(DM 21)

ASSIGNMENT-1

M.Sc. DEGREE EXAMINATION, JUNE 2022.

Second Year

Mathematics

TOPOLOGY AND FUNCTIONAL ANALYSIS

MAXIMUM MARKS :30

ANSWER ALL QUESTIONS

SECTION — A

1. (a) Let X be a second countable space. Prove that any open base for X has a countable subclass which is also an open base.

   (b) Let $f: x \to y$ be a mapping of one topological space into another. Show that f is continuous if and only if $f^{-1}(f)$ is closed in X whenever F is closed in Y if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.

2. (a) Show that a subset of a metric space is bounded if and only if it is non-empty and is contained in sample closed sphere.

   (b) Are the empty set ϕ and full space 'X' in a metric space 'X' are open sets?

3. (a) Show that every separable metric space is second countable.

   (b) Prove that in any metric space X, each closed sphere is a closed set.

4. (a) Prove that a topological space is compact if every sub basic open cover has a finite sub cover.

(b) State and prove the generalized Herine-Bool theorem.

5. (a) Prove that a subspace of the real line $\mathbb{R}$ is connected if and only if it is an interval

(b) Let X be a topological space. If $\{A_i\}$ is a nonempty class of connected subspaces of X such that $\bigcap_i A_i$ is nonempty then Prove that $A = \bigcup_i A_i$ is also a connected subspace of X.

ASSIGNMENT-2

M.Sc. DEGREE EXAMINATION, JUNE 2022.

Second Year

Mathematics

TOPOLOGY AND FUNCTIONAL ANALYSIS

MAXIMUM MARKS :30

ANSWER ALL QUESTIONS

1 (a) Show that one-to-one continuous linear transformation of one Banach space onto another is a homeomorphism.

(b) Let N be a non-zero normed linear space. Then prove that N is a Banach space if $\{x \| x \| = 1\}$ is complete.

2(a) State and prove open mapping theorem

(b) State and prove the uniform boundedness theorem.

3(a) Show that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

(b) Show that a Hilbert space is finite dimensional if and only if every, complete orthonormal set is a basis.

4(a) State and prove Gram-Schmidt orthogonalization process.

(b) If a normed space X is reflexive, then show that it is a Banach space.

5(a) If T_1 and T_2 are normed operators such that each commuter with the adjoint of the operator, then show that $T_1 + T_2$ and $T_1 T_2$ are normal.

(b) Define an orthonormal set in a Hilbert space H. Prove that e_n is the sequence with 1 on the n[th] place and O's elsewhere then $\{e_1, e_2, ... e_n ... 1\}$.

ASSIGNMENT-1

M.Sc. DEGREE EXAMINATION, JUNE 2022.

Second Year

Mathematics

MEASURE AND INTEGRATION

MAXIMUM MARKS :30

ANSWER ALL QUESTIONS

1. (a) Let $\{A_n\}$ be a countable collection of sets of real numbers. Then show that $m^*(\bigcup A_n) \leq \sum m^* A_n$. Hence show that the set [0, 1] is not countable.

   (b) Show that the interval (a,∞) is measurable.

2. (a) Prove that the collection $\mathcal{M}$ of measurable sets in a $\sigma-$aglebra.

   (b) Prove that the outer measure of an interval is its length.

3. (a) State and prove the Egoroff's theorem.

   (b) Let C be a constant and f and g two measurable real valued functions defined on the same domain. Then prove that the functions $f+c$, cf, $f+g$, $g-f$, fg an also measurable.

4. (a) State and prove the dominated convergence theorem.

(b) Let f be a non negative function which is integrable over a set E. Then prove that given $\in > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$ we have $\int_A f < \in$.

5. (a) State and prove Lebesque convergences theorem.

(b) Show that if f is integrable over E, then so is $|f|$ and $\left|\int_E f\right| \leq |f|$. Does the integrability of $|f|$ imply that of f?

ASSIGNMENT-2

M.Sc. DEGREE EXAMINATION, JUNE 2022.

Second Year

Mathematics

MEASURE AND INTEGRATION

MAXIMUM MARKS :30

ANSWER ALL QUESTIONS

1 (a) State and prove the vitali lemma.

(b) Prove that a function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real valued functions on $[a, b]$.

2 (a) If f is integrable on $[a, b]$ and $\int_a^x f(t)\,dt = 0$ for all $x \in [a, b]$, then prove that $f(t) = 0$ a.e. in $[a, b]$.

(b) State and prove Holder inequality.

3 (a) Prove that the L^p spaces are complete.

(b) Let f be an integrable function on $[a, b]$ and suppose that $F(x) = f(a) + \int_a^x f(t)\,dt$. Then prove that $F'(x) = f(x)$ for almost all x in $[a, b]$.

4 (a)Define measure on an algebra $\mathcal{A}$ and prove that if $A \in \mathcal{A}$ then A is measurable with respect to μ^*.

(b) If $E_i \in B$, μ, $E < \infty$ and $E_i \supset E_{i+1}$, then show that $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n$.

5 State and prove the Radon-Nikodym theorem.

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ASSIGNMENT-1

M.Sc. DEGREE EXAMINATION, JUNE 2022.

Second Year

Mathematics

ANALYTICAL NUMBER THEORY AND GRAPH THEORY

MAXIMUM MARKS :30

ANSWER ALL QUESTIONS

SECTION A

1. (a) For all $x \geq 1$ show that $\sum_{n \leq x} d(x) = x \log x + (x - 1)\, x + 0(\sqrt{x})$ for the Euler's constant.

   (b) If $n \geq 1$, then we have $\log n = \sum_{d/n} \wedge (d)$.

2. (a) State and prove Euler' summation formula.

   (b) State and prove the Abel's identity.

3. (a) For $x \geq 2$, show that

   $$\theta(x) = \pi(x) \log x - \int_{2}^{x} \frac{\pi(t)}{t} dt .$$

   (b) State and prove Shapiro's Tauberian theorem.

4. (a) State and prove Selberg's asymptotic formula.

   (b) What is prime number theorem? Explain in detail.

5. (a) Prove that a connected graph G is an Euler graph if and only if all vertices of G are of even degree.

   (b) If a graph has exactly two vertices of odd degree, show that there is a path joining these vertices.

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ASSIGNMENT-2

M.Sc. DEGREE EXAMINATION, JUNE 2022.

Second Year

Mathematics

ANALYTICAL NUMBER THEORY AND GRAPH THEORY

MAXIMUM MARKS :30

ANSWER ALL QUESTIONS

1(a) Prove that every tree has either one or two centers.

(b) Draw a graph that has a Hamiltonian path but does not have a Hamiltonian circuit.

2(a) Prove that the ring sum of two cut-sets in a graph in either a third cut-set or an edge disjoint union of cut-sets.

(b) Show that a graph with n vertices, $(n-1)$ edges and no circuit is connected.

3(a) Prove that the Kuratowski's second graph is non-coplanar.

(b) Show that every circuit has an even number of edges in common with any cut-set.

4Show that the complete graph of five vertices is non-planar.

5 (a) What is vectors and vector spaces? Explain the vector space associated with a graph G.

(b) Prove that the ring sum of two circuits in a graph G is either a circuit or an edge disjoint union of circuit.

ASSIGNMENT-1

M.Sc. DEGREE EXAMINATION, JUNE 2022.

Second Year

Mathematics

RINGS AND MODULES

MAXIMUM MARKS :30

ANSWER ALL QUESTIONS

1. (a) Show that in any distributive Lattice, we also have the dual distribution Law

   $a \vee (b \wedge C) = (a \vee b) \wedge (a \vee C)$.

   (b) The homomorphism $\phi : R \to S$ is an isomorphism if and only if there exists a homomorphism $\psi : S \to R$ such that $\phi \partial \psi$ is an automorphism of $'S'$ and $\psi \circ \phi$ is an automorphism of R.

2. (a) There is one-to-one correspondence between the ideals K and the congruence relations θ of a ring R such that $r - r' \in K \Leftrightarrow r\theta\, r'$.

   This is an isomorphism between the lattice of ideals and the lattice of congruence relations.

   (b) Prove that every proper right ideal in a ring is contained in a maximal proper right ideal.

3. (a) A module is Noetherian if and only if every submodule is funitely generated.

(b) Verify that $Hom_R(A,B)$ is an Abelian group.

4. (a) If the ideal A is contained in the prime ideal B, then there exist minimal elements in the set of all prime ideals P such that $A \subset P \subset B$.

(b) Prove that every ring is a sub direct product of sub directly irreducible rings.

5. (a) Prove that every equivalence class of fractions contains exactly one irreducible fraction, and this extends all fractions in the class.

(b) Prove that, If R is a commutative ring then $Q(R)$ is regular if and only if R is semiprime.

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ASSIGNMENT-2

M.Sc. DEGREE EXAMINATION, JUNE 2022.

Second Year

Mathematics

RINGS AND MODULES

MAXIMUM MARKS :30

ANSWER ALL QUESTIONS

1(a) Prove that the ring R is primitive if and only if there exists a faithful irreducible module A_R.

(b) Prove that R is a Prime ring if and only if $1 \neq 0$ and, for all $\alpha \neq 0$ and $b \neq 0$ in R, there exists $r \in R$ such that $arb \neq 0$.

2(a) Prove that, the following conditions concerning the ring R are equivalent.

(i) O is the only nilpotent ideal of R.

(ii) O is an intersection of prime ideals, that is rad $R = 0$.

(b) Prove that R is semiprimitive if and only if it is a sub direct product of primitive rings.

3(a) If $A \subset BCC$, Show that A is large submodule of C if and only if A is large submodule of B and B is a large submodule of C.

(b) Prove that the radical of a right Aritinian ring is nilpotent.

4(a) Prove that, Every module is isomorphic to a factor module of a projective module.

(b) Show that a ring is right hereditary if and only if every submodule of a projective module is project.

5(a) If R^F is a free module then prove that F_R^* is injective.

(b) Prove that, Every R-module is injective if and only if R is completely reducible.