gcd(s, n) = 1$.
3. The number of generators of a finite cyclic group of order $n$ is $\phi(n)$ where $\phi$ is the Euler function. [$\phi(1) = 1$ and for $n > 1$, $\phi(n)$ is the number of positive integers which are less than $n$ and relatively prime to $n$].
4. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where $p_1, \ldots, p_k$ are primes then
   $$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$
5. Any finite cyclic group of order $n$ is isomorphic to the additive group $(\mathbb{Z}_n, +_n)$ of integers modulo $n$.

Procedures:

Since any finite cyclic group of order $n$ is isomorphic to the additive group $(\mathbb{Z}_n, +_n)$ of integers modulo $n$, it is enough to consider $(\mathbb{Z}_n, +_n)$. $(\mathbb{Z}_n, +_n)$ is generated by 1.
(a) Let \( n \) be the given positive integer. Let \( s \) be the given element of \( \mathbb{Z}_n = \{0, 1, \ldots, n - 1\} \).

Find \( d = \gcd\{s, n\} \). Then the order of the subgroup generated by \( s \) is \( \frac{n}{d} \). For verification, list and count the elements in the subgroup generated by \( s \) and compare with the result already obtained.

(b) Let \( n \) be the given positive integer. For each \( s \in \mathbb{Z}_n \), find \( \gcd\{s, n\} \). Then \( s \) is a generator of \( \mathbb{Z}_n \) iff \( \gcd\{s, n\} = 1 \). List the generators of \( \mathbb{Z}_n \) and verify whether these are generators.

(c) Let \( n \) be the given positive integer. Express \( n \) as \( 1 \equiv \sum_{i=1}^{k} a_i p_i^{\alpha_i} \pmod{p_1 \cdots p_k} \) where \( p_1, \ldots, p_k \) are primes.

Compute \( \phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \).

Then the number of generators of \( \mathbb{Z}_n \) is \( \phi(n) \). Verify this result by finding the generators.

**Example:**

(a) (i) Find the order of the subgroup generated by 3 in \( \mathbb{Z}_8 \).

Here \( n = 8, s = 3 \)

\[ d = \gcd\{n, s\} = \gcd\{3, 8\} = 1 \]

The order of the subgroup generated by 3 is \( \frac{n}{d} = \frac{8}{1} = 8 \)

**Verification:** The subgroup generated by 3 in \( \mathbb{Z}_8 = \{3, 6, 1, 4, 7, 2, 5, 0\} \) is \( \mathbb{Z}_8 \).

Thus order of the subgroup generated by 3 in \( \mathbb{Z}_8 \) is 8.

(ii) Find the order of the subgroup generated by 12 in \( \mathbb{Z}_{15} \).

Here \( n = 15, s = 12 \). \( d = \gcd\{12, 15\} = 3 \)

\( \frac{n}{d} = 5 \). Thus the order of the subgroup generated by 12 in \( \mathbb{Z}_{15} \) is 5.

**Verification:** The subgroup generated by 12 in \( \mathbb{Z}_{15} = \{12, 9, 6, 3, 0\} \). Thus the order of this subgroup is 5.

(b) **Example:** Find the generators of \( \mathbb{Z}_{20} \).

Here \( n = 20 \). The set of generators = \( \{s \mid s \in \mathbb{Z}_{20} \text{ and } \gcd\{s, 20\} = 1\} \)

\[ = \{1, 3, 7, 9, 11, 13, 17, 19\} \]
Verification : 
\begin{align*}
(1) & = \mathbb{Z}_8 \\
(3) & = \{3, 6, 9, 12, 15, 18, 1, 4, 7, 10, 13, 16, 19, 25, 8, 11, 14, 17, 0\} = \mathbb{Z}_{20} \\
(7) & = \{7, 14, 8, 15, 2, 9, 16, 3, 10, 17, 4, 11, 18, 5, 12, 19, 6, 13, 0\} = \mathbb{Z}_{20} \\
(9) & = \{9, 18, 7, 16, 5, 14, 3, 12, 10, 19, 8, 17, 6, 15, 4, 13, 2, 11, 0\} = \mathbb{Z}_{20} \\
(11) & = \{11, 2, 13, 4, 15, 6, 17, 8, 19, 10, 1, 12, 3, 14, 5, 16, 7, 18, 19, 0\} = \mathbb{Z}_{20} \\
(13) & = \{13, 6, 19, 12, 5, 18, 11, 4, 17, 10, 3, 16, 9, 2, 15, 8, 1, 14, 7, 0\} = \mathbb{Z}_{20} \\
(17) & = \{17, 14, 11, 8, 5, 2, 19, 16, 13, 10, 7, 4, 18, 15, 12, 9, 6, 3, 0\} = \mathbb{Z}_{20} \\
(19) & = \{19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0\} = \mathbb{Z}_{20} \\
\end{align*}

(c) Example : Find the number of generators of $\mathbb{Z}_{50}$.

$n = 50 = 5 \times 2 \times 5 = 5^2 \times 2$

The number of generators of $\mathbb{Z}_{20} = \phi(n)$

$$ = n \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{2}\right) = \frac{50}{10} \left(4\right)\left(1\right) = 20 $$

Verification : The set of generators is $\{s \in \mathbb{Z}_{50} | (s, 50) = 1\}$

$$ = \{1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39, 41, 43, 47, 49\} $$

From this also the number of generators is 20.

Note : For practical examination, the examiner shall set 2 questions in (a), one question in (b) and 1 question in (c)

Writer

Smt. N. Rajani


**Experiment - XIII**

**VECTOR DIFFERENTIAL OPERATORS**

**Aim:**
A) To find the gradient of a given scalar function \( \phi(x, y, z) \)

B) To find the divergence of a given vector function \( \vec{F}(x, y, z) \)

C) To find the curl of a given vector function \( \vec{F}(x, y, z) \)

**Definitions:**

(i) **Gradient:** The gradient of a scalar function \( \phi(x, y, z) \) is defined as

\[
\nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}
\]

\( \nabla \phi \) is also written as \( \nabla \phi \)

(ii) **Divergence:** The divergence of a vector function \( \vec{F}(x, y, z) \) is defined as

\[
\nabla \cdot \vec{F} = \hat{i} \frac{\partial F_1}{\partial x} + \hat{j} \frac{\partial F_2}{\partial y} + \hat{k} \frac{\partial F_3}{\partial z}
\]

\( \nabla \cdot \vec{F} \) is also written as \( \nabla \cdot \vec{F} \)

**Formulae used**

1. If \( \vec{F} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} \), then

\[
\nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}
\]

2. If \( \vec{F} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} \), then

\[
\nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
f_1 & f_2 & f_3
\end{vmatrix}
\]

**Procedure:**

(A) Consider the given scalar function \( \phi(x, y, z) \)
Step 1: Find $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, $\frac{\partial \phi}{\partial z}$

Step 2: Write $\nabla \phi$ as $\nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$

(B) Consider the given vector function $\mathbf{\mathbf{f}}(x, y, z) = f_1(x, y, z) \mathbf{i} + f_2(x, y, z) \mathbf{j} + f_3(x, y, z) \mathbf{k}$.

Step 1: Find $\frac{\partial f_1}{\partial x}$, $\frac{\partial f_2}{\partial y}$, $\frac{\partial f_3}{\partial z}$

Step 2: Write $\text{div} \mathbf{\mathbf{f}}$ as $\text{div} \mathbf{\mathbf{f}} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

(C) Consider the vector function $\mathbf{\mathbf{f}}(x, y, z) = f_1(x, y, z) \mathbf{i} + f_2(x, y, z) \mathbf{j} + f_3(x, y, z) \mathbf{k}$

Step 1: Write $\text{curl} \mathbf{\mathbf{f}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

Step 2: Expand the determinant

Example 1: If $\phi(x, y, z) = 3x^2y - y^3z^2$, then find $\text{grad} \phi$ at $(1, -2, -1)$.

$$\frac{\partial \phi}{\partial x} = 6xy, \quad \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2z^2, \quad \frac{\partial \phi}{\partial z} = -2y^3z$$

$$\therefore \text{grad} \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} = 6xy \mathbf{i} + (3x^2 - 3y^2z^2) \mathbf{j} - 2y^3z \mathbf{k}$$

$\text{grad} \phi$ at $(1, -2, -1)$ is $= -12 \mathbf{i} - 9 \mathbf{j} - 16 \mathbf{k}$

Example 2: If $\phi(x, y, z) = x^2 - y^2 + x^2z$, then find $\text{grad} \phi$ at $(1, 1, -2)$

$$\frac{\partial \phi}{\partial x} = 2x, \quad \frac{\partial \phi}{\partial y} = -2y, \quad \frac{\partial \phi}{\partial z} = x^2$$
Vector Calculus

Example 3: If \( \mathbf{\nabla} = \text{grad} (x^3 + y^3 + z^3 - 3xyz) \), then find \( \text{div} \mathbf{\nabla} \).

Let \( \phi(x, y, z) = x^3 + y^3 + z^3 - 3xyz \)

\[
\frac{\partial \phi}{\partial x} = 3x^2 - 3yz, \quad \frac{\partial \phi}{\partial y} = 3y^2 - 3zx, \quad \frac{\partial \phi}{\partial z} = 3z^2 - 3xy
\]

\[\therefore \mathbf{\nabla} = \text{grad} \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z}\]

\[
= (3x^2 - 3yz) \mathbf{i} + (3y^2 - 3zx) \mathbf{j} + (3z^2 - 3xy) \mathbf{k}
\]

\[\text{div} \mathbf{\nabla} = \frac{\partial}{\partial x} (3x^2 - 3yz) + \frac{\partial}{\partial y} (3y^2 - 3zx) + \frac{\partial}{\partial z} (3z^2 - 3xy)
\]

\[= 6x + 6y + 6z
\]

Example 4: If \( \mathbf{\nabla} = x^3 z \mathbf{i} + xy^3 \mathbf{j} + yz^3 \mathbf{k} \), then find \( \text{div} \mathbf{\nabla} \).

Let \( f_1(x, y, z) = x^3 z, \quad f_2(x, y, z) = xy^3, \quad f_3(x, y, z) = yz^3 \)

\[
\therefore \frac{\partial f_1}{\partial x} = 3x^2 z, \quad \frac{\partial f_2}{\partial y} = 3x y^2, \quad \frac{\partial f_3}{\partial z} = 3y z^2
\]

\[\mathbf{\nabla} (x, y, z) = f_1(x, y, z) \mathbf{i} + f_2(x, y, z) \mathbf{j} + f_3(x, y, z) \mathbf{k}
\]

\[\therefore \text{div} \mathbf{\nabla} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}
\]

\[= 3x^2 z + 3xy^2 + 3yz^2
\]
Example 5: If $\vec{A} = 2xz^2\vec{i} - yz\vec{j} + 3xz^3\vec{k}$, then find curl $\vec{A}$ at $(1, 1, 1)$

$$\text{curl } \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \vec{i}(0 + y) - \vec{j}(3x^2 - 4xz) + \vec{k}(0 - 0)$$

$$= y\vec{i} + (4xz - 3z^3)\vec{j}$$

At $(1, 1, 1)$, curl $\vec{A}$ is $= \vec{i} + \vec{j}$

Example 6: If $\vec{F} = 3xyz^3\vec{i} + 4x^3y\vec{j} - xy^2z\vec{k}$, then find curl $\vec{F}$ at $(-1, 2, 1)$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xyz^3 & 4x^3y & -xy^2z \end{vmatrix}$$

$$= \vec{i}(2xyz - 0) - \vec{j}(-y^2z - 9xyz^2) + \vec{k}(12x^2y - 3xz^3)$$

At $(-1, 2, 1)$, curl $\vec{F} = 4\vec{i} + 15\vec{j} + 27\vec{k}$

Suggested Problems:

1. Find grad $\phi$ at $(1, 1, -2)$, where $\phi(x, y, z) = x^3 + y^3 + 3xyz$

2. Find grad $\phi$ at $(2, 1, -2)$, where $\phi(x, y, z) = x^2 + y^2 - z^2$

3. Find div $\vec{F}$ at $(1, 1, 1)$, where $\vec{F}(x, y, z) = x^2y\vec{i} - 2xz\vec{j} + 2yz\vec{k}$

4. Find div $\vec{F}$ at $(1, -1, 1)$, where $\vec{F}(x, y, z) = xy^2\vec{i} + 2x^2yz\vec{j} + 3xyz^2\vec{k}$.

5. Find curl $\vec{F}$ at $(2, -1, 0)$, where $\vec{F}(x, y, z) = (3x^2y - z)\vec{i} + (xz^3 + y^4)\vec{j} - 2x^3z^2\vec{k}$

6. If $u = x^2 + y^2 + z^2$, find curl grad $u$. 
GREEN'S THEOREM AND ITS APPLICATIONS

Aim: To verify Green's theorem for a given vector function in a region in the XY - Plane

Definitions and Theorems used:

1. **Definition (Line Integral)**: An integral which is to be evaluated along a curve is called a line integral.

   Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a smooth curve joining the points $A$ and $B$. Then $d\mathbf{r}$ stands for $\frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right)dt$ which is written as $dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$. If $s$ stands for the arc length of a point on $C$ from a fixed point on $C$, then $\frac{d\mathbf{r}}{dt} = \mathbf{T}$ is a unit vector along the tangent to the curve $C$ at the point $\mathbf{r}$.

   Let $\mathbf{F}(\mathbf{r})$ be a vector point function defined and continuous along $C$. The component of the vector $\mathbf{F}$ along the tangent is $\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}$ along $C$ from $A$ to $B$, written as $\int_{A}^{B} \mathbf{F} \cdot \frac{d\mathbf{r}}{ds} ds = \int_{A}^{B} \mathbf{F} \cdot d\mathbf{r}$ is called the (tangential) line integral of $\mathbf{F}$ along $C$ (from $A$ to $B$).

2. **Green's theorem**: Let $S$ be a closed region in the XY - plane enclosed by a closed curve $C$. If $P$ and $Q$ are differentiable real valued functions of $x$ and $y$ in $S$,

   $$\oint_{C} P \, dx + Q \, dy = \iint_{S} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy,$$

   the line integral being taken along the boundary $C$ of $S$ such that $S$ is on the left, as one moves along $C$.

Procedure:

**Step 1**: Identify the arcs $C_1, C_2, C_3, \ldots, C_r$ of the (piecewise continuous) closed curve $C$ such that the initial point of $C_i$ coincides with the terminal point of $C_{i-1}$ for $i = 2, 3, \ldots, r$ and the terminal point of $C_r$ coincides with the initial point of $C_1$.

**Step 2**: Calculate $\alpha_i = \int_{C_i} P \, dx + Q \, dy$ for $i = 1, 2, \ldots, r$.

**Step 3**: Add $\alpha_1, \ldots, \alpha_r$ to get $\oint_{C} P \, dx + Q \, dy$. 

Step 4: Find \( \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y} \).

Step 5: Evaluate \( \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \) over the region \( S \) bounded by \( C \).

Step 6: Verify whether the values obtained in Step 3 and Step 5 are equal.

Example: Verify Green’s Theorem for \( \mathbf{F} = (xy + y^2) \mathbf{i} + x^2 \mathbf{j} \) in the region bounded by \( y = x \) and \( y = x^2 \).

To verify \( \oint_C P \, dx + Q \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \)

Here \( P = xy + y^2, \, Q = x^2 \)

\( \frac{\partial P}{\partial y} = x + 2y, \quad \frac{\partial Q}{\partial x} = 2x \)

Let \( C_1 \) be the curve from \( O \) to \( A \)

Let \( C_2 \) be the line from \( A \) to \( O \).

Along \( C_1 \): \( y = x^2 \Rightarrow dy = 2x \, dx \)

\( x \) varies from 0 to 1.
\[ \int_{C_1} (Pdx + Qdy) = \int_0^1 \left( x \cdot x^2 + x^4 + x^2 \cdot 2x - 3x^3 + x^4 \right) dx = \int_0^1 \left( \frac{3}{4} x^4 + \frac{x^5}{5} \right) dx \]
\[ = \left[ \frac{3}{4} x^4 + \frac{1}{5} x^5 \right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20} \]

Along \( C_2 \): \( y = x \), \( \Rightarrow \) \( dy = dx \)
\( x \) varies from 1 to 0.
\[ \int_{C_2} (Pdx + Qdy) = \int_1^0 [x \cdot x + x^2 + x^2] dx = \int_1^0 3x^2 dx = (x^3)_1^0 \]
\[ = 0 - 1 = -1 \]
\[ \therefore \ \text{LHS} = \int_C (Pdx + Qdy) = \int_{C_1} (Pdx + Qdy) + \int_{C_2} (Pdx + Qdy) \]
\[ = \frac{19}{20} - 1 = -\frac{1}{20} \]

\[ \text{RHS} = \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_0^1 \int_{x=0}^{x=x^2} (x - 2y) dy dx \]
\[ = \left[ (xy - y^2) \right]_0^{x=x^2} dx = \left[ ((x^2 - x^2) - (x^3 - x^4)) \right] dx = \left( -\frac{x^4}{4} + \frac{x^5}{5} \right)_0^1 \]
\[ = -\frac{1}{4} + \frac{1}{5} = -\frac{1}{20} \]
\[ \therefore \ \text{LHS} = \text{RHS} \]
\[ \therefore \ \text{Green's theorem is verified.} \]

**Suggested Problems:**

1. Verify Green's Theorem for \( \int_C (3x^2 - 8y^2) + (4y - 6xy) dy \), where \( C \) is the boundary of the region enclosed by \( y = \sqrt{x} \) and \( y = x^2 \).
2. Verify Green's Theorem for\[ \int_C (2xy - x^2) \, dx + (x^2 + y^2) \, dy \] where \( C \) is the boundary of the region enclosed by \( y = x^2 \) and \( y^2 = x \).

3. Verify Green's Theorem for\[ \int_C \left( x^2 - xy^3 \right) \, dx + (y - 2xy) \, dy \] where \( C \) is the square with vertices \((0, 0), (2, 0), (2, 2), (0, 2)\).

4. Verify Greens' Theorem for\[ \int_C \left( x^2 + xy \right) \, dx + (x^2 + y^2) \, dy \] where \( C \) is the square formed by the lines \( x = \pm 1 \), \( y = \pm 1 \).

5. Verify Green's Theorem for\[ \int_C \left( x^2 - 2xy \right) \, dx + (x^2y + 3y) \, dy \] where \( C \) is the boundary of the region enclosed by \( y^2 = 8x \) and \( x = 2 \).
GAUSS'S DIVERGENCE THEOREM

**Aim:** To verify Gauss's divergence theorem for given vector function.

**Definitions and Theorems used:**

1. **Definition (Surface Integral):** An integral which is to be evaluated over a surface is called surface integral.

   Let $F$ be a vector function depend in $S$. Let $S$ be a region of the surface. Divide $S$ in to $n$ sub regions of areas $\delta S_1, \delta S_2, \ldots, \delta S_n$. Let $P_i$ be a point and $S_i$ and $N_i$ be the unit normal to $\delta S_i$ at $P_i$. Then $\delta S_i \cdot N_i$ is denoted by $\delta \vec{A}_i$ which is the vector area of $\delta S_i$ i.e. a vector normal to $\delta S_i$ at $P_i$ and having magnitude $\delta S_i$.

   Let $I_n = \sum_{i=1}^{n} F(T_i) \cdot N_i \cdot \delta S_i$. The limit of $I_n$ as $n \to \infty$, if it exists, is called the normal surface integral of $F(\tau)$ over the surface $S$ and is denoted by $\int_S F \cdot d\vec{A}$ or $\int_S F \cdot d\vec{s}$.

2. **Gauss’s Divergence Theorem:** If $F$ is a vector function, continuous and having first order partial derivatives in some domain containing a closed surface and $V$ is the region enclosed by $S$, then $\int_S F \cdot d\vec{N} ds = \int_V \int \text{div} F \cdot dV$, where $\vec{N}$ is the outward drawn unit normal vector at any point of $S$.

**Formula:** If $F = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$, then $\text{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$  \hspace{1cm} (A)

**Procedure:**

**Step 1:** Identity the sub surfaces $S_1, S_2, \ldots, S_n$ which constitute the closed surface $S$, enclosing the region $V$ so that the surface integral $\alpha_i = \int_{S_i} F \cdot \vec{N} ds$ can be computed for $i = 1, 2, \ldots, r$.

   [If $S_i$ is parallel to $XY$ plane / $YZ$ plane / $ZX$ plane, then $\vec{N} = \vec{k}$ or $-\vec{k}$ / $\vec{N} = \vec{j}$ or $-\vec{j}$ according as the outward unit normal vector $\vec{N}$ is in the direction of $\overrightarrow{OZ}$ or $\overrightarrow{ZO}$ / $\overrightarrow{OX}$ or $\overrightarrow{OX}$ / $\overrightarrow{OY}$ or $\overrightarrow{YO}$ respectively.]

   If $S_i$ is not parallel to all the co-ordinate planes, then project $S_i$ into a plane which is not perpendicular to $S_i$. 

Step 2: Compute $\alpha_i$'s, $1 \leq i \leq r$

Step 3: Add $\alpha_1, \ldots, \alpha_r$ to get $\int_S \vec{F} \cdot \vec{N} \, ds$

Step 4: Compute $\text{div} \vec{F}$ using formula (A).

Step 5: Evaluate $\int_V \text{div} \vec{F} \, dv = \int_V \int \text{div} \vec{F} \, dx \, dy \, dz$

Step 6: Verify whether the values in step 3 and step 5 are equal.

Example: Verify Gauss's divergence theorem for $\vec{F} = 2xy \hat{i} + yz^2 \hat{j} + xz \hat{k}$ and $S$ is the surface of the rectangular parallelepiped bounded by $x = 0, y = 0, z = 0 \ x = z, y = 1, 2 = 3$.

Solution:

![Diagram of a rectangular parallelepiped]

Solution:

(i) For $S_1$ : face OANB, $Z = 0$, $\vec{N} = -\hat{k}$, $dS = dx \, dy$

$$\vec{F} \cdot \vec{N} = 0$$
\[ \therefore \int_{S_1} \mathbf{F} \cdot \mathbf{N} \, ds = 0 \]

(ii) For \( S_2 \): Face PMCL:

\[ Z = 3, \quad \mathbf{N} = \mathbf{k}, \quad dS = dxdy \]

\[ \mathbf{F} \cdot \mathbf{N} = xz = 3x \]

\[ \therefore \int_{S_2} \mathbf{F} \cdot \mathbf{N} \, ds = \int \left( \int_{x=0}^{2} 3x \, dx \right) \, dy = \int \left( \int_{y=0}^{2} (3xy) \, dy \right) \, dx = \int 3x \, dx \]

\[ = \left( \frac{3x^2}{2} \right)_{0}^{2} = 6 \]

(iii) For \( S_3 \): Face OBLC: \( x = 0 \), \( \mathbf{N} = -\mathbf{i} \)

\[ dS = dydz \]

\[ \therefore \mathbf{F} \cdot \mathbf{N} = -2xy = 0 \]

\[ \therefore \int_{S_3} \mathbf{F} \cdot \mathbf{N} \, ds = 0 \]

(iv) For \( S_4 \): Face AMPN: \( x = 2 \), \( \mathbf{N} = \mathbf{i} \)

\[ dS = dydz; \quad \mathbf{F} \cdot \mathbf{N} = 2xy = 4y \]

\[ \int_{S_4} \mathbf{F} \cdot \mathbf{N} \, ds = \int \left( \int_{z=0}^{3} 4y \, dy \right) \, dz = \int \left( \int_{y=0}^{3} (2y^2) \, dy \right) \, dz \]

\[ = \int 2 \, dz = \left[ 2z \right]_{0}^{3} = 6 \]

(v) For \( S_5 \): Face OAMC: \( y = 0 \), \( \mathbf{N} = -\mathbf{j} \)

\[ \mathbf{F} \cdot \mathbf{N} = 0, \quad dS = dx \, dz \quad \therefore \int_{S_5} \mathbf{F} \cdot \mathbf{N} \, ds = 0 \]

(vi) For \( S_6 \): Face PLBN: \( \mathbf{N} = \mathbf{j}, \quad y = 1 \)
\[ \therefore \vec{F} \cdot \vec{N} = yz^2 = z^2 \]

\[ ds = dx \, dz \]

\[ \therefore \int_{S_6} \vec{F} \cdot \vec{N} \, ds = \int_{x=0}^{2} \int_{z=0}^{3} z^2 \, dz \, dx = \frac{2}{3} \left( \frac{z^3}{3} \right)_{z=0}^{3} dx = (9x)^2_0 = 18 \]

\[ \therefore \int \vec{F} \cdot \vec{N} \, ds = \int_{S_1} \vec{F} \cdot \vec{N} \, ds + \int_{S_2} \vec{F} \cdot \vec{N} \, ds + \int_{S_3} \vec{F} \cdot \vec{N} \, ds + \int_{S_4} \vec{F} \cdot \vec{N} \, ds + \int_{S_5} \vec{F} \cdot \vec{N} \, ds + \int_{S_6} \vec{F} \cdot \vec{N} \, ds \]

\[ = 0 + 6 + 0 + 6 + 0 + 18 = 30 \quad \text{(1)} \]

\[ \text{div} \vec{F} = \frac{\partial}{\partial x} (2xy) + \frac{\partial}{\partial y} (yz^2) + \frac{\partial}{\partial z} (x^2) = 2y + z^2 + x \]

\[ \therefore \int \text{div} \vec{F} \, dv = \int_{V} \int_{y=0}^{1} \int_{z=0}^{3} (2y + z^2 + x) \, dz \, dy \, dx \]

\[ = \int_{x=0}^{2} \int_{y=0}^{1} \left( 2yz + \frac{z^3}{3} + zx \right)_{z=0}^{3} \, dy \, dx \]

\[ = \int_{x=0}^{2} \int_{y=0}^{1} (6y + 9 + 3x) \, dy \, dx \]

\[ = \int_{x=0}^{2} (3y^2 + 9y + 3xy)_{y=0}^{1} \, dx \]

\[ = (3 + 9 + 3x) \, dx \]
\[
\int_{0}^{2} (12 + 3x) \, dx = \left( 12x + \frac{3x^2}{2} \right)_{0}^{2} = 24 + 6 = 30 \quad \text{-------- (2)}
\]

\[
\Rightarrow \int_{S} \mathbf{F} \cdot \mathbf{N} \, ds = \int_{V} \text{div} \mathbf{F} \, dV \quad \text{(From (1) & (2))}
\]

\[
\therefore \text{Gauss divergence Theorem is verified.}
\]

**Suggested Problems :**

1. Verify Gauss's divergence theorem for \( \mathbf{F} = (x^3 - yz) \mathbf{i} - 2x^2y \mathbf{j} + zk \) and S is the surface bounded by the Coordinate planes and \( x = y = z = a \).

2. Verify Gauss's divergence Theorem for \( \mathbf{F} = 4xz \mathbf{i} - (y^2 - j) + yz \mathbf{k} \) over the cube bounded by \( x = 0, y = 0, z = 0, x = 1, y = 1, z = 1 \).

3. Verify Gauss's divergence theorem for \( \mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k} \) taken over the cube \( 0 \leq x, y, z \leq 1 \).

4. Verify Gauss's divergence Theorem for \( \mathbf{F} = (x^2 - yz) \mathbf{i} + (y^2 - zx) \mathbf{j} + (z^2 - xy) \mathbf{k} \) taken over the rectangular parallelopiped \( 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c \).

5. Verify Gauss's divergence theorem for \( \mathbf{F} = 2xy \mathbf{i} - yz \mathbf{j} + x^2 \mathbf{k} \) and S is the surface of the cube bounded by the co-ordinate planes and the plans \( x = a, y = a, z = a \).
Experiment - XVI

STOKE'S THEOREM

Aim: To verify Stoke’s theorem for a given vector function.

1. Definition (Line Integral): An integral which is to be evaluated along a curve is called a line integral.

Let \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \) be a smooth curve joining the points A and B. Then \( d\mathbf{r} \) stands for
\[
\left( \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \right) dt
\]
which is written as \( dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} \). If \( s \) stands for the arc length of a point on \( C \) from a fixed point on \( C \), then \( \frac{d\mathbf{r}}{dt} = \mathbf{T} \) is a unit vector along the tangent to the curve \( C \) at the point \( r \).

Let \( \mathbf{F}(r) \) be a vector point function defined and continuous along \( C \). The component of the vector \( \mathbf{F} \) along the tangent is \( \mathbf{F} \cdot \frac{d\mathbf{r}}{ds} \) along \( C \) from A to B, written as
\[
\int_{A}^{B} \mathbf{F} \cdot \frac{d\mathbf{r}}{ds} \, ds = \int_{A}^{B} \mathbf{F} \cdot d\mathbf{r}
\]
is called the (tangential) line integral of \( \mathbf{F} \) along \( C \) (from A to B).

2. Definition (Surface Integral): An integral which is to be evaluated over a surface is called surface integral.

Let \( \mathbf{F} \) be a vector function depend in \( S \). Let \( S \) be a region of the surface. Divide \( S \) into \( n \) sub regions of areas \( \delta S_1, \delta S_2, \ldots, \delta S_n \). Let \( P_i \) be a point and \( S_i \) and \( \mathbf{N}_i \) be the unit normal to \( \delta S_i \) at \( P_i \). Then \( \delta S_i \cdot \mathbf{N}_i \) is denoted by \( \delta A_i \) which is the vector area of \( \delta S_i \) i.e. a vector normal to \( \delta S_i \) at \( P_i \) and having magnitude \( \delta S_i \).

Let \( I_n = \sum_{i=1}^{n} \mathbf{F}(P_i) \cdot \mathbf{N}_i \cdot \delta S_i \). The limit of \( I_n \) as \( n \to \infty \), if it exists, is called the normal surface integral of \( \mathbf{F}(r) \) over the surface \( S \) and is denoted by \( \iint_{S} \mathbf{F}(r) \cdot d\mathbf{A} \) or \( \iint_{S} \mathbf{F} \cdot \mathbf{N} \, ds \).

3. Stoke’s Theorem: Let \( S \) be a surface bounded by a closed non-intersecting curve \( C \). If \( \mathbf{F} \) is any differentiable vector point function over \( S \), then \( \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} \text{curl} \, \mathbf{F} \cdot \mathbf{N} \, ds \), where \( \mathbf{N} \) is the outward drawn unit normal vector to \( S \) and \( C \) is the boundary of \( S \) which is traversed in the positive direction (in the sense that if a person walking on the boundary of \( S \) in this direction with his head pointing in the direction of outward drawn normal \( \mathbf{N} \) to \( S \), has the surface on his left).
Formula: If $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$, then

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

--------------- (B)

Procedure:

Step 1: Identify the arcs $C_1, C_2, \ldots, C_r$ of the (pieurose continuous) closed curve such that the initial point of $C_i$ coincides with the terminal point of $C_{i-1}$ for $i = 2, 3, \ldots, r$ and the terminal point of $C_r$ coincides with the initial point of $C_1$.

Step 2: Calculate $\alpha_i = \int_{C_i} F_1 \, dx + F_2 \, dy + F_3 \, dz$, for $i = 1, 2, 3, \ldots, r$.

Step 3: Add $\alpha_1, \alpha_2, \ldots, \alpha_r$ to get $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Step 4: Evaluate $\text{curl } \mathbf{F}$ using formula (B).

Step 5: Identify the sub surfaces $S_1, S_2, \ldots, S_r$ so that $\int_S \text{curl } \mathbf{F} \cdot \mathbf{N} \, dS$ can be obtained as the sum $\sum_{i=1}^{r} \int_{S_i} \text{curl } \mathbf{F} \cdot \mathbf{N} \, dS$.

[If $S_i$ is parallel to XY plane / YZ plane / ZX plane, then $\mathbf{N} = \mathbf{k}$ or $-\mathbf{k}$ / $\mathbf{N} = \mathbf{j}$ or $-\mathbf{j}$ according as the outward unit normal vector $\mathbf{N}$ is in the direction of $\mathbf{OZ}$ or $\mathbf{ZO}$ / $\mathbf{OX}$ or $\mathbf{OX}$ / $\mathbf{OY}$ or $\mathbf{YO}$ respectively. If $S_i$ is not parallel to all the co-ordinate planes, then project $S_i$ into a plane which is not perpendicular to $S_i$.]

Step 6: Verify whether the values in step 3 and step 5 are equal.

Example: Verify Stoke’s Theorem for

$$\mathbf{F} = (2x - y) \mathbf{i} - yz^2 \mathbf{j} - y^2 z \mathbf{k}$$

and $S$ is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and $C$ is its boundary.
Solution: We have to verify $\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} \text{curl} \mathbf{F} \cdot \mathbf{N} \, dS$. The boundary $C$ of $S$ in the circle $x^2 + y^2 = 1, z = 0$ in the XY-plane with parametric equations: $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$.

$\mathbf{F} = (2x - y) \mathbf{i} - yz^2 \mathbf{j} - y^2z \mathbf{k} = (2x - y) \mathbf{i} = (2\cos t - \sin t) \mathbf{i}$

$\mathbf{F} \cdot d\mathbf{r} = (2x - y) \, dx = (2\cos t - \sin t)(-\sin t) \, dt = (-2\sin t \cos t + \sin^2 t) \, dt$

$LHS = \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{2\pi} (-2\sin t \cos t + \sin^2 t) \, dt = \left[ -\sin 2t + \left(\frac{1 - \cos 2t}{2}\right) \right]_{0}^{2\pi} = \left[ \frac{\cos 2t}{2} + \frac{1}{2}t - \frac{1}{2}\sin 2t \right] = \pi$

Let $R$ be the projection of $S$ in the XY-plane

$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} = \mathbf{i}(-2yz + 2yz) - \mathbf{j}(0 - 1) + \mathbf{k}(0 + 1) = \mathbf{k}$

$\text{RHS} = \iint_{S} \text{curl} \mathbf{F} \cdot \mathbf{N} \, dS = \iint_{R} \mathbf{k} \cdot \mathbf{N} \, ds = \int_{R} \int_{R} dx \, dy$, since $\mathbf{k} \cdot \mathbf{N} \, ds = dx \, dy$

$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \, dy \, dx = \int_{x=-1}^{1} 2\sqrt{1-x^2} \, dx = 4 \int_{0}^{1} \sqrt{1-x^2} \, dx$

$= 4 \left[ \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right]_{0}^{1} = 4 \cdot \frac{\pi}{2} = \pi$

$\therefore LHS = RHS \therefore$ Stoke's Theorem is verified.
Suggested Problems:

1. Verify Stoke’s Theorem for \( \mathbf{F} = -y\mathbf{i} + x\mathbf{j} \) and \( S \) is the circular disc \( x^2 + y^2 \leq 1, z = 0 \).

2. Verify Stoke’s Theorem for \( \mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k} \) and \( S \) is the upper half surface of the sphere \( x^2 + y^2 + z^2 = 9 \) and \( C \) is its boundary.

3. Verify Stoke’s Theorem for \( \mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} \) integrated round the square in the plane \( z = 0 \), whose sides are along the lines \( x = 0, y = 0, x = a, y = a \).

4. Verify Stoke’s Theorem for \( \mathbf{F} = xy\mathbf{i} + xy^2\mathbf{j} \) where \( C \) is the square in the XY - plane with vertices \( (1,0), (-1,0), (0,1), (0,-1) \).

5. Verify Stoke’s Theorem for \( \mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} - (x + z)\mathbf{k} \) and \( C \) is the boundary of the triangle with vertices \( (0,0,0), (1,0,0), (1,1,0) \).