MATHEMATICAL METHODS

M.A. ECONOMICS
Semester – I, Paper – V

Director, I/c.
Prof. V. Venkateswarlu
M.A., M.P.S., M.S.W., M.Phil., Ph.D.

CENTRE FOR DISTANCE EDUCATION
ACHARYA NAGARJUNA UNIVERSITY
Nagarjuna Nagar 522 510

Ph: 0863-2346222, 2346208
0863- 2346259 (Study Material)
Website www.anucde.info
E-mail: anucdedirector@gmail.com

MATHEMATICAL METHODS

MODULE 1:

M.A. ECONOMICS
SEMESTER -I
PAPER V: MATHEMATICAL METHODS
Concept of Function, Types of Functions
function -Limit and continuity of a function
Applications in Economics.

MODULE 2:

Graphical Representation of
Concept of Straight line
Concept of derivative - Rules of differentiation - Interpretation of revenue, cost, demand, supply, functions; Elasticities and their types

MODULE 3:

Multivariable functions; Rules of partial differentiation; Problems of maxima and minima in single and multiple variables; Simple problems in market equilibrium;

MODULE 4:

Total derivatives, Indifference curve analysis etc., Concept of integration; Simple rules of integration; Application to consumer's surplus and producer's surplus.

MODULE 5:

Matrix Theory Matrices and Determinants Inverse of Matrix
Cramer's Rule to Solve the System of Simultaneous Equation System
Output analysis; Meaning-Types-Assumptions; Review exercises.

REFERENCE:

1. Vohara, Quantitative Techniques, Tata McGraw Hill, 2nd ed., 2001.
2. D.C.Sancheti and V.K.Kapoor, Business Mathematics.
3. S.C. Gupta Fundamentals of statistics.
4. K.Chandra Sekhar, Business of statistics.

MATHEMATICAL METHODS

(Table of Contents)

LESSON - 1
FUNCTIONS: THEIR TYPES

Lesson Outline

After reading this lesson, you will be able to:

i) Explain what a set is, what a Cartesian product is, and their importance;
ii) Discuss in detail the concepts of relation and function;
iii) Understand various types of 'Polynomial Functions' and their applications in economics;
iv) Write equations for various types of functions and draw their graphs;

1.1 INTRODUCTION

One of the most important applications of mathematical methods in economics is functions. This is because whenever two variables are related in a cause-and-effect manner, we must recognize that there is a functional relationship between the two variables. We know that the utility of a good depends on the quantities of the good consumed. The demand and supply of a good depend on its price. The cost of production of a good depends on its level of output. The profits of a firm depend on the cost of production and revenue of that good. In this way, all these economic variables are functionally related to each other. To understand the concept of functional relationship between variables, we need to understand the concept of a function. But to understand functions and their various types, we need to know what a set is, the concepts of Cartesian product, relation, and function. Therefore, before undertaking the concept of functions and their applications in economics, let's briefly understand these concepts.

1.2 CONCEPTS OF SET, CARTESIAN PRODUCT, RELATION

We know that a set is a well-defined and well-distinguished collection of objects. Examples of sets are 'the set of all positive integers', 'the set of all negative integers', 'the set of vowels in English alphabets', 'the set of numbers obtained on the faces by rolling two unbiased dice numbered from 1 to 6', etc.

As indicated in the last example, if there are two different sets obtained through two different experiments, we can form ordered pairs by taking the first element from the first set and the second element from the second set. These are also called 'ordered pairs'.
This set containing ordered pairs is called the Cartesian product set - P.

Example:
The numbers in the first set are X={1,2,3,4,5,6}
Y={1,2,3,4,5,6}

Based on these two sets, the Cartesian product set (P) can be formed as follows:
P = X Y =
(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)
{(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)}
((4, 1) (4, 2) (4,3) (4, 4) (4, 5) (4,6)}
(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6))
((6, 1) (6, 2) (6,3) (6, 4) (6, 5) (6,6))

The above Cartesian product set contains 6×6=36 ordered pairs. These 36 ordered pairs can be represented as dots or points in a four-quadrant X-Y plane. From this entire product set, subsets can be derived based on certain conditions. Ordered pairs that satisfy the given condition are a part or a subset of the Cartesian product set. This subset is called a relation. For example, let's take the condition that the sum of the values on the first die and the second die must be greater than 9. In symbols, this can be stated as x+y>9. The ordered pairs that satisfy this given condition are (4,6), (5,5), (5,6), (6,4), (6,5) (6,6). Let's consider these ordered pairs as a set and call them a relation. This relation is a subset of the Cartesian product set.

$P=X*Y$
$R\subseteq P=\{X*Y\}/x+y>9=\{(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\}$
In the above example, except for the X-value 4, for all other X-values, more than one Y-value is associated. For example, for the X-value of 5, there are two Y-values like (5, 5), (5, 6). Similarly, for the X-value of 6, there are three Y-values like (6,4), (6, 5), (6, 6).
Let's take another condition, X=Y. The ordered pairs that satisfy this condition are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6). This is a subset of the Cartesian product set.

$R\subseteq P=\{X*Y\}/x=y=\{(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)\}$
These ordered pairs can be represented in a four-quadrant X-Y plane.
Let's discuss the nature and importance of this relation, which is different from the previous relation.

1.3 FUNCTION: CONCEPT, TYPES

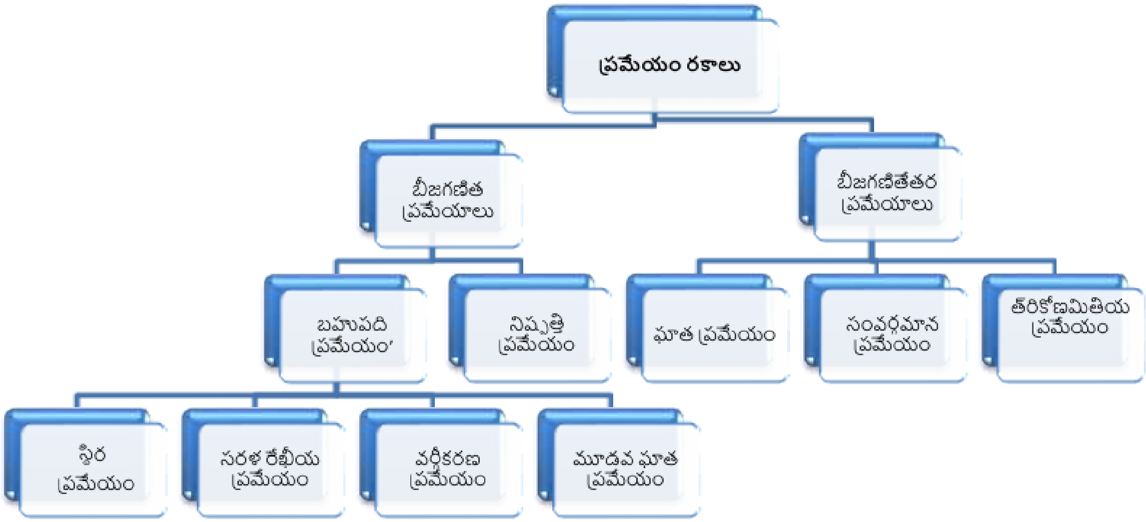
In the above relation, which is a subset of the Cartesian product set, every Y-value is associated with one X-value. This subset of the Cartesian product set is called a "Function". Therefore, in a relation, if for every X-value or for more than one X-value, there is only one (unique) Y-value, then that specific relation is called a "Function". Otherwise, it is merely a relation. So, a function is a special case of a relation. The relation in the first example is not a function. It is only a relation. On the other hand, the relation in the second example is a function. A function denoted by Y=f(X) is also called a mapping or transformation. These two terms refer to the act of associating one value with another.

In the function expression y=f(x), the letter 'f' can be interpreted as the function rule, by which the set X can be mapped (transformed) into the set Y. We can also write it as $f{:}x \rightarrow y$. The function expression y=f(x) is only a general statement relating X and Y. It does not indicate the specific rule of the function.

To explain the concept of a function, let's take the example of a flour mill. From the flour mill, we can obtain various types of products such as rice grits, rice flour, and rice vermicelli. Here, the input (X) is rice, and the output (Y) can be rice grits, rice flour, or rice vermicelli. To obtain different types of products from rice, we need to adjust and apply the wheel on the left side of the machine to different positions. The various processes, such as tightening or loosening the wheel, are called function rules. Let's examine the different types of functions that represent different rules.

Functions can be broadly divided into two categories: Algebraic Functions and Non-Algebraic Functions. The latter are also called Transcendental Functions. Algebraic functions can again be divided into 'Polynomial Function' and 'Rational Function'. The word 'Bahu' (బహు) means many. 'Padi' (పది) refers to terms. In other words, if a function has many terms, it is called a 'Polynomial Function'. If two polynomial functions are in a ratio to each other, it is called a "Rational Function". The word 'Rational' is used only in the sense of 'ratio'. It does not convey any meaning related to 'rationality in human behavior' which we discuss extensively in economics. Transcendental functions take three forms: Exponential Function, Logarithmic Function, and Trigonometric Function. A polynomial function is a common type of function. It takes many forms such as Constant Function, Linear Function, Quadratic Function, and Cubic Function. These different types of functions are systematically represented in the figure below:

Figure - 1.1
Types of Functions
[Flowchart/Diagram showing the classification of functions]



- Functions
  - Algebraic Functions
    - Polynomial Function
      - Constant Function
      - Linear Function
      - Quadratic Function
      - Cubic Function
    - Rational Function
  - Non-Algebraic Functions (Transcendental Functions)
    - Exponential Function
    - Logarithmic Function
    - Trigonometric Function

1.3.1 POLYNOMIAL FUNCTION

$Y = a + aX1 + a2X2\text{^}2 + a3X3\text{^}3 + a4X4 + ...$

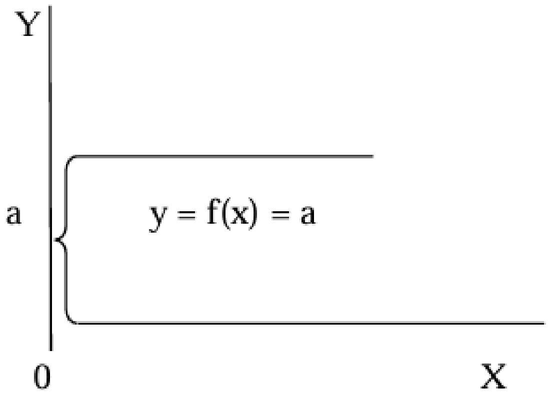
In this, each term has a coefficient ao, a1, etc., and a variable with a non-negative positive integer exponent. Based on the highest exponent of the variable X in the first two terms, X^0 = 1 and X^1 = X, there are four sub-categories of polynomial functions: constant function, linear function, power function, and cubic function. The highest exponent of a polynomial is called the degree of the polynomial function. Let's discuss these four types of polynomial functions along with their respective algebraic and graphical notations and their economic applications.

1.3.1.1 CONSTANT FUNCTION

A polynomial function that has only one element is called a 'constant function'. The functional relationship form of a constant function is $Y = f(X) = a$. For example, $Y = 10$ or $Y = 200$. In this function, the value of Y remains constant regardless of the values of X. In the X-Y plane, such a function appears as a straight line parallel to the X-axis. 'a' is called the intercept term. The geometric form of a constant function is shown in Figure 1.2.

Figure - 1.2 Constant Function

పటం - 1.2 స్థిర ప్రమేయం



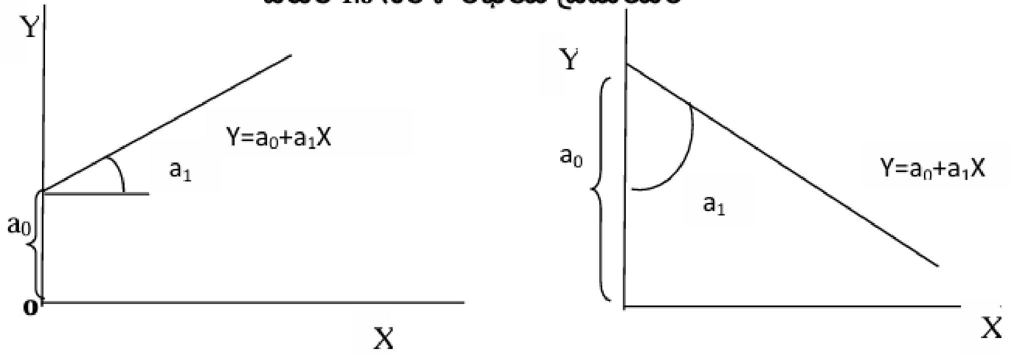
The constant function has many applications in economics. In microeconomics, perfectly elastic demand curve and supply curve are the best examples. The merger of AR and MR curves in a perfectly competitive market is another best example. Autonomous investment or government investment not dependent on interest rates is another example of a constant function from macroeconomics.

1.3.1.2 LINEAR FUNCTION

If the ratio of change between two variables is constant, then there is a linear relationship between them. A linear relationship is also called a straight line relationship. The functional relationship form of a linear function is Y = a0 + a1X. This is also called the first order of the first polynomial function. The geometric form of a linear function is given in Figure 1.3.
In the equation, a0 is called the Y-intercept or constant. a1 is called the coefficient or slope of the curve. In Figure 1.3 (A), if the slope a1 > 0, we get an upward sloping curve. If the slope a1 is negative, as in Figure 1.3 (B), we get a downward sloping curve. In both diagrams above, the coefficient values are positive as both curves appear in the first quadrant. In a later lesson, you will be able to study the equation of a straight line using the equation Y = MX + C. This will be similar to the current linear equation or linear function. By assigning different values to X and applying the function rule, we can obtain the corresponding Y-values. This is explained using a simple example.

Figure 1.3 Linear Function

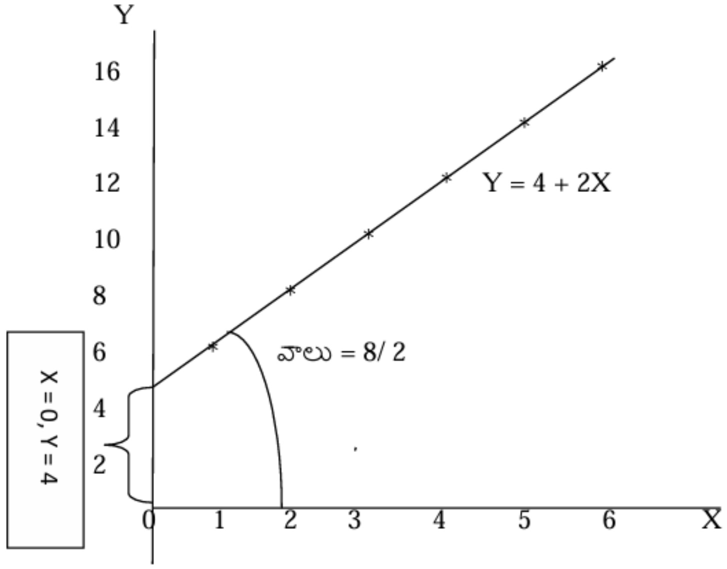
పటం 1.3 సరళ రేఖీయ ప్రమేయం



(A) (B)

Let Y = a0 + a1X, Y = 4 + 2X. As shown in the table, by substituting X values into the equation and calculating the values, we get the corresponding Y values on the left side of the equation. For example, if X = 3, Y = 4 + 2(3) = 4 + 6 = 10. Similarly, we can calculate other values of Y. In the XY plane, by plotting these X, Y values, we can obtain a straight line. As shown in the diagram below, the slope of the straight line is 2 and the intercept value (when X = 0) is 4.

X	1	2	3	4	5	6
Y	6	8	10	12	14	16



Economic Applications: The straight line has many applications in economics. In microeconomics, linear demand and supply functions, average and marginal revenue curves in a perfectly competitive market, and monopoly are the best examples of straight lines. Linear consumption function, linear saving function, linear investment function, average propensity to consume (APC), and average propensity to save (APS) are the best examples of straight lines in macroeconomics.

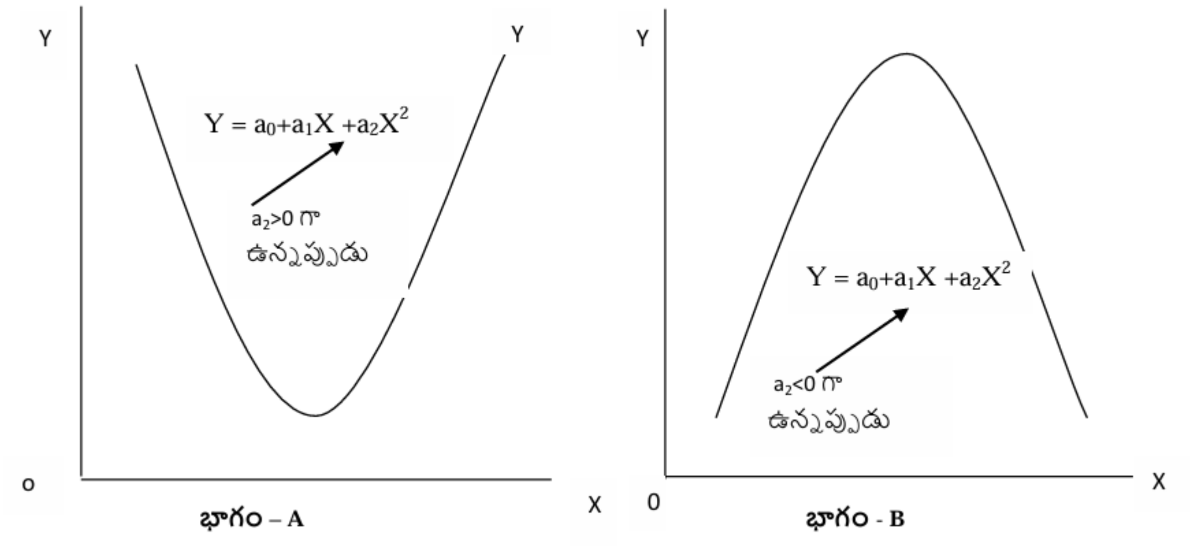
1.3.1.3 POWER FUNCTION (QUADRATIC FUNCTION)

A second-degree polynomial function like Y = aX^2 + bX + c or Y = a0 + a1X + a2X^2 is called a 'power function' in 'X'. If the power function is set to zero, it is called a quadratic equation. Like any other function, a power function only states the function rule. It takes the shape of a parabola with a single peak or valley.

The general shape of a power function curve is as shown in Part A and Part B.

Figure 1.4 Power Function

పటం 1.4 ఘాత ప్రమేయం



Example:

Let Y = X^2 + 4X - 5 be a power function. By giving different values to X and applying the specific rule of the power function, the Y-values are obtained as shown in the table below.

X	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
Y	16	7	0	-5	-8	-9	-8	-5	0	7	16

For example, when X takes the value -5,

Y = X^2 + 4X - 5 = (-5)^2 + 4(-5) - 5 = 25 - 20 - 5 = 25 - 25 = 0

Similarly, when X takes 2,

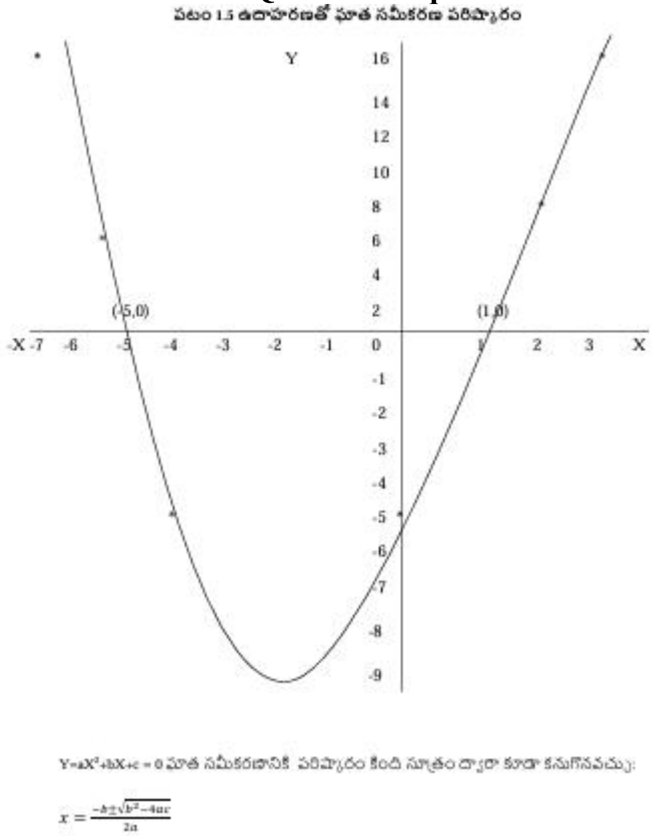
Y = X^2 + 4X - 5 = (2)^2 + 4(2) - 5 = 4 + 8 - 5 = 12 - 5 = 7

Similarly, other values are calculated. By substituting these x, y values into the diagram, we get a power curve with a parabolic shape.

As already mentioned, when the power function is equal to zero, i.e., when Y=0, the power function becomes a quadratic equation. The solution to the quadratic equation is found. Finding the solution to a quadratic equation means identifying the specific X-values when Y=0. That is, identifying the points where the parabola intersects the horizontal axis in the figure. In the function Y = f(x) = 0, there are two such intersections. These are the (X,Y) coordinate points, (1,0) and (-5,0). Therefore, the solution values for the above power function are -5 or +1.

X

Figure 1.5 Solution to Quadratic Equation with Example



-3

The solution to the quadratic equation Y = aX^2 + bX + c = 0 can also be found using the following formula:

X=2a−b±b2−4ac

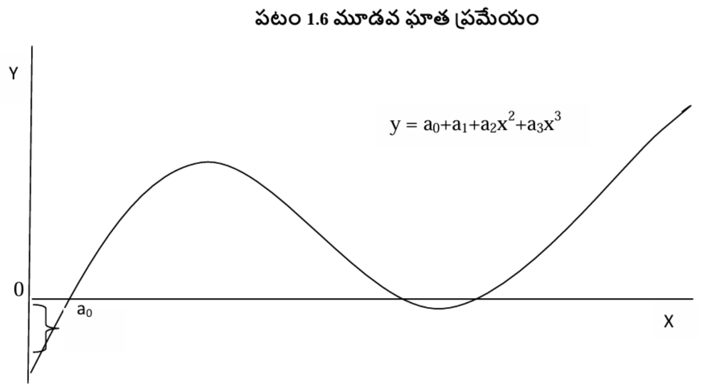
1.11: **Economic Applications**

In traditional cost theory, U-shaped average (AC) and marginal (MC) cost curves, the total product curve of a variable factor, and the average revenue curve of a factor in a perfectly competitive market are some other examples of power functions. In economics, the peak and valley points of a power function are used to determine maximum profits or maximum revenue or minimum cost, etc.

1.3.1.4 CUBIC FUNCTION

Y = a0 + a1X + a2X^2 + a3X^3 or Y = ax^3 + bx^2 + cx + d
The functional form of the function. In this, the degree of the polynomial is raised to its third level. The cubic function curve is given in Figure 1.6.

Figure 1.6 Cubic Function



Y = a0 + a1 + a2x^2 + a3x^3
X
Depending on the parameters (a0, a1, a3), signs, and magnitude of the cubic function, the shape of the cubic function curve depends on a0, a1, a3.

ECONOMIC APPLICATIONS OF CUBIC FUNCTION

Various phases of business cycles, and trends in time series of many economic variables such as income, expenditures, savings, and investment can be represented with the help of a cubic function.

1.3.1.5 RATIONAL FUNCTION

If any two polynomial functions are expressed as a ratio of one another, it is called a rational function. According to this definition, every polynomial function will necessarily be a rational function, because dividing a polynomial by a constant function of 1 also results in a rational function.

Example 1: Y = 3/X
In the above example, the rational function is expressed as the ratio of a constant function to a linear function.
Example 2: Y = (X-1) / (X^2 + 2X + 4)
In the above example, the rational function is expressed as the ratio of a linear function to a power function.
The graph of a rational function is as shown in Figure 1.7.
Figure 1.7 Graph of Rational Function
Price,
Rational Function, Y = a/X
0
Quantity of Goods

APPLICATIONS OF RATIONAL FUNCTION IN ECONOMICS

In microeconomics, a special demand curve with unit elasticity, a rectangular hyperbola, is the best example of a rational function.

Average Fixed Cost Curve: AFC x Q = TFC or Q = a/P or PQ = 'a' is another example of a rational function.

1.4 SUMMARY

In this lesson, we learned about the concepts of relations and functions. To understand the concept of function, it is necessary to understand the concepts of 'set', 'Cartesian product', and 'relation', so we briefly learned about these concepts. We saw that a function is a special type of relation in which there is only one (unique) Y-value for each X-value or for more than one X-value. In a function, X-values are transformed into Y-values through a mapping process. The term mapping refers to a specific rule.

Which transforms X-values into Y-values. Based on the mapping rule, there are many types of functions. Functions can be broadly divided into algebraic and non-algebraic functions. Algebraic functions are of two types: polynomial functions and rational functions. There are four main types of polynomial functions as applied to economics: constant function, linear function, power function, and cubic function. In this lesson, we have seen the functional forms or equations of these four types of polynomial functions along with their graphical representation and economic applications. In addition to these functions, we have also mentioned a special type of function widely used in economics, which is the rational function. We have learned in detail about its functional form, geometric representation, and economic applications.

1.5 SAMPLE EXAMINATION QUESTIONS

I. Answer the following questions in 10 lines each.

1. Define a set. Give examples for a set.
2. What is a Cartesian product set? How can you obtain it?
3. What is a relation? Give two examples.
4. Define a function. Differentiate it from a relation.

II. Answer the following questions in 30 lines each.

1. Define a function. Discuss various types of functions with examples.
2. What is a polynomial function? Discuss different types of polynomial functions with their functional forms, geometric representation, and economic applications.
3. What is a power function? Discuss its functional forms, geometric representation, and economic applications.
4. Explain the concept of a power function. Find the solution to Y = X^2 + 4X - 5 using geometric and formula methods.

1.6 GLOSSARY

1. Set: A well-defined, well-distinguished collection of elements.
2. Cartesian Product Set: A set of ordered pairs formed by taking the first element from the first set and the second element from the second set.
3. Relation: A subset of a Cartesian product set that satisfies a given condition.
4. Function: A special type of relation where there is only one unique Y-value for each X-value or for more than one X-value.

1.7 SUGGESTED BOOKS

1. Alpha Chiang: Fundamental Methods of Mathematical Economics
2. R. G. D. Allen: Mathematical Analysis for Economists
3. Mehta and Medhani: Mathematics for Economists.

LESSON - 2
LIMITS OF FUNCTIONS - CONTINUITY

Lesson Outline

2.0 EXPECTED LEARNING OUTCOMES OF THE LESSON

After successfully learning this lesson, you will be able to:
i) Understand the concept of limits;
ii) Grasp practical examples of limits;
iii) Analyze theorems of limits;
iv) Apply limit theorems to estimate common problems on limits;
v) Demonstrate the application of the concept of continuity of functions to practical situations.

2.1 INTRODUCTION:

In the previous lesson, we discussed the concept of functions derived from the Cartesian product. A function expresses the relationship between independent and dependent variables. We also saw that depending on the function rule, the graph of the function can take various shapes such as a straight line, parabola, or exponential curve. In this lesson, we will learn new concepts such as limits and continuity of functions. These concepts will help us understand a very important concept called "differentiation" of functions.

1.2 CONCEPT OF LIMIT WITH ILLUSTRATIONS:

Simply put, "the limit of a function" is the ultimate or final point of the value of the dependent variable (function) when the independent variable approaches a given value. For a student who has not studied mathematics, understanding the definition of the limit of a function is very difficult. Therefore, before giving the definition of the limit of a function, let's understand the limit through some practical examples. This will help you better understand the concept of limit when we define it.
Example - 1: Let's assume y=1−x1.
In the above function, let's substitute some hypothetical values for x and then evaluate the values of the dependent variable (y) (function).

2.3 LIMITS OF FUNCTIONS, CONTINUITY

Table - 2.1

X	1	2	3	4	5	6	∞
Y	0	0.5	0.66	0.75	0.80	0.83	→ 1

Table - 2.2

X	-1	-2	-3	-4	-5	-6	-7	→ -∞
Y	2	1.5	1.33	1.25	1.20	1.17	1.14	→ 1

As can be observed in the first table, as the x-sequence increases indefinitely from 1 through integer values, the y-sequence also increases from zero and approaches the limit (L) '1'. This corresponds to the idea that as x approaches infinity (∞), the function y=1−x1 approaches 1. In symbols, we write this as:
Lt.(y=1−x1)=1.
x→∞

In the second table, the x-sequence decreases indefinitely through integer values. So, the y-sequence decreases from 2 and approaches the limit (L) '1'. This corresponds to the idea that as the X sequence approaches negative infinity (-∞), the function y=1−x1 approaches 1. In symbols, we write this as:
Lt.(y=1−x1)=1
x→−∞

If x approaches N from 1, 2, 3... and the y-function approaches a finite number L, then we call L the left-hand limit of y. In symbols, we write this as:
Lt.(y=1−x1)=1.
x→N−

N- indicates that X-values approach N from values less than N. On the other hand, when X-values approach N from values greater than N, if y obtains the limit L, we call L the right-hand limit of y. In symbols, we write it as:
Lt.(y=1−x1)=1.

x→N+
N+ indicates that X-values approach N from values greater than N. We write it as:
Lt.(y=1−x1)=1.
x→N
only when both limits have a common value L.
In the above examples, note that both sequences have a common limit value of 1.

Example 2: Let's assume the function is y=x2+3x−2.
Let's substitute hypothetical values for x and calculate the corresponding y values as obtained below:

Table - 2.3

X	1	2	3	4	5	6	∞
Y	2	8	16	26	38	52	∞

Table - 2.4

X	-1	-2	-3	-4	-5	-6
Y	-4	-4	-2	2	8	16

As can be observed in the tables, in the case of Table-2.3, as X approaches infinity, the corresponding y values also approach infinity. Similarly, as can be seen in Table-2.4, when given a sequence of values where x decreases towards negative infinity, the corresponding y-values also approach infinity. Therefore, when x approaches infinity or negative infinity, the function y=x2+3x−2 approaches infinity. In other words, in both cases, the y sequence does not reach any finite number L. So, it can be said that the function y=x2+3x−2 does not have an L-limit.

Example 3:
Let's assume the function is y=x23. We substitute hypothetical values for x and calculate the corresponding y values as obtained below:

2.5 Tables

Table - 2.5

X	1	2	3	4	5	∞
Y	3	1.5	1	0.75	0.60	0

Table - 2.6

X	1	1/2	1/3	1/4	1/5	1/6	1/7	0
Y	3	6	9	12	15	18	21	∞

In Table - 2.5, as the X values increase from one to infinity, the corresponding y values gradually decrease from 3 and approach zero. Similarly, in Table - 2.6, as the x values decrease from 1 to zero, the corresponding y-values approach infinity.
Lt.(y=x3)=0.

$x \to \infty$
$Lt.(y=x3)=\infty$.
$x \to 0$

It should be noted that infinity is not a number and we cannot calculate it. To calculate the limit of a sequence and apply it to other mathematical applications, it is essential to express the idea of the limit in finite quantity. For example, in the function $y=x1+1$, when $x \to \infty$, the limit of the function is 1. In this limit process, the next number after 1 is 2, the next number after 2 is 3, and so on. If we are dealing with Integers, we know what the next number is. But if we are dealing with the Real Number System, we don't know what the next number is in the real number system. This is because in the real number system, which takes all possible values including decimal points or fractions, we don't know what the next number is. The reason for this is that the real number system is dense and also uncountable. To overcome this problem, mathematicians of the 18th century photographed this moving process and obtained a still image. They analyzed this still image of the moving process in terms of finite quantities.

2.3 FORMAL DEFINITION OF LIMIT:

A function $f(x)$ has a limit if, for every given $\epsilon>0$, no matter how small, we can find a positive δ dependent on ϵ such that for all values of x in the interval $|x-x1|<\delta$, the inequality $|f(x)-L|<\epsilon$ is satisfied, except possibly at the point $x=x1$.

2.4 ILLUSTRATIVE DEFINITION

Let's assume a rocket, $f(x)$, is approaching the moon (L). Let's assume the rocket, $f(x)$, can reach the moon (L) at 6:00 PM (x1). Then the difference $|x-x1|=\delta$ can be considered as the remaining time for the rocket $f(x)$ to land on the moon (L).
Let's assume we want to take a picture of the rocket at a distance of $\epsilon=10{,}000$ miles from the moon (L). $|f(x)-L|$ represents the distance between the rocket and the moon. Then this is ($|f(x)-L|=10{,}000$ miles).

How many minutes before 6:00 PM (x1) should the picture be taken? If it is 5 minutes, then the time difference is $|x-x1|=\delta=5$ minutes.
If the distance is even smaller, at $\epsilon=10$ miles, then $\delta=2$ seconds. If the distance is $\epsilon=1$ mile, then the time will be $\delta=0.4$ seconds. It should be understood that the closer the picture is taken to the moon, the clearer the picture will be.
Even though this is a moving process, since we want to take a photo, the rocket ($f(x)$) and the moon (L) will end up as a static picture at a distance of ϵ during time δ, determined by time (δ) and distance (ϵ) in this manner.

2.5 LIMIT THEOREMS:

There are many theorems on limits. When we evaluate the limit of a function, we can use some well-established or well-proven limit theorems. These theorems can actually simplify the task of evaluating many complex limits.

2.5.1 Theorems involving a single function:

When we have a single function, $y=f(x)$, the following limit theorems can be used.

2.5.1.1 Theorem - I:
If y=ax+b, then
Lt.y=aN+b
x→N
Example: Let's assume y=5x+7.
Lt.y=a+b=5(2)+7=10+7=17
x→2

2.5.1.2 Theorem - II:

If y=f(x)=b, then
Lt.y=b
x→N
According to this theorem, the limit of a constant function is that constant. It should be noted that this theorem is a special case of the first theorem when a=0.
Example: Let's assume y=7.
So, y=7
x→0

2.5.1.3 THEOREM - III:

If y=f(x)=y=x, then

Lt.yx→N=N
Example: If y=f(x)=y=x3, then
Lt.y=23=8
x→2
If y=f(x)=y=xk, then
Lt.y=Nk
x→N

2.5.2 THEOREMS INVOLVING MORE THAN ONE FUNCTION

If two functions, y1=f(x) and y2=g(x), related to the same independent variable X, have limits L1 and L2 when x→N, then the following theorems apply:

2.5.2.1 THEOREM IV - SUM-DIFFERENCE LIMIT THEOREM:

This theorem states that the limit of the sum or difference of two functions is the sum or difference of their respective limits. That is,
Lt.(y1+y2)=L1+L2
x→N
Lt.2y1=Lt.(y1+y1)=L1+L2=2L1
x→N

2.5.2.2 THEOREM V - LIMIT PRODUCT THEOREM:

The limit product theorem states that the limit of the product of two functions is the product of their respective individual limits. That is,
Lt.(y1·y2)=L1·L2

x→N
Applying this to the square of a given function:
Lt.(y1·y1)=Lt.(y1)2=L1·L1=L12
x→N

2.5.2.3 THEOREM VI - QUOTIENT LIMIT THEOREM:

The limit of the quotient of two functions is equal to the quotient of their respective individual limits. Naturally, the limit L2 is restricted to be non-zero. Otherwise, the quotient is undefined.
Lt.(y2y1)=L2L1 where L2□=0
x→N

2.5.2.4 THEOREM VII - POLYNOMIAL LIMIT:

We know that a polynomial is a function that has a single independent variable raised to a power of n. The following is a general polynomial function:

Y=a0+a1x1+a2X22+a3x33+a4X44+⋯+anXnn
The limit of this polynomial is

Lt.[f(x)±g(x)±h(x)±........]

2.6 PRACTICAL EXAMPLES OF LIMIT THEOREMS:

Let's look at some common numerical examples of limits. In these examples, some can be solved directly, while others require the use of limit theorems.

2.6.1 Example - 1.
Find the limit of the function y=2+x1+x as x→0.
Solution: By substituting the value 0 for X and then simplifying, we get:

Lt.2+x1+x as x→0=2+01+0=21=0.5.

2.6.2 Example - 2.

Find the limit of the function y=1−x1−x2 as x→1.
Solution: By directly substituting the value I for x in the above equation, we encounter the problem of division by zero.

Lt.1−x1−x2 as x→1=00
This is an indeterminate form.
Therefore, we cannot allow x=1. The appropriate method here is to modify the given equation in the denominator. So, x→1 implies x□=0. Hence the term (1−x) is not zero, and dividing after modifying the given equation is a legitimate and correct method as follows:
Lt.1−x1−x2 as x→1=Lt.1−x12−x2 as x→1=Lt.(1−x)(1+x)(1−x)=Lt.(1+x)=1+1=2.
x→1

2.6.3 Example - 3.

y=x+12x+5

Find the limit of the function as $x \to \infty$.
Solution: The variable X appears in both the numerator and the denominator. If we allow X to be in both the numerator and the denominator, the result will be a ratio between two infinitely large numbers. This does not convey a clear meaning. So, first divide the numerator by the denominator so that X does not appear in the numerator.
Dividing 2x+5 by x+1 gives us the following result:
x+1)2x+5(2 2x+2 Changing the signs for both terms gives us (-) (-)
3

So, y=2+x+13
Now, applying the limit to the function:
Lt.(y)=Lt.(2)+Lt.(x)+Lt.(1)Lt.(3)
as $x \to \infty$ as $x \to \infty$ as $x \to \infty$ as $x \to \infty$
$2+(3/\infty)=2+0=2$ we get
So the limit for this function is 2.

2.6.4 Example 4. Find the limit of the function y=x−2x2−4 when the value of x approaches

Solution: This is a common problem. We can solve it by using factorization of the numerator or by using the formula a2−b2=(a+b)(a−b). The numerator can be written as:

y=x−2x2−22
Using the formula a2−b2=(a+b)(a−b), it can be written as:

y=(x−2)(x+2)(x−2)
Canceling (x-2) in both the numerator and the denominator, we get (x+2) in the numerator. Applying the limit to the above term as $x \to 2$:
Lt.(x+2)=Lt.(x)+Lt.(2)=2+2=4
$x \to 2$ $x \to 2$ $x \to 2$

2.6.5 Example - 5.

Find the limit of the function y=x2−1x2+1 when the value of X approaches ∞**.**

Solution: Directly applying the limit rule to the problem suggests division between two large numbers (∞), which does not give a meaningful result. Some mathematical tricks need to be done to make x not appear in the numerator or denominator. If all terms in the function are divided by x2:

y=x2/x2−1/x2x2/x2+1/x2
We get y=1−1/x21+1/x2.
When x approaches infinity, applying the limit to all terms in the numerator and denominator:
Lt.(y)=Lt.(1)−Lt.(1/x2)Lt.(1)+Lt.(1/x2)=Lt.(y)=1−01+0
$x \to \infty$
Because x is infinitely large, the limit of 1/x2 is equal to zero.

2.6.6 Example 6.

Find the limit of the function y=x−3x2−5x+6 when the value of X approaches 3.

Solution: We can solve this problem by factoring the terms in the numerator and then applying the limit to all terms in the numerator and denominator when the value approaches 3.

y=x−3x2−5x+6
y=x−3x2−2x−3x+6

y=x−3x(x−2)−3(x−2)

y=(x−3)(x−2)(x−3)
Canceling (x-3) in both the numerator and denominator, we are left with only (x-2). When the value approaches 3, we apply the limit to (x-2). That is, by substituting 3 in place of x:
Lt.y=Lt.(x−2)=3−2=1.
x→3 x→3

2.6.7 Example - 7.
Find the limit of the function y=x2−x−64x2−17x+15 when the value of x is near 3.
Solution: By factoring the terms in both the numerator and denominator, and then applying the limit to all terms in the numerator and denominator,

y=x2−x−64x2−17x+15=x2−3x+2x−64x2−12x−5x+15
Take 4x as a common term in the first two terms of the numerator, and -5 in the next two terms. Similarly, take X as a common term in the first two terms of the denominator, and 2 in the next two terms.
y=x(x−3)+2(x−3)4x(x−3)−5(x−3)
y=(x+2)(x−3)(4x−5)(x−3)
Canceling (x-3) in both the numerator and denominator, we get y=(x+2)(4x−5).
When the value is near 3, applying the limit to all terms of the function, numerator, and denominator of y=(x+2)(4x−5):

Lt.y=Lt.(x+2)Lt.(4x−5)
--- PAGE 1 ---
Acharya Nagarjuna University
Lt.y=(3+2)(4(3)−5)
x→3 x→3

Lt.y=((5)(12−5)=7/5.
2.12)
Centre for Distance Education 3
2.6.8 Example 8: When x is 2, find the limit of the function y=4x2+3x−10.
Solution: Since the value of X approaches 2, directly apply the limit to the function terms.

Lt.(y)=Lt.(4x2+3x−10)x→2x→2

=Lt.4(x2)+Lt.3(x)−Lt.(10)x→2x→2x→2

=4(22)+3(2)−10

=4(4)+3(2)−10

=16+6−10=22−10=12.
2.6.9 Example - 9. When x→ 1, find the limit of the function y=[(2x2+4x+1)(x−4)].

Solution: When x→1, applying the limit to the function y=[(2x+4x+1)(x−4)], we get:
Lt.(y)=Lt.[(2x2+4x+1)(x−4)]
x→1 x→1

= =[Lt.2(x2)+Lt.4(x)+Lt.(1)].[Lt.(x−4)]x→1x→1

=Lt.2(12)+Lt.4(1)+Lt.(1).Lt.(1−4)

=[2(1)+4(1)+1][−3]

=[2+4+1][−3]

=7(−3)=−21
2.6.10 Example 10. Assume that the interest rate charged depends on the amount of capital borrowed. However, there is a specific minimum interest rate at the lower limit, which will never decrease. Let the minimum interest rate be two percent. Then the interest rate function can be written as:

r=2++Ka
Here r is the interest rate, K is the borrowed capital, and a is the minimum interest rate that never decreases (constant). When the total borrowed capital approaches infinity, the function can be written as:
Lt.(r)=Lt.2+Lt.(Ka)
K→∞=K→∞ K→∞
=2+0 (because a is a constant)

2.7 APPLICATIONS OF DIFFERENTIATION TO EVALUATE LIMITS

Sometimes, the concept of differentiation is used to estimate the limit of a specific function. We will learn in a later lesson that when a function reaches a finite value, the expected limit value of the function's Difference Quotient is its derivative. Before introducing and understanding the concept of differentiation of a function involving one independent variable or more than one independent variable, and their rules, we present the three basic rules of differentiation without proof:
(Power Function) , n.n(n−1)

i. y=xn
ii. The derivative of a constant is zero.
iii. If a power function is multiplied by a constant, i.e.,
y=c.xn, the derivative is next-1)
Let's use these three basic rules of differentiation to estimate the limits of functions.
2.7.1 Example 11: Evaluate:
Lf.(x−a)(xn−an)
xa

Solution: Directly applying the limit to the given function results in division by zero. So, let's first differentiate the function terms and then apply the limit. Differentiating the numerator and denominator terms of the function, we get:

dxd(x)−dxd(a)dxd(xn)−dxd(an)
d

The derivative of x" means it passes to the minimum and indicates change.
It becomes zero. The derivative of x or x' in the special case of x is 1 (because it applies the rule n.n(n·1)=1.1(1·1)=10=1). Thus,

(1−0)(nxn−1−0)=1nxn−1=nxn−1
Now, applying the limit to the above function,

Lt.(nx^{n-1})=na^{(n-1)}\\ x\rightarrow a\end{aligned}
2.7.2 Example 12: Evaluate:

Lt.(x−3)(x3−27)
x 3

Solution: Directly applying the limit to the function mentioned above results in division by zero. So, let's first differentiate the function terms and then apply the limit. By differentiating the terms in both the numerator and denominator using the method discussed above,

=dxd(x)−dxd(3)dxd(x3)−dxd(27)
=(1−0)(3x3−1−0)=3x3−1=3x2.
Now, in the context of x→3, applying the limit to the function 3x',
=3.(3)2=3(9)=27.

2.8 CONTINUITY OF A FUNCTION:

The concept of the limit of a function and its evaluations are closely related to the concept of "continuity of a function." The concept of "differentiation," which is widely used in economic applications, is very useful for understanding the continuity of a function.

2.8.1 Continuity - Definition

A function, y=f(x), is said to be continuous at a point N in the domain of x (the set of values X takes in a given context) when, as x approaches N, it has a limit, and when x=N, this limit is also equal to f(N) (equal to the value of the function (y)). More specifically, the term "continuity" must satisfy three conditions.
i) The point N must be in the domain of the function;
ii) When x→N, the function must have a limit;
iii) The limit must be equal to the value of f(N).

When discussing the limit of a function, it is appropriate to note that the ordered pair (N, L) was not considered. But now in the discussion of function continuity, we have specifically included this ordered pair of points. In fact, in the third condition of continuity, it is stated that for the function to be considered continuous at that specific point N, the graph of the function must contain the ordered pair (N,L). To explain the concept of continuity of a function, let's consider a set of four diagrams. Let's see which of these satisfies the conditions for function continuity.

As observed from Figure 2.1 (a), all the necessary conditions for function continuity are met at point N. Point N is in the domain of the function. When x→N, the function y has a limit L. The limit is also equal to the value of the function at N. Thus, the function indicated by that curve is continuous at N.

Similarly, the function indicated in Figure 2.1 (b) is also continuous at N. This is because L is the limit of the function as X approaches the value N in the domain of the function. Furthermore, L is also the value of the function at N. For a function to be continuous, the curve does not need to be smooth at x = N. As shown in Figure 2.1(b), it has a sharp point. Nevertheless, it is considered continuous.

The function curve shown in Figure 2.1(c) encountered discontinuity at N. This is because there is no limit at that point. Therefore, the second condition required for continuity at this point is violated. However, in the domain of the function (0,N), as well as in the domain after discontinuity (N, ∞), the conditions for continuity are met. But there is a discontinuity in the range of the function's Lp, L_2 spreads.

Clearly, the function indicated in Figure - 2.1(d) is discontinuous at x= N. In this case, discontinuity occurred because N was excluded from the function's domain, contrary to the first condition of continuity.

From the previous discussion, it is clear that a sharp point, as in Figure 2.1(b), is consistent with continuity. But gaps in the function's domain or range are not acceptable and are not consistent with the function's continuity.

LESSON – 3
STRAIGHT LINE - ITS APPLICATIONS IN ECONOMICS

Lesson Outline

3.0 EXPECTED OUTCOMES OF THE LESSON

After successfully completing this lesson, you will be able to:

i) Understand the concepts of a linear function, a straight-line equation, and the intercept and slope;

ii) Know how to construct a straight line that passes through given points;

iii) Analyze the linear demand curve, linear supply curve, and equilibrium state;

iv) Apply the linear function of the average revenue curve and marginal revenue curve in microeconomics;

v) Demonstrate the applications of concepts like the average consumption function, average savings function, investment function, etc., in macroeconomics.

3.1 INTRODUCTION

In the second lesson, we learned in detail about the concepts of the limit and continuity of functions. These concepts help us understand a very important concept called the derivative of functions. The most important among polynomial functions is the linear function. A linear function has many applications in economics. In this lesson, we will learn in detail about the concept of a linear function, its construction, and its many applications in economics.

3.2 THE STRAIGHT LINE

In the equation ax+by+c=0, if a, b, c are constants, that equation is a first-degree equation. We can derive functions from it:

y=b−ax−bc

x=a−by−a−c

All the points that satisfy these equations lie on the same straight line. Therefore, the graph of the equation ax+by+c=0 is a straight line. Furthermore, it can be proven that the equation for any straight line is in this form. The equation ax+by+c=0 can be called the straight-line equation or the general form of a straight line. For example, if 2x+3y−7=0 is a straight-line equation, then y=2x+3 and x=3y−5 are also straight-line equations. An equation for a straight line can be in various forms depending on the space it occupies.

3.2.1 INTERCEPT OF THE LINEAR CURVE

If a straight line intersects the x and y axes at points A and B, we call OA and OB the "intercepts" that the straight line makes on the x and y axes.

Figure 3.1: Both a and b are positive

If OA = a and OB = b, it can be proven that the equation for the straight line AB is ax+by=1. This type of equation is called the intercept form of an equation. For example, if a = 5 and b = 7, the equation is 5x+7y=1. That is, 7x+5y−35=0. These intercepts a, b do not necessarily have to be positive as shown in Figure 3.1. One or both of a and b can also be negative. Study the following diagrams.

Figure -3.2: In the intercepts, a is positive, b is negative

Here, as shown in Figure 3.2, the intercepts a is positive and b is negative.

Figure 3.3: In the intercepts, 'a' is negative, 'b' is positive.

As shown in Figure 3.3, in these "intercepts," 'a' is negative and 'b' is positive.

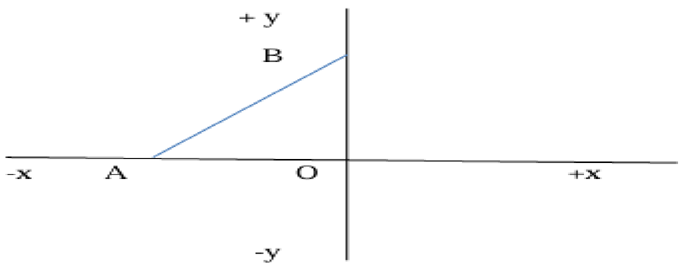
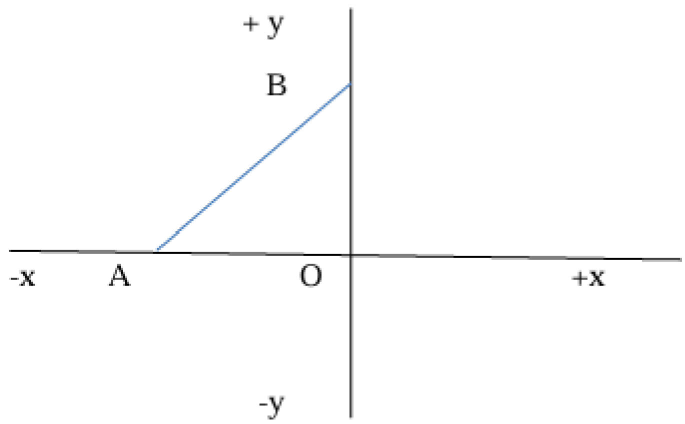


Figure 3.4: In the intercepts, both 'a' and 'b' are negative.
As shown in Figure 3.4, both 'a' and 'b' intercepts are negative.



In economics, negative values are generally meaningless. For example, there is no negative demand, negative supply, or negative investments. Therefore, the x and y intercepts are both positive and lie in the first quadrant.

3.2.2 (Slope of the Linear Curve)

Let's consider two points P and Q on the straight line explained above. Assume that the lines drawn parallel to the X-axis from P and parallel to the Y-axis from Q intersect at N. Then, the vertical distance NQ indicates how high the straight line AB is above the horizontal distance PN. Therefore, the ratio NQ/PN measures the height of the straight line AB for one unit of horizontal distance. We should remember the fact that this ratio remains constant no matter where the points P and Q are taken on the straight line AB. For example, let's take two other points P1 and Q1 on the straight line AB. Assume that the lines drawn parallel to the X-axis from P1 and parallel to the Y-axis from Q1 intersect at N1. Then, the triangles PQN and P1Q1 N1 are similar triangles. Therefore, it can be proven that the ratio NQ/PN = N1Q1/P1N1. This constant ratio with respect to the X-axis is called the slope of the line AB.
Figure 3.5: Slope of a Straight Line

The slope of a straight line determines how steep the line is. Also, as the steepness of a straight line increases, its slope value also increases. These details can be explained through Figure 3.5.
Figure 3.6: Steepness of a Straight Line and its Slope Value
As shown in Figure 3.6, as the steepness of the straight line increases, its slope value also increases. That is,

OMMP1<OMMP2<OMMP3<OMMP4
3.2.3
If a straight line makes an angle θ with the X-axis, its slope is called tan θ.

tanθ=OAOB
The slope of the X-axis is zero, as it makes a zero angle with itself. Therefore, the slope of any line parallel to the X-axis is zero. As explained earlier, as the steepness of a straight line increases, its slope value also increases. When the steepness of the straight line increases and

merges with the Y-axis, the slope value reaches 90 degrees. That is, the X-axis makes a 90-degree angle with the Y-axis.

The points discussed so far regarding the slope of a straight line can be briefly summarized as follows:

1. The slope of a straight line is generally described only with respect to the X-axis.
2. If a straight line slopes upwards from left to right, its slope is positive, and if it slopes downwards from left to right, its slope is negative.
3. As the steepness of a straight line increases, its slope value also increases.
4. The slope of the X-axis is zero, and the slope of the Y-axis is infinite.

3.2.4 STRAIGHT LINE PASSING THROUGH GIVEN POINTS

If a straight line passes through two points $(x1,y1)$ and $(x2,y2)$, its slope is given by $x2-x1y2-y1$. Its equation can be proven as $(y-y1)=x2-x1y2-y1(x-x1)$. If the slope is denoted by 'm', the equation is $(y-y1)=m(x-x1)$.
Figure 3.7: Straight Line Passing Through Given Points

For example, if a straight line passes through the points (2,5) and (4,8), its equation will be:
$(y-5)=4-28-5(x-2)$
$(y-5)=23(x-2)$
$2(y-5)=3(x-2)$
$2y-10=3x-6$
$-3x+2y-10+6=0$
$-3x+2y-4=0$
$3x-2y+4=0$

3.3 APPLICATIONS OF STRAIGHT LINES IN ECONOMICS

Straight lines have many applications in economics. We will briefly learn about some of their main applications.

3.3.1 CONSUMPTION FUNCTION

In economics, the consumption function is assumed to have a linear form. The functioning of consumption arises from people's actual behavior. The general form of the consumption function is $C=f(Yd)$. Its linear form is $C=a+bYd$. In this equation, C represents the quantity of consumption, and Yd represents disposable income. For example, it can be considered as $C=500+0.8Yd$.
Figure 3.8: Consumption Function
We can observe from Figure 3.8 that consumption remains positive even when income is zero. This quantity is identified as 'a' in the figure. The slope of the straight line represents the Marginal Propensity to Consume. It clarifies the extent to which consumption increases if income increases by one unit. If the given consumption function is $C=500+0.8Y$, then if income increases by 100 rupees, consumption will increase by 80 rupees.

3.3.2 DEMAND FUNCTION

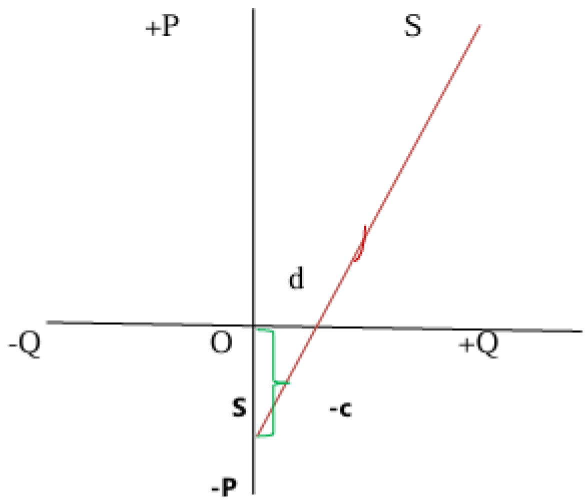
Similar to the consumption function in economics, the demand function is also assumed to have a linear form. With a single independent variable (price), the demand equation is written as:

Qd=a−bP, where a>0 and b<0.
Here, P represents price and Q represents the quantity demanded. In the straight line representing the demand equation, the y-intercept value is positive, and the slope is negative. The negative slope value indicates the law of demand, which states that there is an inverse relationship between price and the quantity demanded of a good when other factors affecting demand are constant.
Figure 3.9: Demand Straight Line

3.3.3 SUPPLY FUNCTION

When other factors influencing supply are constant, supply refers to the quantity of goods that firms are willing to supply at various market prices. That is, supply depends only on the price of the good. In economics, the supply function also has a linear form. The supply straight line has a negative vertical intercept and a positive slope. Similar to the demand function, the supply function equation is also written with a single independent variable (price) as follows. This type of economic model is called a Partial Equilibrium Model.
Qs=−c+dP, where c<0 and d>0.

Figure 3.10: Supply Straight Line



3.3.4 Partial Equilibrium State

In a free economic system, demand and supply forces interact to determine the equilibrium price and quantity of a good. At that price, the quantity demanded and the quantity supplied are equal. Unless external forces influence it, no one is interested in changing from such a state on their own. Such a state is called 'equilibrium state' in economics.
In our model, there are three equations:

1. Qd=a−bP
2. Qs=−c+dP
3. Qd=Qs (Equilibrium condition)

Similarly, there are three unknown variables: Qd, Qs, and P. In equilibrium, since Qd=Qs, we denote them jointly as Q. Then,

$Q=a-bP----1$

$Q=-c+dP----2$

$Qd=Qs----3$

Substituting the first and second equations into the third equation:

$a-bP=-c+dP$

$-bP-dP=-c-a$

Changing the signs on both sides (multiplying both sides by -1):

$bP+dP=c+a$ or

$(b+d)P=c+a$

The equilibrium price, $Pe=b+da+c$

Thus, the equilibrium price is given in terms of parameters (or constants). Similarly, the equilibrium quantity can also be expressed in terms of parameters.

By substituting the parameters related to the equilibrium price into either of the above two equations, we obtain the equilibrium quantity. If we take the demand equation and substitute the parameters,

$Q=a-bP$

$=a-b(b+da+c)$

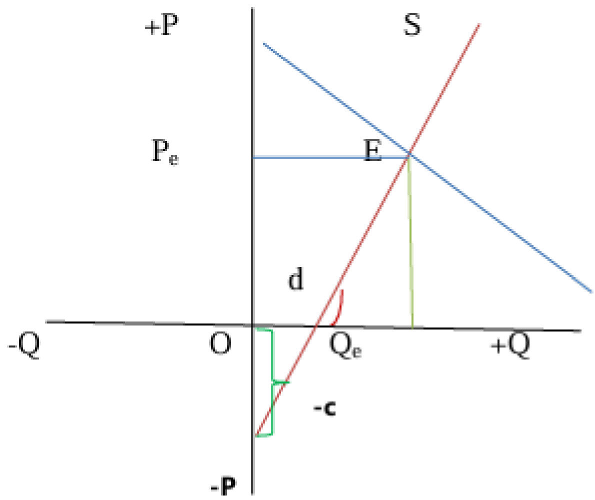
$=b+da(b+d)-b(a+c)$

$=b+dab+ad-ab-bc$

$Qe=b+dad-bc$

Figure 3.11: Equilibrium State

Thus, the equilibrium quantity is also expressed in terms of parameters.



Example: 1

Given the following parameter values, calculate the equilibrium price and equilibrium quantity.

$a=400$, $b=-2$, $c=-80$, $d=34$

Solution:

Equilibrium Price: $Pe=b+da+c=-2+34400-80=32320=10$

Quantity: $Qe=b+dad-bc=-2+34(400)(34)-((-2)(-80))=3213600-160=3213440=420$

Example: 2

In a partial linear market model where $Qd=Qs$, given $Qd=8-2P$ and $Qs=-4+4P$. Find the values using the elimination method for variables.

Solution:

1. $Qd=a-bP=8-2P$, Demand Equation ---1
2. $Qs=-c+dP=-4+4P$, Supply Equation ---2
3. $Qd=Qs$, Equilibrium State ---3

At equilibrium, $Qd=Qs$.

Substituting the first and second equations into the third equation:

$8-2P=-4+4P$

$-2P-4P=-4-8$

$6P=12$

$Pe=12/6=2$

Substituting the value of P into the demand equation:

$Q=8-2P$

$Qe=8-2(2)=8-4=4$

Substituting the value of P into the supply equation:

$Q=-4+4P$

$Q=-4+4(2)=-4+8=4$

Thus, $Qd=Qs=4$.

3.4 LINEAR GENERAL EQUILIBRIUM MARKET MODEL

Every good generally has many substitutes and complementary goods. They also influence the demand and supply of those goods. Therefore, it is necessary to consider the prices of many other goods along with the price of that good. In the partial equilibrium model we discussed earlier, the condition necessary for equilibrium is that excess demand in the market should be zero. That is, $Qd-Qs=0$. When considering multiple goods markets, to achieve equilibrium, excess demand in every market must be zero. That is, $Qdi-Qsi=0$, for $i=1,2,3,\ldots,n$. In this, the prices of all goods and the quantities of all goods are determined simultaneously. This type of market demand model is called a General Equilibrium Model.

Consider the following linear general equilibrium market model:

$Qd1-Qs1=0$ ---1, Excess demand is zero in the first good's market

$Qd1=a0+a1P1+a2P2$ ---2, Demand function for the first good

$Qs1=b0+b1P1+b2P2$ ---3, Supply function for the first good

$Qd2-Qs2=0$ ---4, Excess demand is zero in the second good's market

$Qd2=\alpha0+\alpha1P1+\alpha2P2$ ---5, Demand function for the second good

$Qs2=\beta0+\beta1P1+\beta2P2$ ---6, Supply function for the second good

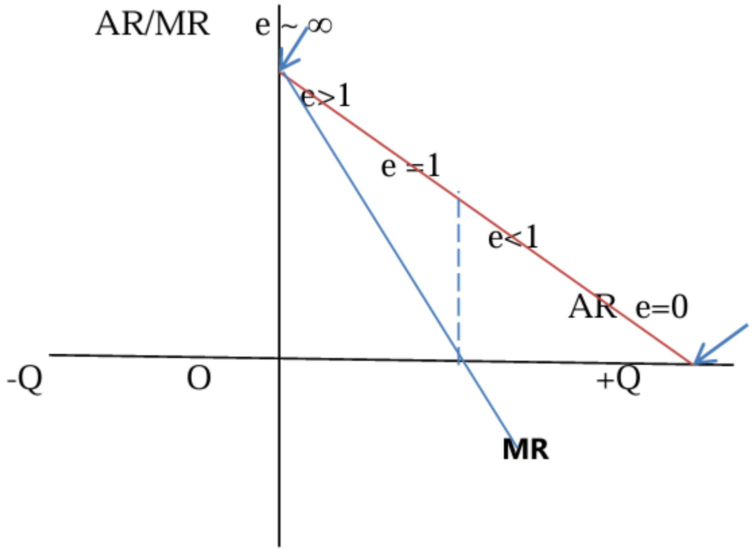
Substituting the second and third equations into the first equation:

$(a0+a1P1+a2P2)-(b0+b1P1+b2P2)=0$

$a0+a1P1+a2P2-b0-b1P1-b2P2=0$

$(a0-b0)+(a1-b1)P1+(a2-b2)P2=0$——————7

Substituting the fifth and sixth equations into the fourth equation:



Then the equations become:
c0+c1P1+c2P2=0
c1P1+c2P2=−c0·−−−−9
γ0+γ1P1+γ2P2=0 or γ1P1+γ2P2=−γ0−−−−−−−10
Writing this system of linear equations in matrix form:
[c1γ1c2γ2][P1P2]=[−c0−γ0]
A X = B

This formulated model can be solved much more easily by Cramer's Rule than by the elimination method for variables. You will learn about Cramer's Rule in detail in Chapter 14.

According to Cramer's Rule:
P1=Det.AD1
P2=Det.AD2

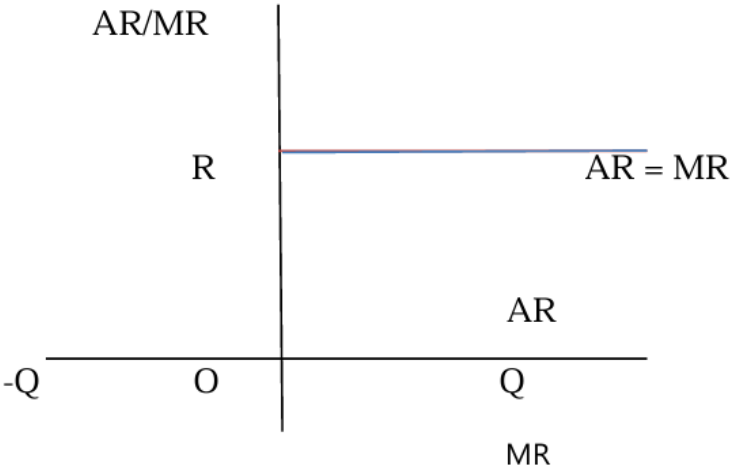
Here, D1 is the determinant obtained by substituting the matrix of constant terms on the right side of the equation in place of the values in the first column of the coefficient matrix. D2 is the determinant obtained by substituting the matrix of constant terms on the right side of the equation in place of the values in the second column of the coefficient matrix.
Determinant of the coefficient matrix:
Det. A=c1γ1c2γ2=c1γ2−c2γ1
D1, D2, Determinant of Matrices

3.14
Det .D1=|−c0−γ07c2γ2c2|A|=−c0γ2+c2γ0
Det. D2=|c1γ1|γc0γ0|A|=−c1γ0+c0γ1
Therefore p1=Det.AD1=c1γ2−c2γ1−c0γ2+c2γ0=P2=Det.AD2

3.5 STRAIGHT AVERAGE, MARGINAL REVENUE CURVES:

According to economic theory, the average revenue curve has a straight shape. We know that average revenue depends on the quantity of the good. That is, C =f(Q). By dividing the total revenue by the quantity of the good, we get the average revenue. The change in total revenue when an additional unit of a good is sold is called marginal revenue. In this way, marginal revenue also depends on the quantity of goods sold. In a perfectly competitive market, a firm does not have the power to determine the market price, so it continues to sell at the market price. Therefore, the firm's average revenue (AR) and marginal revenue (MR) curves are parallel to the X-axis. Moreover, the MR curve merges with the AR curve.

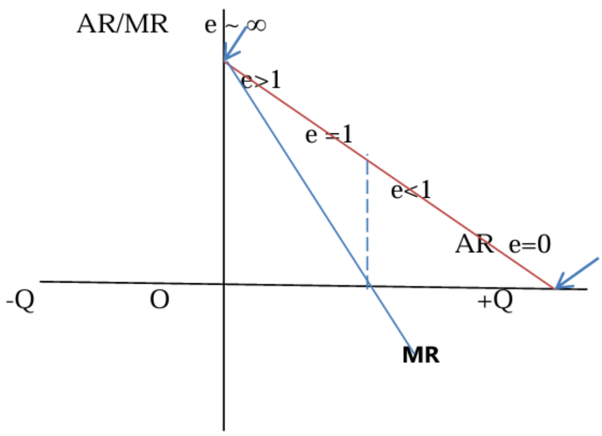
Figure 3.12 Average and Marginal Revenue Curves in a Perfectly Competitive Market



However, in a monopoly, both AR and MR curves slope downwards from top to bottom, having a negative slope. At the midpoint of the straight line, price elasticity is equal to one. Above the midpoint, below the axis, elasticity is greater than one, and on the vertical axis, elasticity is infinite. Below the midpoint, above the horizontal axis, elasticity is less than one.

and greater than zero. At the horizontal axis, elasticity is zero. These details can be observed in Figure 3.13.

Figure 3.13 Average and Marginal Revenue Curves in a Monopoly



Similarly, in macroeconomics, it is assumed that savings and investment curves also have a straight line shape.

Figure 3.14 Savings Curve

పటం 3.14 పొదుపు రేఖ

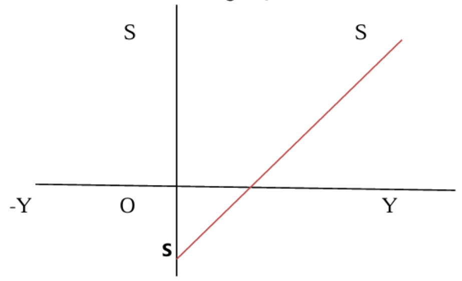
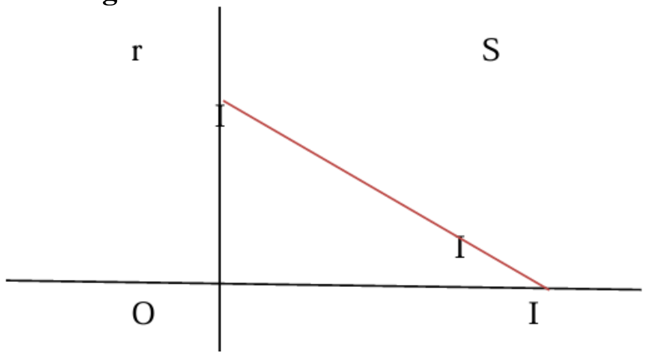


Figure 3.15 Investment Curve



As shown in Figure 3.14 and Figure 3.15, the savings curve has a positive slope, while the investment curve has a negative slope.

3.6 SUMMARY:

In this lesson, we have learned in detail about the mathematical concepts and properties of a straight line. Specifically, we studied the concepts of intercept and slope. We learned about these concepts in detail through diagrams. We also learned how to draw a straight line through given points. Straight lines have many applications in economics. Important among these are the demand curve, supply curve, consumption curve, savings curve, investment curve, etc. We have illustrated these through diagrams.

3.7 GLOSSARY

1. Straight line : సరళ రేఖ
2. Intercept : అంతర ఖండం
3. Slope : వాలు
4. Horizontal axis: క్షితిజ అక్షం
5. Vertical axis : ఊర్ధ్వ అక్షం

6. Demand curve : డిమాండ్ రేఖ

7. Supply curve : సప్లై రేఖ

8. Consumption curve : వినియోగ రేఖ

9. Savings Curve : పొదుపు రేఖ

10. Investment curve : పెట్టుబడి రేఖ

11. Average Revenue Curve : సగటు రాబడి రేఖ

12. Marginal Revenue Curve : ఉపాంత రాబడి రేఖ

3.8 Sample Examination Questions

3.8.1 Short Answer Questions

1. Define the equation of a straight line.
2. Define the concepts of intercept and slope.
3. Explain the shapes of the average revenue curve and marginal revenue curve in a perfectly competitive market.
4. Explain the average and marginal propensity functions through a diagram.

3.8.2 Essay Type Answer Questions

1. Explain the concepts of intercept and slope through a diagram.
2. Derive the equation of a straight line passing through points (1,4) and (3,1).
3. $Qd=a-bP$ $Qs=-c+dP$, $Qd=Qs$
4. Find the quantities of goods.

4. Using matrix theory, find the equilibrium price and quantity for the following simple general equilibrium market model.

$Qd1-Qs1=0$

$Qd1=a0+a1P1+a2P2$

$Qs1=b0+b1P1+b2P2$

$Qd2-Qs2=0$

$Qd2=\alpha0+\alpha1P1+\alpha2P2$

$Qs2=\beta0+\beta1P1+\beta2P2$

3.9 SUGGESTED READING

1. Alpha Chiang (2017), Fundamental Methods of Mathematical Economics, 4th Edition,
2. New Delhi: McGraw Hills.
3. R. G. D. Allen, (2014), Mathematical Analysis for Economists, New Delhi: Trinity Press.
4. B.C. Mehta and G.M.K. Madnani, Mathematics for Economists, New Delhi: Sultan
5. Chand & Sons.

LESSON - 4
DIFFERENTIATION

Table of Contents :

4.0 OBJECTIVES, GOALS :

After reading this lesson, we will gain an understanding of the following topics.
Definition of Differentiation
Types of Differentiation and its application in Microeconomics

4.1 DIFFERENTIATION - DEFINITION :

In economics, we study the relationships between various economic variables. For example, in microeconomics, we study the relationship between the demand for a good and its price. This is called the demand function.

In these functions, it is customary to refer to one variable as the independent variable and the other as the dependent variable. For example, in the demand function, if y represents the quantity demanded of a good and X represents the price of that good, then the demand function can be represented by the equation $y=f(x)$. It is natural that if there is a change in the independent variable X, there will also be changes in the dependent variable y. The dynamic patterns of these changes can be found through differentiation.

If the function $y=f(x)$ is continuous within a specified range, and if a change of Δx occurs in X, resulting in a change of Δy in y, then if the limit can be found as shown below, that limit is called the derivative.

$\lim \Delta x \to 0 \Delta x f(x+\Delta x) - f(x)$
This derivative is also sometimes called the differential coefficient. It is written as $dxdy, f'(x), y'$.
Some important derivatives:
$y=xn$ then $dxdy=n \cdot xn-1$

y=ex then dxdy=ex

y=log x then dxdy=x1

y=ax then dxdy=axlog x

If y=c is a constant, then dxdy=0
Differentiation formulas when the given functions are single-power functions.

1. y=(a+bx)n
/

y=(a+bx)n=n(a+bx)n−1b

If y=aA+bx is a general exponential function, then dxdy=dxd(aA+bx)=aA+bx·loga(b)

If y=ea+bx is an exact exponential function, then zdxdy=dxd(ea+bx)=ea+bx·b

If y=loge(a+bx) is a logarithmic function, then dxdy=dxd(logc(a+bx))=a+bx1·b

4.2 THEOREMS OF DIFFERENTIATION :

4.2 Differential Theorems: 4.3 Differentiation
Before learning the differentiation theorems, let's learn how to find the derivative of a function with a constant coefficient in front. Let's assume the function is in the form of y=c·f(x), where C is a constant and f(x) is a function.
∴dxdy=c·dxd(f(x)) =c·f′(x)
Therefore, when a function has a constant coefficient, you can take it out and differentiate the remaining function. For example: y=f(x)=5x5

4.2.1 Sum Rule: The derivative of the sum of two or more functions is equal to the sum of the derivatives of those individual functions.
y=u+v+w dxdy=dxd(u+v+w)=dxd(u)+dxd(v)+dxd(w)
Similarly, if y=u+V−W, then dxdy=dxd(u)+dxd(v)−dxd(w)

Examples: Find the derivatives of the following functions.

1. y=x3+x4 dxdy=dxd(x3)+dxd(x4) =3·x3−1+4·x4−1 =3x2+4x3
2. y=45x3−76x5+3x−2 dxdy=45dxd(x3)−76dxd(x5)+3·dxd(x−2)
3. =453x3−1−765·x4−1+3·−2x−2−1 =453x2−765·x3−6x−3 =415x2−730·x3−6x−3

Differentiate this function: y=(3 x−2)1/2 Since this function is in the form (ax+b)n,

y=(ax+b)n=n·(ax+b)n−1 dxdy=dxd(3x −2)1/2 =21(3 x−2)21−1·3 =23

(3 x−2)21−1 =23 (3 x−2)21−2 =23 (3 x−2)−21

4. y=4+x3−42x−7 +4(2 x−1)3/2 The above equation can be written as: y=4+3x−1−5(2x−7)21+4(2 x−1)3/2 Differentiating the above equation with respect to x: dxdy=dxd[4+3x−1−5(2x−7)21+4(2 x−1)23]

5. =dxd(4)+3·dxd(x−1)−5dxd(2x−7)21+4dxd(2x −1)23 =0+3·−1x−1−1−5[21 (2x−7)21·2]−1+42[23(2x −1)23·2] =0−3x−2−5[22(2x−7)−21]+62 (2x −1)1/2 =−3x−2−5(2x−7)−21+62 (2x −1)21

6. y=2x+1 −42x−1 +|−2x 1 The above function can be written as: y=(2x+1)21=(2x−1)41+(1−2x)−21 Differentiating the above function with respect to X: dxdy =dxd[(2x+1)21−(2x−1)41+(1−2x)−21] =dxd[2x+1]21−1·2−41(2x−1)411·2+2−1(1−2x)−21 −1(−2) =22(2x+1)−21−21(2x−1)43+(1−2x)−23 =22(2x+1)21−2−42(2x−1)41−4+−2−2 (1−2x)2−1−2 =(2x+1)−21−21(2x−1)4−3+(1−2x)2−3

Exercise: Find dxdy for the following functions.

1. y=4x2+x+8
2. y=2x2+3x4
3. y=x61
4. y=x+x4−xy2+5x
5. y=ex+x3−x−5+4x

Answers:

1. 8x+1
2. 4x+12x3
3. −x76
4. 1−x24−x814+57 log 5
5. ex+3x2+5x−6+4xlog 4

4.2.2 Product Rule: Let y=u·v where u and v are functions of X. Then dxdy=u·dxd(v)+vdxd (u) Similarly, if y = u, v, w, then dxdy=uv·dxd(w)+u wdxd(v)+v wdxd(u)
Examples: (1) y=5x2(1−3x) Since the given function is in the form of u, we use the product rule. dxd(uv)=u·dxd(v)+v·dxd(u) ∴dxd(y)=dxd[5x2(1−3x)]

(2) y=(x+x1)(x −x 1) (4.7) =5x2·dxd(1−3x)+(1−3x)dxd(5x2) =5x2·[dxd(1)−3dxd (x)]+(1−3x)5·dxd(x2) =5x2[0−30]+(1−3x)10x where dxd(1)=0 =5x2(−3)+(1−3x)10x =−15x2+10x−30x2 =10x−45x2

dxdy=dxd[(x+x1)(x −x 1)] =(x+x1)dxd[x −x 1]+(x −x 1)dxd(x+x1)

We can write x1 as x−1, x as x21 and x 1 as x−21. $=(x+\frac{1}{x})[\frac{d}{dx}(_{x}/2})-\frac{d}{dx}(_{-\frac{1}{2}})]+(\sqrt{x}+\frac{1}{\sqrt{x}})[\frac{d}{dx}(x)+\frac{d}{dx}(x^{-1})]$

=(x+x1)[21·x21−1−2−1·x−21−1]+(x +x 1)(1+−1 x−1−1) =(x+x1)(21·x21−2−2−1 x2−1−2)+(x +x 1)(1−x−2) =(x+x1)[21·x2−1+21x−3]+(x +x 1)(1−x−2)

(3) =(x+x1)[2x 1+23x 1]+(x +x 1)(1−x21) =(x2+1)(2x+3)+(x2+3x)(2x) =2x3+3x2+2x+3+2x3+5x2 =4x3+7x2+2x+3
Find the derivative of y=(4x2+2x)(8x3+3x2). y=(u.v) so dxdy=u·dxd(v)+vdxd(u) dxdy=dxd [(4x2+2x)(8x3+3x2)] =(4x2+2x)dxd(8x3+3x2)+(8x3+3x2)dxd(4x2+2x) =(4x2+2x)[8dxd (x3)+3dxd(x2)]+(8x3+3x2)[4dxd(x2)+2dxd(x)] =(4x2+2x)[8·3x2+3·2x]+(8x3+3x2)(4·2x+2) =(4x2+2x)(24x2+6x)+(83+3x2)(8x+2) =[96x4+24x3+48x3+12x2]+[64x4+16x3+24x3+6x2] =96x4+24x3+48x3+12x2+64x4+16x3+24x3+6x2 =160x4+112x3+18x2

Exercise: Find dxdy for the following functions.

1. y=(x2+3)(2x2+7)
2. y=(7x−8)4(5x−1)3
3. y=x5(2x2+1)
4. y=(x5+x2)(x3+x)
5. (3x2−5)(3−5x3)

Answers:

1. 2x[4x2+13]
2. (7x−8)3(5x−1)2[245x−148]
3. 4x2(x4+2x2+1)
4. (x5+x2)(3x2+1)+(x3+x)(5x4+2x)
5. 3x(6+25x−25x3)

4.2.3 Quotient Rule: Let y=vu where u and v are functions of x. Then dxdy=v2v·dxd(u)−u·dxd (v)

Examples: (1) y=x2+2x2+1 dxdy=v2v·dxd(u)−u·dxd(v) This is in the form y=vu. So y=x2+2x2+1 dxdy=dxd[x2+2x2+1] =(x2+2)2(x2+2)[dxd(x2+1)−(x2+1)dxd(x2+2)] =(x2+2)2(x2+2)[dxd(x2)+dxdx(1)]−(x2+1)dxd(x2)−1dxd(2) =(x2+2)2(x2+2)(2·x2−1+0)−(x2+1)[2x2−1+0]
(2) =(x2+2)2(x2+2)2x−(x2+1)2x =(x2+2)22x3+4x−(2x3+2x) =(x2+x)22x3+4x−2x3−2x =(x2+2)24x−2x=(x2+2)22x

Find dxdy for y=1−2x1+2x . This is in the form of y=vu. dxdy=v2v·dxd(u)−udxd(v) dxdy

=(1−2x)21−2x dxd(1+2x)−1+2x dxd(1−2x) =(1−2x)1−2x dxd

(1+2x)21−1+2x dxd(1−2x)21 $=\frac{\sqrt{1-2x}[\frac{1}{2}(1+2x)^{\frac{1}{2}}\cdot2-\sqrt{1+2x}[\frac{1}{2}(1-2x)^{\frac{1}{2-

1}\cdot2]]}{(1-2x)}$ =(1−2x)1−2x 22(1+2x)212−1+2x (−22(1−2x)21−2)

=(1−2x)1−2x (1+2x)−21+1+2x (1−2x)−21 =1−2x[1−2x (1+2x 1)+1+2x

(1−2x 1)] =(1−2x)1+2x 1−2x (1−2x)+(1+2x) =1+2x 1−2x

(1−2x)1−2X+1+2X=1+2x (1−2x)232

Exercise: Find dxdy for the following functions.

1. y=a+xa−x
2. y=1+x2x3
3. y=1+x 1−x
4. y=x−3(x+1)(2x−1)
5. y=4x2+33x+2

Answers:

1. (a+x)22a
2. (1+x2)2x2(3+x2)
3. x (1+x)21
4. (x−3)22(x2−6x+1)
5. (4x2+3)2−(12x2+16x−9)

4.2.4 Chain Rule: If y is a function of u, and u is a function of x, i.e., y=f(u), u=ϕ(x), then it becomes a function of a function. Then dxdy=dudydxdu

Examples: (1) y=t2−3t+2, t=x2−5 Find dxdy. ∴dtdy=dtd(t2−3t+2)

=dtd(t2)−3dtd(t)+dtd(2) =2t−3+0 =2t−3 t=x2−5 dxdt=dxd(x2−5) =dxd(x2)−dxd(5) =2x−0=2x (2) y=z2+1, z=t2+1, t=x2+1 dxdy=dzdy·dtdz·dxdt y=z2+1 ∴dzdy=dzd(z2+1) =dzd(z2)+dzd(1) =dzd(z2)+dzd(1) =2z+0=2z z=t2+1 dtdz=dzd(t2+1) =dzd(t2)+dzd(1) =2t+0=2t t=x2+1

dxdt=dxd(x2+1) =dxd(x2)+dxd(1) =2x+0=2x dxdy=dzdy·dtdz·dxdt =2z·2t·2x =8ztx =8(t2+1)(x2+1)x =8((x2+1)2+1)(x2+1)x =8(x4+2x2+1+1)(x2+1)x =8(x4+2x2+2)(x2+1)x

Exercise: Find dxdy for the following.

1. (x2+2x+3)5
2. y=e2x+5
3. y=3x−4
4. y=2−3x1
5. y=1−2xx

Answers:

1. 10(x+1)(x2+2x+3)4
2. 2e2x+5
3. 12(3x−4)3
4. (2−3x)23
5. (1−2x)21

3.2.5 Logarithmic Differentiation: If a function is in the form of y=f(x)ϕ(x), we can find dxdy by taking logarithms on both sides.

If a function is in the form of y=f(x)ϕ(x), we can find dxdy by taking logarithms on both sides. If y=f(x)ϕ(x) and we take log on both sides, we get logy=ϕ(x)logf(x), which is in the form of a product rule. The derivative of logy is y1dxdy, so we can find the derivative using the product rule.

y1dxdy=ϕ′(x)logf(x)+ϕ(x)f(x)f′(x) dxdy=y[ϕ′(x)logf(x)+ϕ(x)f(x)f′(x)] dxdy =f(x)ϕ(x)[ϕ′(x)logf(x)+ϕ(x)f(x)f′(x)]

Example: (1) If y=x2x+3, find dxdy. Take log on both sides: logy=(2x+3)logx y1dxdy =(2x+3)x1+logx·2 dxdy=y[x2x+3+2logx] dxdy=x2x+3[x2x+3+2logx]
(2) If y=xx2, find dxdy. This can be written as y=xx2, so we take log on both sides. logy=x2logx y1dxdy=x2·x1+logx·2x dxdy=y[x+2xlogx] dxdy=xx2[x+2xlogx]

Exercise: (1) (2x+3)4x+5 (2) xx (3) (4x+5)2x+5 (4) (3x+4)2x+2 (5) xxx
Answers: (1) (2x+3)4x+5[4log(2x+3)+2x+32(4x+5)] (2) xx(1+logx) (3) (4x+5)2x+5[2log(4x+5)+4x+54(2x+5)] (4) (3x+4)2x+2[2log(3x+4)+3x+43(2x+2)] (5) xxxxx[logx+logx·logx+1]

4.3 DIFFERENTIATION

Before learning the theorems of differentiation, let's learn how to find the derivative when a function has a constant coefficient. The function is in the form y=c·f(x), where C is a constant and f(x) is a function.

∴dxdy=c·dxd(f(x))

=c·f′(x)
Therefore, when a function has a constant coefficient, it is sufficient to take it out and find the derivative of the remaining function.

Example: y=f(x)=5x5
4.2.1 Sum rule: The derivative of the sum of two or more functions is equal to the sum of the derivatives of those individual functions.

y=u+v+w
.

dxdy=dxd(u+v+w)=dxd(u)+dxd(v)+dxd(w)
Similarly, if y = u + V -W

dxdy=dxd(u)+dxd(v)−dxd(w)
Example: Find the derivatives for the following functions.

y=x3+x4

dxdy=dxd(x3)+dxd(x4)

=3·x3−1+4·x4−1

=3x2+4x3

y=45x3−76x5+3x−2

dxdy=45dxd(x3)−76dxd(x5)+3·dxd(x−2)

=453x3−1−765·x4−1+3·−2x−2−1
=453x2−765·x3−6x−3
=415x2−730·x3−6x−3

y=(3x−2)1/2 Differentiate this function.
This function is in the form (ax+b)n so

y=(ax+b)n=n·(ax+b)n−1

dxdy=dxd(3x−2)1/2

=21(3x−2)21−1·3

=23(3x−2)21−1

=23(3x−2)21−2

=23(3x−2)−21

4. y=4+x3−42x−7+4(2x−1)3/2
The above equation can be written as.

y=4+3x−1−5(2x−7)21+4(2x−1)3/2
Differentiating the above equation with respect to x

dxdy=dxd[4+3x−1−5(2x−7)21+4(2x−1)23]
=dxd(4)+3·dxd(x−1)−5dxd(2x−7)21+4dxd(2x−1)23
=0+3·−1x−1−1−5[21(2x−7)21·2]−1+42[23(2x−1)23·2]
=0−3x−2−5[22(2x−7)−21]+62(2x−1)1/2
=−3x−2−5(2x−7)−21+62(2x−1)21

y=2x+1−42x−1+|−2x1
The above function can be written as.
y=(2x+1)21=(2x−1)41+(1−2x)−21
Differentiating the above function with respect to X

dxdy=dxd[(2x+1)21−(2x−1)41+(1−2x)−21]
=dxd[2x+1]21−1·2−41(2x−1)411·2+2−1(1−2x)−21−1(−2)
=22(2x+1)−21−21(2x−1)43+(1−2x)−23
=22(2x+1)21−2−42(2x−1)41−4+−2−2(1−2x)2−1−2
=(2x+1)−21−21(2x−1)4−3+(1−2x)2−3
Example: Find dy/dx for the following functions.
Exercise

1. y=4x2+x+8

y=2x2+3x4

3. y=x61

4. y=x+x4−xy2+5x

5. y=ex+x3−x−5+4x
Answers:

8x+1

4x+12x3

−x76

1−x24−x814+57 log 5

ex+3x2+5x−6+4xlog 4

4.6
4.2.2 Product rule: Let y=u,v∈S6 u, v be functions of X. Then

dxdy=u·dxd(v)+vdxd(u)
Similarly, if y = u, v, w
Example:

dxdy=uv·dxd(w)+u vdxd(v)+v wdxd(u)

(1) y=5x2(1−3x)
The given function is in the form u, so using the product rule
dxd(uv)=u·dxd(v)+v·dxd(u)
∴dxd(y)=dxd[5x2(1−3x)]

Mathematics 4.7
( ) ( ) ( ) 2 2 d d 5x 1 3x 1 3x 5xdx dx= · − + −
( ) ( ) ( ) ( ) 2 2 d d d 5x 1 3 x 1 3x 5 xdx dx dx⎡ ⎤= · − + − · ⎢ ⎥⎣ ⎦
[ ] ( ) 2 2 1 5x 0 30 1 3x 5 2x − = − + − ·
where ( ) d 1 0dx =
( ) ( ) 2 x 5x 3 1 3x 10= − + −
2 2 15x 10x 30x= − + −
2 10x 45x= −
(2)
1 1
y x x
x x
⎛ ⎞⎛ ⎞= + −⎜ ⎟⎜ ⎟⎝ ⎠⎝ ⎠
dy d 1 1 x x
dx dx x x
⎡ ⎤⎛ ⎞⎛ ⎞∴ = + −⎢ ⎥⎜ ⎟⎜ ⎟⎝ ⎠⎝ ⎠⎣ ⎦
1 d 1 1 d 1
x x x x
x dx dx x x x
⎡ ⎤⎛ ⎞⎛ ⎞ ⎛ ⎞= + − + − +⎢ ⎥⎜ ⎟⎜ ⎟ ⎜ ⎟⎝ ⎠ ⎝ ⎠⎣ ⎦⎝ ⎠
Note:
1
x
can be written as x−1, and x can be written as x1/2, and x1 can be written as x−1/2.

( ) ( ) ( ) ( ) 11 1 2 21 d d 1 d dx x x x x x x dx dx dx dx x− − ⎛ ⎞⎡ ⎤⎛ ⎞⎡ ⎤= + − + + + ⎢ ⎥ ⎜ ⎟ ⎢ ⎥ ⎜ ⎟
⎢⎣ ⎥⎦⎝ ⎠⎣ ⎦⎝ ⎠
( )
1 1 1 1 1 1 2 2 1 1 1 1 x x x x 1 1 x
x 2 2 x
− − −
− −
⎡ ⎤− ⎛ ⎞⎛ ⎞= + ⋅ − ⋅ + + + −⎢ ⎥ ⎜ ⎟⎜ ⎟ ⎢⎝ ⎠⎣ ⎦⎝ ⎠
( )
12 12
2 2 2 1 1 1 1 x x x x 1 x
x 2 2 x
− − −
−
⎛ ⎞− ⎛ ⎞⎛ ⎞⎜ ⎟= + ⋅ − + + −⎜ ⎟ ⎜ ⎟⎜ ⎝ ⎠⎝ ⎠⎝ ⎠
( ) 31 2 2 2 1 1 1 1 x x x x 1 x
x 2 2 x
−−
− ⎛ ⎞⎡ ⎤⎛ ⎞= + ⋅ + + + −⎜ ⎟ ⎢ ⎥ ⎜ ⎟ ⎜⎝ ⎠⎣ ⎦⎝ ⎠

4.82
1 1 1 1 1 x x 1
x 2 x 23 x x x
⎛ ⎞⎡ ⎤⎛ ⎞⎛ ⎞= + + + + −⎜ ⎟ ⎢ ⎥ ⎜ ⎟ ⎜ ⎟⎜ ⎝ ⎠⎝ ⎠⎣ ⎦⎝ ⎠
( )( ) ( )( ) 2 2 x 1 2x 3 x 3x 2x= + + + +
3 2 3 2 2x 3x 2x 3 2x 5x= + + + + +
3 2 4x 7x 2x 3= + ++
(3) ( )( ) 2 3 2 y 4x 2x 8x 3x = + + Find the derivative.
If ( ) y u.v = then ( ) ( ) dy d d u v v udx dx dx= ⋅ +
( ) ( ) 2 3 2 dy d 4x 2x 8x 3xdx dx⎡ ⎤= + +⎣ ⎦
( ) ( ) ( ) ( ) 2 32 32 2 d d 4x 2x 8x 3x 8x 3x 4x 2xdx dx= + + + + + +
( ) ( ) ( ) ( ) ( ) ( ) 2 3 2 3 2 2 d d d d 4x 2x 8 x 3 x 8x 3x 4 x 2 xdx dx dx dx⎡ ⎤⎡ ⎤= + + + + + + ⎢
⎥⎢ ⎥⎣ ⎦⎣ ⎦
( ) ( )( ) 2 2 3 2 4x 2x 8 3x 3 2x 8x 3x 4 2x 2⎡ ⎤= + ⋅ +⋅ + + ⋅ +⎣ ⎦
( )( ) ( )( ) 2 2 3 2 4x 2x 24x 6x 8 3x 8x 2= + + + + + +
4 3 3 2 4 3 32 96x 24x 48x 12x 64x 16x 24x 6x⎡ ⎤⎡ ⎤= +++ + +++⎣ ⎦⎣ ⎦
4 3 2 160x 112x 18x= + +
Exercise: Find dy/dx for the following functions.
1. ( )( ) 2 2 y x 3 2x 7 = + + 2. ( ) ( ) 4 3 y 7x 8 5x 1 = − −
2. ( ) 5 2
Here's the English translation of the Telugu text from the provided PDF:
Deer
=dtd(t2)−3dtd(t)+dtd(2) =2t−3+0 =2t−3 t=x2−5 dxdt=dxd(x2−5) =dxd(x2)−dxd(5) =2x−0=2x

4.12
(2) y=z2+1, z=t2+1, t=x2+1
dxdy=dzdy⋅dtdz⋅dxdt
y=z2+1 ∴dzdy=dzd(z2+1) =dzd(z2)+dzd(1) =dzd(z2)+dzd(1) =2z+0=2z

z=t2+1 dtdz=dzd(t2+1) =dzd(t2)+dzd(1) =2t+0=2t
=x2+1
dxdt=dxd(x2+1) =dxd(x2)+dxd(1) =2x·0
dxdy=dzdy·dtdz·dxdt =2z·2t·2x =8ztx =8(t2+1)(x2+1)x
4.13
=8[((x2+1)2+1)(x0+1)x](:t=x2+1) =8[(x4+2x2+1)+1(x2+1)x] =8x[x−4+2x2+2][x2+1]
Exercise: Find dxdy for the following functions.
Answers:
1. y=(x2+2x+3)5
2. y=e2x+5

3. y=3x−4

4. y=2−3x 1

5. y=1−2x1+x
6. 10(x2+2x+3)4(x+1)

7. 23·2−3x 1
8. 2·e2x+5

9. 23·3x−4 1

10. 231+x (1−2x)31
dxdy

3.2.5 Logarithmic Differentiation: If a function is in the form y=f(x)ϕ(x), we can find the derivative using logarithmic differentiation. If we take (natural) logarithms on both sides of the above function:

(Because, on the right side, log y = ϕ(x) log f(x). Since y on the left side is a function of X, the derivative of log y is y1. Therefore, since y is a function of x, according to the chain rule, the derivative of y is dxdy. We need to find the derivative of the function using product differentiation.)
∴y1dxdy=ϕ(x)f(x)1f′(x)+logf(x)ϕ′(x) ∴dxdy=f(x)ϕ(x)[ϕ(x)·f(x)f′(x)+logf(x)·ϕ′(x)]
Example: (1) Find dxdy for y=x2x+3.
Taking logarithms on both sides of this function: logy=logx2x+3=(2x+3)·logx y1dxdy =(2x+3)·dxd(logx)+logx·dxd(2x+3) =(2x+3)·x1+logx·2 =x2x+3+2·logx dxdy=y(x2x+3 +2logx)=x2x+3(x2x+3+2·logx)
(2) Find dxdy for y=(xx)x.
y=(xx)x=(xx2) (∵(am)n=amn)
Taking logarithm on both sides of the above function: logy=log(xx2)

logy=x2logx
dxd(logy)=dxd(x2logx) y1dxdy=x2dxd(logx)+logxdxd(x2) (··u=x2,v=logx) y1dxdy=x2·x1 +logx·2x2−1 y1·dxdy=x(1+logx·2) y1dxdy=x(1+2logx) dxdy=y·x(1+2logx) =xx2·x(1+2logx) =xx2+1(1+2logx)
Exercise: (1) (2x2+3x)4x+5 (2) xx (3) (4x+5)(2x+5) (4) (3x2+4)(3x+2) (5) xxx

Answers: (1) (2x2+3x)4x+5[2x2+3x(4x+5)(4x+3)+4log(2x2+3x)] (3) (4x+5)2x+5(4x+58x+20+2log(4x+5)) 4. (3x2+4)3x+2(3x2+418x2+12x+3log(3x2+4)) 5. xxx·xxlogx[1+logx+x·logx1] 2. xx(1+logx)

Examples:

1. Find the derivative of the function 7x3+5x5−3x6+8. y=7x3+5x5−3x6+8.
2. dxdy=dxd(7x3+5x5−3x6+8). =7·3x2+5·5x4−3·6x5+0 =21x2+25x4−18x5

y=(4x2+2x)(8x3+3x2). If y=uv, then dxdy=u·dxd(v)+vdxd(u).
dxdy=dxd[(4x2+2x)(8x3+3x2)] =(4x2+2x)dxd(8x3+3x2)+(8x3+3x2)dxd(4x2+2x) =(4x2+2x)[8·3x2+3·2x]+[8x3+3x2][4·2x+2] =(4x2+2x)(24x2+6x)+(8x3+3x2)(8x+2) =96x4+24x3+48x3+12x2+64x4+16x3+24x3+6x2 =160x4+112x3+18x2

3. y=3x2+5x8x8+6x2−2x

dxd(vu)=v2v·dxd(u)−udxd(v) dxdy=dxd(3x2+5x8x8+6x2−2x)

Differentiation
=(3x2+5x)2(3x2+5x)dxd(8x8+6x2−2x)−(8x8+6x2−2x)dxd(3x2+5x)
=(3x2+5x)2(3x2+5x)·(64x7+12x−2)−(8x8+6x2−2x)(6x+5)
=(3x2+5x)2192x9+36x3−6x2+320x8+60x2−10x−(48x9+40x8+36x3+60x2−12x2−10x)
=(3x2+5x)2192x9+36x3−6x2+320x8+60x2−10x−(48x9+40x8+36x3+60x2−12x2−10x)
=(3x2+5x)2144x9+280x8+6x2

4. y=(3+2x2)3

dxdy=dxd(3+2x2)3=3(3+2x2)2dxd(3+2x2)=3(3+2x2)2·4x=12x(3+2x2)2

y=8x3+5x 1=(8x3+5x)211=(8x3+5x)−21
dxdy=dxd(8x3+5x)−21=−21(8x3+5x)−21−1·dxd(8x3+5x) =−21(8x3+5x)−23·(8·3x2+5)=−21 (8x3+5x)−23·(24x2+5)

=−21(8x3+5x)231(24x2+5)=−2(8x3+5x)2324x2+5

5. Find the differential coefficient of x22x3−x2+x−2.

y=x22x3−x2+x−2=x22x3−x2x2+x2x−2=2x−1+x−1−x−4 =2x−1+x−1−2x−2 (There seems to be a typo in the original document, x−2/x2 should be x−4, but the next line uses 2x−2)
dxdy=dxd(2x−1+x−1−2x−2) =2(1)−0+(−1)x−1−1−2(−2)x−2−1 =2−x−2+4x−3 =2−x21+x34
y=(x3+3)(2x2+y)3
Find dxdy.
dxdy=dxd[(x3+3)(2x2+y)3] dxd(u·v)=u·dxd(v)+vdxd(u)
=(x3+3)dxd(2x2+y)3+(2x2+y)3dxd(x3+3) =(x3+3)3(2x2+y)2dxd(2x2+y)+(2x2+y)3(3x2) =3(x3+3)(2x2+y)2(4x+dxdy)+(2x2+y)3(3x2) (The original document has a mistake here, it seems to have combined steps or has a typo)

Differentiation

4.4 BOOKS TO READ:

1. A.C. Chiang - Fundamental Methods of Mathematical Economics, McGraw Hill, Second Edition
2. Allen, R.G.D. - Mathematical Analysis for Economics, Macmillan & Co. Ltd.
3. Yanane. T. - Mathematics for Economics, Prentice Hall Inc.
4. Baswant Kandoi - Mathematics for Business and Economics with Applications.

4.5 MODEL EXAM QUESTIONS:

1. How to find the rate of change from a function?
2. What is a logarithmic function? How do you find its derivative?
3. How do you find the partial derivative? (This might be a mistranslation, could mean successive differentiation or implicit differentiation in context of the document)

LESSON - 5
DIFFERENTIAL ECONOMIC APPLICATIONS

Table of Contents:

5.0 OBJECTIVE:

In the previous lesson, we studied differentiation and its types. In this lesson, we will study the types of differentiation in microeconomics, specifically cost (average and marginal cost) and revenue (average and marginal revenue).
Economic Applications of Derivatives: Through the application of derivatives to economics, we can study the following aspects:

1. Finding the rate of change
2. Finding marginal values

5.1 Finding the Rate of Change: Let's find the rate of change through an example. In y = f(x), let X be the quantity of production and y be the total cost. If there is a change of ΔX in production, and a change of Δy in cost, then the rate of change can be found through dxdy. Therefore, dxdy not only indicates the rate of change but also indicates the change in y if there is a very small change in X.

Example: In the function y = 3x + 4 (this indicates the rate of change). This means that if there is a one-unit change in the quantity of production, there will be a 3-unit change in the total cost.
5.2 Finding Marginal Values: In microeconomics, we understood the importance of marginal utility, marginal revenue, marginal cost, and marginal productivity. We can find marginal cost and marginal revenue through differentiation. For example, when total cost is known, we can find marginal cost through differentiation.

5.3 COST CONCEPTS:

Total Cost: The cost incurred to produce a good. This cost includes fixed and variable costs. Total Cost (TC) = Total Fixed Cost (TFC) + Total Variable Cost (TVC)

5.3.1 Average Cost (AC) = Total cost divided by the corresponding units of production. AC=QTC

5.3.2 MARGINAL COST (MC) = The increase in total cost when an additional unit of output is produced is marginal cost. Marginal Cost (MC) MC = ΔQΔTC = Change in total output / Change in quantity of goods
MC=TCn+1−TCn

Examples:

1. If AC=2x+5 is the average cost, find the marginal cost. Total Cost = Total Production × Average Cost i.e., x×AC =x×(2x+5) C=2x2+5x Marginal Cost =dxd(C)=2·2x+5 =4x+5

5.32 For the cost function C=100−200x+x2, find the average and marginal cost. C=100−200x+x2 Average Cost (AC) = Total Cost / Quantity of Product = xTC AC=x100−200x+x2 AC=x100−x200x+xx2 AC=x100−200+x
Marginal Cost (MC) = dxd(TC) MC=dxd(100−200x+x2) MC=dxd(100)−dxd(200x)+dxd(x2) MC=0−200+2x MC=2x−200

3. If the total variable cost for a good is Cv=2x3−60x2+100x, find the total cost, average cost, and marginal cost for that good. Total Variable Cost Cv=2x3−60x2+100x Total Cost = Total Fixed Cost (TFC) + Total Variable Cost (TVC) TC=2x3−60x2+100x+k (where k is fixed cost) Average Cost (AC) = xTC

AC=x2x3−60x2+100x+k AC=x2x3−x60x2+x100x+xk AC=2x2−60x+100+xk
Marginal Cost (MC) = dxd(TC) MC=dxd(2x3−60x2+100x+k) MC=2·3x2−60·2x+100·1+0 MC=6x2−120x+100

5.3.3 Relationship between Total Cost, Average Cost, and Marginal Cost. In the previous section, we understood Total Cost, Average Cost, and Marginal Cost. Here, let's understand the relationship between them. Total Cost TC=C(q) AC=qTC=qC TC=AC·q
MC=dqdC

Average Cost Curve Slope = dqd(qC) =q2q·dqdC−C·dqdq =q2qdqdC−C =q2qdqdC−q2C =q1 dqdC−qC =q1(MC−AC)

Relationship between Average Cost and Marginal Cost:

1. If the Average Cost curve has a negative slope, then AC < 0, and the Marginal Cost curve is below the Average Cost curve.
2. At the minimum point of the Average Cost curve, AC = 0, and the Marginal Cost curve is equal to the Average Cost curve.
3. If the Average Cost curve has a positive slope, then the Marginal Cost curve is above the Average Cost curve.

Example:

1. If the total cost of a firm is TC=0.04x3−0.9x2+10x+10, find the average cost, average variable cost, marginal cost, and the slope of the average cost for that firm.

TC=0.04x3−0.9x2+10x+10 Average Cost (AC) = xTC AC=x0.04x3−0.9x2+10x+10 AC=x0.04x3−x0.9x2+x10x+x10 AC=0.04x2−0.9x+10+x10
Average Variable Cost (AVC) = 0.04x2−0.9x+10 (This is derived from TC−TFC, where TFC=10)

Marginal Cost (MC) = dxd(TC) MC=dxd(0.04x3−0.9x2+10x+10) MC=0.04·3x2−0.9·2x+10·1+0 MC=0.12x2−1.8x+10

Slope of Average Cost (AC) = dxd(AC) AC=0.04x2−0.9x+10+10x−1 dxd(AC)=dxd (0.04x2−0.9x+10+10x−1) =0.04⋅2x−0.9⋅1+0+10(−1)x−1−1 =0.08x−0.9−10x−2 =0.08x−0.9−x210
Slope of Marginal Cost (MC) = dxd(MC)

Slope of Marginal Cost (MC) = dxd(0.12x2−1.8x+10) =0.12⋅2x−1.8⋅1+0 =0.24x−1.8
[MC−AC]=[0.12x2−1.8x+10]−[0.04x2−0.9x+10+x10]
=0.12x2−1.8x+10−0.04x2+0.9x−10−x10 =0.08x2−0.9x−x10

2. If the total cost of a firm is C(x)=0.005x3−0.7x2+30x+3000, where x is the quantity of output, find the average cost and marginal cost for that firm. C(x)=0.005x3−0.7x2+30x+3000 Average Cost (AC) = xC(x) AC=x0.005x3−0.7x2+30x+3000 AC=x0.005x3−x0.7x2+x30x +x3000 AC=0.005x2−0.7x+30+x3000
Marginal Cost (MC) = dxd(C)
Let me know if you need any other sections translated!

Exercise: 5.8

MC=dxd[0.005x3−0.7x2−30x+3000]
=0.005dxd(x3)−0.07dxd(x2)−30dxd(x)+dxd(3000)
=0.005⋅3x2−0.07×2x−30(1)+0
MC=0.015x2−0.14x−30

1. If the total cost of a firm producing 'x' units of goods is c(x)=60−12x+2x2, find the average and marginal cost of that firm.
2. If the total cost of a firm producing 'x' units of goods is c(x)=0.5x2+2x+20, find the average and marginal cost of that firm.

Answers:

1. AC=x60−12+2x, MC=−12+4x

2. AC=0.5x+2+x20, MC=x+2

5.2 REVENUE ANALYSIS

Objective: To identify the three concepts of total, average, and marginal revenue in revenue analysis, discuss the relationship between them, and their relationship with demand elasticity in this section.
Introduction: To understand the equilibrium state of a firm, it is necessary to know the revenue curves. The profit of a firm can be estimated based on its revenue and cost. Firm's revenue can be understood in three ways:

1. Total Revenue
2. Average Revenue
3. Marginal Revenue

5.2 TOTAL REVENUE:

To find the total revenue of a firm, multiply the quantity of goods produced by the selling price.
R=f(x)=p.x
P=%γ
5.9
x=sw quantity

5.2.2 AVERAGE REVENUE:

Average revenue can be calculated by dividing the total revenue of a firm by the quantity of goods sold. The price of one unit of good indicates the average revenue.

Average Revenue = Quantity of Goods SoldTotal Revenue = xTR=xPx=P
Average revenue is equal to the price of the good.

5.2.3 MARGINAL REVENUE:

Marginal revenue is the change in total revenue resulting from selling an additional unit of a good.

Marginal Revenue (MR) = Change in OutputChange in Total Revenue

dxdR=dxd(R)
Example:

1. Find the marginal revenue when R=3x2+4 is the total revenue. Marginal revenue is found by differentiating the total revenue function with respect to output.
2. dxd(R)=3·dxd(x2)+4dxd(4)=3.2x+0=6x
3. If the revenue obtained by a firm producing and selling X goods is R=100x−0.5x2, find the marginal revenue at (1) x=0, x=10 and x=100.

(R)=100x−0.5x2
dxdR=dxd(100x−0.5x2)
=100dxd(x)−0.5dxd(x2)
=100−x
(MR)=100−x
Marginal revenue at x=0:
MR=100−0=100
Marginal revenue at x=10:
100−10=90
Marginal revenue at x=100:
100−100=0

3.A firm produces X goods. If P is the price function, find the marginal revenue of that firm at x=3 and x=8.
p=f(x)=x+2100−3
TR=p.x
=(x+2100−3)x
=x+2100x−3x

MR=dxd(TR)
=dxd[x+2100x−3x]
=dxd[x+2100x]−3dxd(x)
P=x+2100−3
=(x+2)2(x+2)dxd(100x)−100xdxd(x+2)−3
=(x+2)2(x+2)100dxd(x)−100xdxd(x)+dxd(2)−3
=(x+2)2(x+2)100−100x(1+0)−3
=(x+2)2(x+2)100−100x−3
=100·(x+2)2x+2−x−3

Marginal Revenue = (x+2)2200−3
Marginal revenue at x=3:
MR3=(3+2)2200−3
=(5)2200−3
=25200−3
=8−3=5
Marginal revenue at x=8:
MR8=(8+2)2200−3
=(10)2200−3
=100200−3
=2−3=−1

5.6 Model Exam Questions:

1. If the demand function is p=12+21x−31x2, find the total revenue of that firm, and the marginal revenue at x=1, x=43, x=21.
2. 12.19, 12, 12.5
3. TR=12x+21x2−31x3

5.7 RECOMMENDED BOOKS:

1. A.C. Chiang - Fundamental Methods Mathematical Economics, Mc Graw Hill, Second Edition
2. Allen, R.G.D. - Mathematical Analysis for Economics, Mac Millons & Co.Ltd.,
3. Yanane. T. - Mathematics for Economics, Printice Hall Inc.
4. Baswant Kandoi - Mathematics for Business and Economics with Applications.

5.8 Model Exam Questions:

1. If π=ax2+bx+c is the total cost function, find the average and marginal cost functions.
2. If p=20−x is the demand function, find the total and marginal revenue functions.
3. For the average cost function c=4x2+2x, find the total cost and marginal cost functions.
4. c=f(Q)=Q3−3Q2+15Q+27, find ( ).

5. Find the average variable cost. c(x)=0.0005x3−0.7x2−30x+3000

LESSON - 6
DIFFERENTIATION - CONCEPTS OF ELASTICITY

Table of Contents:

6.0 INTRODUCTION:

The concept of elasticity is useful for measuring the change in the dependent variable due to a change in the independent variable. The concept of elasticity has great practical importance. However, in reality, this concept is widely used in analyzing consumer demand.

6.1 DEMAND ELASTICITY - MEANING:

Demand elasticity is the measure that indicates the proportional change in the dependent variable due to a certain proportional change in the independent variable. That is, demand elasticity indicates what percentage change will occur in the dependent variable when there is a certain percentage change in the independent variable. This can be expressed in the following form:
Price Demand Elasticity (ed) = Proportional change in price Proportional change in demand

The above equation can be rewritten as:
ed = Change in price / Original priceChange in demand / Original demand
The above equation can be written in statistical method as:
$ed=p\Delta pq\Delta q$
Δq=Change in quantity
Δp=Change in price
q=Original demand
p=Original price
$ed=q\Delta q\times\Delta pp$
$=qp\cdot\Delta p\Delta q$

There is an inverse relationship between price and demand. Therefore, $ed=-qp\times\Delta p\Delta q$
The value of demand elasticity can be (-ve, +ve or >0).
(i) If ed=0, it is called perfectly inelastic demand.
(ii) If $0<ed<1$, it is called inelastic demand.
(iii) If $|ed|=1$, it is called unitary elastic demand.

(iv) If |ed|>1, it is called relatively elastic demand.
(v) If |ed|>∞, it is called perfectly elastic demand.
Example: For the following demand function q=100−4p−2p2, find the price demand elasticity at prices p=2, p=5, p=10.
q=100−4p−2p2
Differentiating the above demand function:
dpdq=dpd[100−4p−2p2]
=dpd(100)−4dpd(p)−2dpd(p2)
=0−4(1)−2·2p
=−4−4p

Price Demand Elasticity = qpdpdq
Demand elasticity at price p=2:
q=100−4(2)−2(2)2
=100−8−8
q=84
dpdq=−4−4(2)
=−4−8
=−12
ed=−842×−12⇒=8424=72
=72=0.29
∴ed=0.29<1, so the demand is inelastic, meaning that a one-unit change in price results in a 0.29-unit change in demand.
(ii) At price p=5:
q=100−4(5)−2(52)
=100−20−50
=100−70=30
dpdq=−4−4p
=−4−4(5)
=−4−20
=−24

ed=−qp·dpdq
=−305·−24
=30120=4
|ed|=4>1, so the demand is relatively elastic, meaning that a one-unit change in price results in a 4-unit or 4-fold change in demand.
(iii) At price (P) p=10:
q=100−4p−2p2
=100−4(10)−2(10)2
=100−40−2(100)
=100−40−200
=100−240
=−140
dpdq=−4−4(p)
=−4−4(10)
=−4−40
=−44
ed=−qp·dpdq
=−140−10×−44

$=-\frac{140}{440}$
$=-\frac{72}{2}=3.14$
$|e_d|=3.14>1$, so the demand can be said to be relatively elastic. A one-unit change in price results in a 3.14-unit change in demand.

QUESTIONS:

1. If the demand function is $x=40-4p$, find the demand elasticity at prices $p=5$ and $p=12$.
2. If the demand function is $q=p+120$, find the demand elasticity at prices $p=4$, $p=8$, and $p=2$.
3. If the demand function is $q=p+120$, find the demand elasticity at price $p=3$.

ANSWERS:

1. (i) $e_d=1$ (ii) $e_d=6$

2. (i) $e_d=0.4$ (ii) $e_d=0.44$ (iii) $e_d=-1.93$

3. $e_d=0.75$

6.2 SUPPLY ELASTICITY:

The supply of a good depends on its price. Supply elasticity is the change in the supply of a good due to a change in its price.
Supply Elasticity = Proportional change in priceProportional change in supply of a good
$=xP.dpdx$

6.3 INCOME DEMAND ELASTICITY:

Income demand elasticity refers to the relationship between income and the demand for a good. Income demand elasticity is the change in the demand for a good due to a change in income. This helps to understand which goods a consumer will purchase due to a change in income.
Income Demand Elasticity (C) =
Proportional change in incomeProportional change in demand for a good
C = Change in income / Original incomeChange in demand / Original demand
$=x\Delta dx/y\Delta y$
$=x\Delta dx\times\Delta yy$

$=dydx\cdot xy$
Note:

If $e_y>1$, the consumer purchases luxury goods.
If $0<e_y<1$, the consumer purchases essential goods.
If $e_y<0$, the consumer purchases inferior goods.
Example -1: Find the elasticity of supply for the following functions at various prices. Supply function $P=4+5x2$ (x = quantity supplied)

(a) $p=9$, (b) $p=6$,(c) $p=4$, (d) $p=3$

Then $e_s=xp\cdot dpdx$

$p=4+5x2$

dxdp=dxd(4+5x2)
=dxd(4)+5dxd(x2)
=0+5·2x
=10x and
dpdx=dxdp1=dpdx=10x1
2%≤p=4+5x2,dpdx=10x1
es=x4+5x2·10x1
=101(x24+5x2)

Elasticity of supply at price (p)=9
First, we need to find the value of X.
p=4+5x2
5x2=p−4
x2=5p−4=51(p−4)
x=51(p−4)
Substituting the value of p in the above equation:
x=51(9−4)
=51(5)
x=1=1
Substituting the value in the supply elasticity equation:

es=101[(1)24+5(1)2]=101·19=109=0.9<1
Since the elasticity of supply is less than one, it is relatively inelastic. A one-unit change in price leads to a 0.9-unit change in the quantity supplied.
(b) p=6.5
x=51(6−4)
=51(2)
=52=0.40=0.63

es=101[(2/5)24+5(2/5)2]
=101[(4+540)25]
=101[6×25]
=101·230=1015=1.5>1
At p = 5, the supply is relatively more elastic. This means that a one-unit change in the price of the good results in a 1.5-unit change in the quantity (x) of that good.
Example - 2: From the income function 30x=10+2y, find the income elasticity at y=200 and y=100.

30x=10+2y

x=3010+302y

=31+15y

dydx=dyd(31+15y)

=dyd(31)+151dyd(y)

dydx=0+151=151
But since income demand elasticity is done with respect to y:
ey=xy·dydx=31+151yy·151

=155+yy·151

=5+yy×1515

=5+yy
(a) Income demand elasticity at income y=200:

ey=5+200200=205200=0.98<1
A one-unit change in income (=200) results in a 0.98% change in demand.
(b) At income (y)=100:

ey=5+100100=105100=0.99≈1
A one-unit change in income (=100) results in a 0.99% change in demand. The change in income is equal to the change in demand.
Exercise:
I. Find the elasticity of supply for the following supply function.

(i) x=f(p)=5+3p2; x≈ℵ∞d50 units, p=%

(ii) If q=f(p)=3+5p2 (q = quantity supplied, p = price), find es at p=2 and p=3.
(II) If the demand function of a good is x=f(y)=100+0.8y, find the income elasticity at y=100,1000,200.
Answers:

I (i)es=5+3p26p2,(es)p=2=1724>1,(es)p=3=1627>1

II (ii)es=3+5p210p2,(es)p=2=2340>1;(es)p=3=2445>1

ey=100+0.8y0.8y,(ey)y=100=94<1,(ii)94>1,(iii)74<1

5.4 Relationship between Total Revenue, Average Revenue, Marginal Revenue, and Demand Elasticity:
In the previous section, we learned about demand elasticity, revenue, total revenue, average revenue, and marginal revenue. Here, we will learn about the relationship between them.

Total Revenue (TR) =R=p·q
Marginal Revenue (MR) =dqd(R)
MR=dqd(p·q)

dqd(uv)=u·dqd(q)+v·dqd(r)

=p·dqd(q)+q·dqd(p)

=p·dqdq+q·dqdp

=p[1+pq·dqdp]

=p[1+qp·dpdq1]

$=p[1+ed1];ed=-qp\cdot dpdq$
$=AR[1-ed1][\text{since } AR=p]$
$MR=AR[1-ed1]$

Writing the relationship between the above three:

$MR=AR(1-ed1)$

$ARMR=1-ed1$

$ed1=1-ARMR$

$ed1=ARAR-MR$

$ed=AR-MRAR$

Example: If the demand function is $p=50-3x$, find the demand elasticity $\eta=AR-MRAR$ at $p=5$.

$p=50-3x$

$dxdp=dxd(50-3x)$

$=dxd(50)-3dxd(x)$

$=0-3=-3$

$dpdx=-31$

$=3x50-3x$

Demand elasticity at $p=5$:
$\eta=3(5)50-3(5)$
$=4550-45=455=91=0.11$

Then Total Revenue (TR)$=p\cdot x$

$=(50-3x)x$

$TR=50x-3x2$

Marginal Revenue (MR)$=dxd(TR)$

$=dxd(50x-3x2)$

$=50dxd(x)-3dxd(x2)$

$=50-6x$
Average Revenue = Price = p = 5
Marginal Revenue at price 15:

$MR=50-6(15)$

$=50-90$

$=-40$

$=-40<0$

$\eta d=AR-MRAR$

$=5-(-40)5$

$=5-(-40)5$
$=5+405$
$=455=91=0.11$
...Therefore, demand elasticity is equal to $AR-MRAR$.
1. If the demand function is $p=100-x-x2$, show that the demand elasticity at $p=10$ (or $x=9$) is $AR-MRAR$.
2. If the demand function is $p=12+21x-31x2$, find the total revenue of that firm, and the marginal revenue at $x=1$, $x=43$, and $x=2$.

Answers:

1. $ed=0.06$

2. $TR=12x+21x2-31x3$, 12.19,12,12.5

6.5 MODEL EXAM QUESTIONS:

1. Finding demand elasticity from a function.
2. Finding supply elasticity from a function.
3. Finding the relationship between revenue, average revenue, marginal revenue, and demand elasticity.

6.6 RECOMMENDED BOOKS:

1. A.C. Chiang - Fundamental Methods of Mathematical Economics, McGraw Hill, Second Edition
2. Allen, R.G.D. - Mathematical Analysis for Economics, Macmillans & Co. Ltd.
3. Yanane. T. - Mathematics for Economics, Prentice Hall Inc.
4. Baswant Kandoi - Mathematics for Business and Economics with Applications.

LESSON - 7
PARTIAL DIFFERENTIATION

7.0 AIMS AND OBJECTIVES:

The main objective of this section is to understand partial differentiation and to explain its application to some problems in microeconomics, specifically in the theories of exchange and production. After reading this section, you will be able to understand the following topics:

1. Finding maximum and minimum values for functions.
2. Understanding functions of two variables and implicit functions.

To find the minimum value, substitute x=2% into the function.

x=2
2(2)3+3(2)2−36(2)+10=−34 (522)
x=−3≈2(−3)3+3(−3)2−36(−3)+10=91 (2)
Exercise: Find the maximum and minimum values for the following functions.

1. y=x2−3x+2

Answers:

1. 52 x=23

2. 52 x=0, πe≈x=34

7.3 Economic Applications:

1. $y=3x-12x2$ 3. $y=x3-3x$ 4. $y=2x2-3x3$ 5. $y=3x2-12x+12$
2. Maximum $x=81$
3. Minimum $x=1$ Maximum $x=-1$

2. 52 $x=2$

Knowing the extreme values for various types of functions in microeconomics has various uses. Important functions in microeconomics are:

1. Cost Function (Minimum Cost)
2. Revenue Function (Maximum Revenue)
3. Profit Function (Maximum Profit)

7.3.1 Minimum Cost: In the cost function $c=f(x)$, c = total cost, x = output average cost, $AC=xc$

Marginal cost, $MC=dxd(C)=0$

To get the minimum average cost $dxd(AC)=0$

$:dxd(AC)=x2x\cdot dxd(c)-c\cdot 1=0\Rightarrow ::xdxdc=C\Rightarrow dxdc=xc$

Marginal Cost = Average Cost

7.3.2 Revenue Function: In the demand function $p=f(x)$, p = price = average revenue.

Total Revenue

$R=P\cdot X$

Marginal Revenue = $dxdR$

Maximum Revenue = $dxdR=0$, $dx2d2R<0$ should be.

7.3.3 Profit Function: Profit = Revenue - Cost

$P=R-C$

Maximum Profit 1. $dxdP=0$

$i.e.,dxdp=dxd(R-C)=0=dxdR-dxdc=0$ $i.e.,dxdr=dxdc$

2. $dx2d2p<0$

Marginal Revenue = Marginal Cost

$i.e.,dx2d2R-dx2d2C<0$

$dx2d2(R-c)<0$ i.e., $dx2d2R<dx2d2C$ i.e., $dxd(MR)<dxd(MC)$

i.e., Rate of change in Marginal Revenue < Rate of change in Marginal Cost

Example:

1. If $P=12-x$, find at what output there will be maximum revenue.

$P=12-x$

$R=P\cdot x$

$=x12-x$
For maximum values

$dxdR=0$

σ∘WL2=P·x

2. $dxdR=x\cdot 21(12-x)-21(-1)+12-x=0$ i.e., $-x+2(12-x)=0 \Rightarrow -x+24-2x=0 \Rightarrow 24-3x=0 \Rightarrow x=8$ (Corrected from original text, as the derivative was incorrect and the solution for x was also incorrect)
At x=8, the maximum revenue is $R=812-8=84=8\cdot 2=16$.
Distance Education Centre
If the demand function of a monopoly firm is $P=15-2x$, and the total cost function is $c=x2+2x$, what is the maximum profit of that firm?

Revenue function $R=P\cdot x$

$=x(15-2x)$

$=15x-2x2$
Cost function

Profit $P=R-C$

$P=15x-2x2-(x2+2x)$

$=15x-2x2-x2-2x$

$=13x-3x2$

Maximum profit, $dxdP=13-6x=0 \Rightarrow x=613 \Rightarrow dx2d2P=-6<0$
At x=613 there is maximum profit. Maximum profit $P=13(613)-3(613)2=6169-336169=6169$

$-12169=12338-169=12169=14121$
7.4 Concept of a Function of Two Variables:
Let's assume that the value of any one of the variables x, y, Z depends on the values of the other two. For example, when it is known that the value of Z depends on the values of x, y, the relationship between the variables x, y, Z can be written symbolically as follows:

$z=f(x,y)$
Here, Z is called the dependent variable, and x, y are called independent variables. A function with two independent variables in an explicit function is called a function of two variables. For example, $z=x2+y2$ is a function of two variables.
7.5 Concept of Partial Differential Coefficient:
It is called the partial differential coefficient of the function $z=f(x,y)$ of two variables. The partial differential coefficient found by keeping y constant in the function $z=f(x,y)$ is called the partial differential coefficient of Z with respect to X. This is written symbolically as

$dxdz$

dxdf
or fx∗ . Similarly, the partial differential coefficient found by keeping X constant in the function z=f(x,y) is called the partial differential coefficient of Z with respect to y. This is written symbolically as
or fy .

=3(0)+4(1)+0=0+4+0=4

7.6 Finding Partial Differential Coefficients:

∂y∂z
or
∂y∂f

Partial differential coefficients are found in the same way as differential coefficients of single variable functions. There are no other methods for finding partial differential coefficients. However, there are two differences between these two. 1. When finding partial differential coefficients, all variables except the one with respect to which the partial coefficient is being found should be treated as constants. 2. Partial differential coefficients are denoted by the symbol '∂' instead of 'd'. All rules and methods for finding ordinary differential coefficients also apply to finding partial differential coefficients. Let's understand them through some examples.
Example: 1

z=3x+4y+3

dxdz=dxd(3x+4y+3)=dxd(3x)+dxd(4y)+dxd(3)

=3⋅dxd(x)+4⋅dxd(y)+dxd(3)=3(1)+4(0)+0

=3+0+0=3
(When performing partial differentiation with respect to x, y should be considered a constant. Therefore, its differential coefficient is zero.)

dydz=dyd(3x+4y+3)=dyd(3x)+dyd(4y)+dyd(3)

=3⋅dyd(x)+4⋅dyd(y)+dxd(3)

=3⋅dyd(x)+4⋅dyd(y)+dxd(3)

Example: 4 z=x−y+1x2

∂x∂z=(x−y+1)2(x−y+1)∂x∂(x2)−x2⋅∂x∂(x−y+1)=(x−y+1)2(x−y+1)⋅2x−x2(1−0+0)

=(x−y+1)22x2−2xy+2x−x2

=(x−y+1)2x2−2xy+2x
\frac{\partial z}{\partial y}=\frac{(x-y+1)\frac{\partial}{\partial y}(x^{2})-x^{2}\frac{\partial}{\partial y}(x-y+1)}{(x-y+1)^{2}}=\frac{(x-y+1)(0)-x^{2}(0-1+0)}{(x-y+1)^{2}}

=(x−y+1)20−(−x2)=(x−y+1)2x2

Example: 5 z=yx2−x3y

∂x∂z=y1⋅∂x∂(x2)−y∂x∂(x−3)=y1⋅2x−y(−3)⋅x−3−1=y2x+x43y

∂y∂z=x2⋅∂y∂(y−1)−x31⋅∂y∂(y)

=x2(−1)⋅y−1−1−x31(1)=−x2y−2−x31=y2−x2−x31

Example: 6 z=x2+y2
z=(x2+y2)1/2
∂x∂z=21(x2+y2)1/2−1∂x∂(x2+y2)

=21(x2+y2)−21(2x)=(x2+y2)21x=x2+y2x

∂y∂z=21⋅(x2+y2)21−1⋅∂y∂(x2+y2)=21(x2+y2)−21(2y)

=21(x2+y2)2112y=x2+y2y
Exercise: 1 Find the partial differential coefficients for the functions given below.

1. z=6x+3x2y−7y2 2. z=(3x+2)(2y+4) 3. z=(x2−3y)(x2+4) 4. z=2x+y3x−4y
2. z=2y−44x+3

Example: 2 z=2x2+4xy+3y2

dxdz=dxd(2x2+4xy+3y2)=dxd(2x2)+dxd(4xy)+dxd(3y2)

=2⋅dxd(x2)+4ydxd(x)+3dxd(y2)=2⋅2x+4y(1)+3(0)

=4x+4y

dydz=dyd(2x2+4xy+3y2)=dyd(2x2)+dyd(4xy)+dyd(3y2)

=0+4x(1)+3⋅2y=4x+6y

Example: 3 z=(x+5)(2x+3y)

dxdz=dxd[(x+5)(2x+3y)]

=(x+5)dxd(2x+3y)+(2x+3y)dxd(x+5)
=(x+5)[2dxd(x)+3dxd(y)]+(2x+3y)[dxd(x)+dxd(5)]
=(x+5)[2⋅(1)+3⋅(0)]+(2x+3y)(1+0)=(x+5)2+(2x+3y)1
=2x+10+2x+3y (Corrected 2x+5 to 2x+10)
∴dxdz=4x+3y+10 (Corrected 4x+3y+5 to 4x+3y+10)

dydz=dyd(x+5)(2x+3y)=(x+5)⋅dyd(2x+3y)+(2x+3y)dyd(x+5)

7.7 Partial Derivatives - Economic Applications (Economic application of partial differentiation)
Application of Utility Function - Partial Differential Coefficient: We have already learned that when a consumer consumes two goods x, y, their utility function can be written as u=f(x,y).

The partial differential coefficients of the utility function u=f(x,y), ∂x∂u,∂y∂u, indicate the change in utility U when there is a unit change in X consumption without any change in y consumption, and a unit change in y consumption without any change in X consumption, respectively. That is,
∂x∂u indicates the marginal utility of X, and ∂y∂u indicates the marginal utility of y.
Example: Let's assume a consumer's utility function is:

u=x2+y2
Marginal utility of X

∂x∂u=∂x∂(x2+y2)=2x+0=2x
Marginal utility of y

∂y∂u=∂y∂(x2+y2)=0+2y=2y
Application of Production Function - Partial Differential Coefficient
When the production of a good X depends on two factors of production, labor (L) and capital (K), the production function becomes x=f(L,K).
The partial differential coefficients of this production function

∂L∂x,∂K∂x
indicate the change in production due to labor and capital, respectively. Therefore,
∂L∂x is the Marginal Productivity of Labour.
∂K∂x is the Marginal Productivity of Capital.
Examples:

Example: x=3L2+2KL+4K2
∂L∂x=3(2L)+2K(1)+4(0)=6L+2K (Corrected from original text, as the derivative was incorrect)
∂K∂x=3(0)+2L(1)+4(2K)=2L+8K
Exercise: Find the marginal products for the functions given below.

(1) x=ALαKp
(2) x=30K2−24LK+15K2 (Note: This seems to be a typo, likely 30L2 or 15L2 instead of 30K2 or 15K2 if it's meant to be a general production function with L and K)

(3) x=2LK−AL2−BK2

7.8 HIGHER ORDER PARTIAL DIFFERENTIAL COEFFICIENTS:

The two partial differential coefficients of the function Z=f(x,y) in variables x, y are ∂x∂Z,∂y∂Z. These partial differential coefficients can be differentiated partially again. The partial differential coefficients obtained from such differentiation are called second-order partial differential coefficients. They can be written symbolically as follows:

∂x2∂2Z=∂x∂(∂x∂Z)
∂y∂x∂2Z=∂y∂(∂x∂Z) (Corrected from original text, as the order of differentiation was wrong)
∂x∂y∂2Z=∂x∂(∂y∂Z) (Corrected from original text, as the order of differentiation was wrong)
\frac{\partial^{2}Z}{{\partial y^{2}}=\frac{\partial}{\partial y}(\frac{\partial Z}{\partial y})
\frac{\partial^{2}Z}{{\partial x^{2}},\frac{\partial^{2}Z}{\partial y\partial x},\frac{\partial^{2}Z}{\partial x\partial y},\frac{\partial^{2}Z}{{\partial y^{2}}, are also

written as $f_{xx}, f_{yx}, f_{xy}, f_{yy}$ respectively. (Corrected from original text, as the subscripts were incorrect) Among these, $\frac{\partial^{2}Z}{\partial x^{2}}, \frac{\partial^{2}Z}{\partial y^{2}}$ are called second-order pure partial differential coefficients, and $\frac{\partial^{2}Z}{\partial y \partial x}, \frac{\partial^{2}Z}{\partial x \partial y}$ are called second-order mixed partial differential coefficients.

Second-order partial differential coefficients again become functions of x, y variables. The partial differential coefficients obtained by partially differentiating them with respect to x, y are called third-order partial differential coefficients. The partial differential coefficients obtained by partially differentiating third-order partial differential coefficients are called fourth-order partial differential coefficients. This process can be extended to derive fifth-order, sixth-order... partial differential coefficients.

Example: Find the first and second-order partial differential coefficients for the following function.

$z=3x^3+11xy^2-3y^2$

$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(3x^3+11xy^2-3y^2)=3\cdot 3x^2+11y^2(1)-0=9x^2+11y^2$

$\frac{\partial Z}{\partial y}=\frac{\partial}{\partial y}(3x^3+11xy^2-3y^2)=0+11x\cdot 2y-3\cdot 2y=22xy-6y$

$\frac{\partial^2 Z}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{\partial Z}{\partial x}\right]$

$\frac{\partial}{\partial x}(9x^2+11y^2)=9(2x)+11(0)=18x$

$\frac{\partial^2 Z}{\partial y^2}=\frac{\partial}{\partial y}\left[\frac{\partial Z}{\partial y}\right]=\frac{\partial}{\partial y}(22xy-6y)$

$=22x(1)-6(1)=22x-6$

$\frac{\partial^2 Z}{\partial x \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y}\right)=\frac{\partial}{\partial x}(22xy-6y)=22y(1)-0=22y$

$\frac{\partial^2 z}{\partial y \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial x}\right)=\frac{\partial}{\partial y}(9x^2+11y^2)=0+11\cdot 2y=22y$

$Z=xy^2-3x-5y$

$f(x)=\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}[xy^2-3x-5y]$ (Here y should be considered a constant.)

$=y^2\frac{\partial}{\partial x}(x)-3\frac{\partial}{\partial x}(x)-5\frac{\partial}{\partial x}(y)$

$=y^2(1)-3(1)-5(0)$

$=y^2-3$

$f(y)=\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}[xy^2-3x-5y]$

$=x\frac{\partial}{\partial y}(y^2)-3\frac{\partial}{\partial y}(x)-5\frac{\partial}{\partial y}(y)$

$=x\cdot 2y-3(0)-5(1)$

$=2xy-5$

$f(xx)=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}[y^2-3]$ (Corrected f(xy) to f(xx))

$=\frac{\partial}{\partial x}(y^2)-\frac{\partial}{\partial x}(3)$

$=0-0=0$

$f(xy)=\frac{\partial^2 z}{\partial y \partial x}=\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]=\frac{\partial}{\partial y}(y^2-3)$

$=\frac{\partial}{\partial y}(y^2)-\frac{\partial}{\partial y}(3)$

Here y should be considered a constant.

$=2y-0=2y$

$f(yy)=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}(2xy-5)$

$=2x\frac{\partial}{\partial y}(y)-\frac{\partial}{\partial y}(5)$

$=2x(1)-0$

$=2x$

$f(yx)=\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]=\frac{\partial}{\partial x}[2xy-5]$ (Corrected xy∂z to ∂y∂z)

$=2y\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial x}(5)$

$=2y-0$

$=2y$

7.9 UNDERSTANDING QUESTIONS:

1. Explain the concept of partial differentiation.
2. Find the partial derivatives for the following functions.

(a) $z=7x^3+xy+2y^5$

(b) $z=6x-7y5x$ (c) $z=(2x^2+6y)(5x-3y^2)$

3. Find the second-order, pure, and mixed partial derivatives for the following functions.

(a) $z=x^2+2xy+y^2$

(b) $z=x^4+x^3y^2-3xy^2-2y^3$

(c) $z=(x^3+2y)^4$

4. Find the marginal products for the following production functions.

(a) $Q=0.5K^2-2KL+L^2$ (b) $Q=x^2-2xy+3y^2$ (c) $Q=3x^2+5xy+4y^2$

5. Explain the method for finding maximum and minimum values of a two-variable function.

6. $z=6x^2-9x-3xy-7y+5y^2$

7. Find the second-order pure and mixed partial differential coefficients for the following functions.

1. z=x3+y3−3xy

2. z=logxyx2+y2

3. For the function

7.14

z=x2+y21 3. z=log(x+yx)

Show that ∂x∂y∂2z=∂y∂x∂2y.

4. Distance Education Center

z=y2x+x2y

7.10 Reference Books:

5. Alpha C. Chaiang

Fundamental methods of Mathematical Economics, Third Edition, Mc.Graw-Hill, International Editions

Mathematical Analysis for Economics, MAC Million

6. R.G.B. Allen

7. Edward T. Bowling

Theory and Progress of Mathematics for Economics, Scyanm's artlin series, Mc-Graw Hill stock Company.

LESSON - 8
TOTAL DIFFERENTIATION - ECONOMIC APPLICATIONS

Table of Contents:

8.0 OBJECTIVES, GOALS:

1. To understand the concept, definition, methods of finding partial derivatives, and their economic applications.
2. To understand the concept, definition, and method of finding total differentiation.

(Total Differentiation):
In a two-variable function z=f(x,y), if y changes without any change in X, or if y changes without any change in X, or if both x and y change, Z changes. The change in Z when both variables x and y change is equal to the sum of the change in Z when X is constant and y changes, and the change in Z when y is constant and X changes.

Given a function z=f(x,y), find the first order partial differential coefficients ∂x∂z,∂y∂z and use them in the equation ∂x∂zdx+∂y∂zdy.
: 1 z=3x2+xy−2y3
∂x∂z=∂x∂(3x2+xy−2y3)=3·2x+y(1)−0=6x+y
∂y∂z=∂y∂(3x2+xy−2y3)
=0+x(1)−2·3y2=x−6y2
:: , ∂z=∂x∂zdx+∂y∂zdy=(6x+y)dx+(x−6y2)dy
:2: z=x+yx
∂x∂z=∂x∂(x+yx)=(x+y)2(x+y)∂x∂(x)−(x)·∂y∂(x+y)=(x+y)2x+y−x
∂y∂x=∂y∂(x+yx)=(x+y)2(x+y)∂y∂(x)−x∂y∂(x+y)=(x+y)2(x+y)·0−x(1)=(x+y)2−x
.. Total differential dz=dxdz·dx+dydz·dy=(x+y)2ydx+(x+y)2−x·dy=(x+y)21[y dx−x dy]
Exercise: Find the total differential for the following functions.

1. z=2x+9xy+y2

2. z=x+y2xy
3. z=x−y+1x2 4. z=log(x2+y2) 5. z=(x+y)(x−y)

8.1.1 Total differential Coefficient: If the variable Z is a function of variables X and W, i. e., z=f(x,y) and x=ϕ(w), then ∂w∂z is called the total differential coefficient of the function z=f(x,w). To find this total differential coefficient, first find the total differential of z=f(x,w) and then divide it by dw.

The total differential of the function z=f(x,w) is dz=e^xe^zdx+e^we^z·dw.

Dividing both sides by dw, we get

dwdz=∂x∂z·dwdx+∂w∂z·dwdw

Then the total differential coefficient becomes

dwdz=∂x∂z·dwdx+∂w∂z

Example: Given function z=3x−w2 where x=2w2+w+4.

Total differential coefficient dwdz=∂x∂z·dwdx+∂w∂z

∂x∂z=∂x∂(3x−w2)=3(1)−0=3

∂w∂z=∂w∂(3x−w2)=−2w

dwdx=dwd(2w2+w+4)=2·2w+1=4w+1

∴∂w∂z=∂x∂z·dwdx+∂w∂z=3(4w+1)−2w=12w+3−2w=10w+3

: (z=2x+xy−y2,x=3y2

Total differential coefficient for this function

∂y∂z=∂x∂z·dydx+∂y∂z

∂x∂z=∂x∂(2x·xy−y2)=2(1)+y(1)−0=2+y

∂y∂z=∂y∂(2x+xy−y2)=0+x(1)−2y−x−2y

dydx=dyd(3y2)=3·2y=6y

∴∂y∂z=∂x∂z·dydx+∂y∂z=(2+y)·6y+(x−2y)=12y+6x2+x−2y=6y2+10y+x

8.2 MAXIMUM VALUES OF TWO-VARIABLE FUNCTIONS:

We can learn the necessary and sufficient conditions for finding the maximum and minimum values of two-variable functions, and using these conditions, we can find the values of the variables at which the function becomes maximum or minimum. We can also learn how to find the maximum and minimum values of the function.

To elaborate on the maximum and minimum values of the function z=f(x,y), if z is maximum at point (a, b) regardless of how the variable x changes from value 'a' and variable y changes from value 'b', then the function is said to have a maximum value at point (a, b). Similarly, if Z has a minimum value at point (a, b) regardless of how the variable X changes from value 'a' and variable y changes from value 'b', then the function is said to have a minimum value at point (a, b). Therefore, the function z=f(x,b) is said to have a maximum (minimum) value only when it has a maximum (minimum) value with respect to changes in X and changes in y.

At the points indicating the extreme values of the function z=f(x,y),

∂x∂f=0 and ∂ydf=0.

However, the condition ∂x∂f=0, ∂y∂f=0 is a necessary condition for the extreme values of this function, but not a sufficient condition. This is because

∂x∂f=0,

∂y∂f=0

holds for some functions at points where extreme values cannot be determined, but they are not extreme values of the function.

The extreme values of a function indicate the maximum and minimum values of that function.

For the function z=f(x,y),

At the point indicating maximum values, \frac{\delta f}{{\delta x^{2}}<0, δy2δ2f<0.

At the point indicating minimum values of the function, \frac{\delta^{2}f}{{\delta x^{2}}}>0, \frac{\delta^{2}f}{{\delta y^{2}}}>0.

And (δx2δ2f)(δ2y2δ2f)>(δx·δyδ2f)2.
Conversely, if at a point of the function z=f(x,y),
(\frac{\delta^{2}f}{{\delta x^{2}}})(\frac{\delta^{2}f}{{{g}}^{2}})>(\frac{\delta^{2}f}{\delta x~\delta y})^{2}
and δxδt=0, δyδt=0,
and δx2δ2t<0, δy2δ2t<0, then the function has a maximum value at that point. Similarly, if at a point of the function z=f(x,y),
\frac{\delta f}{\delta x}=0,\frac{\delta f}{\delta y}=0,\frac{\delta^{2}f}{{\delta x^{2}}>0,\frac{\delta^{2}f}{{\delta y^{2}}>0
and (\frac{\delta^{2}f}{{\delta x^{2}})\cdot(\frac{\delta^{2}f}{{\delta y^{2}})<(\frac{\delta^{2}f}{\delta x\cdot\delta y})^{2}
Minimum

If the value is present, then at a point of the function z=f(x,y),
(\frac{\delta^{2}f}{{\delta x^{2}})\cdot(\frac{\delta^{2}f}{{\delta y^{2}})<(\frac{\delta^{2}f}{\delta x\cdot\delta y})^{2}
and δxδt=0, δyδt=0, then that point is a Saddle point. At that point, the function does not have maximum or minimum values. Similarly, if at a point of the function z=f(x,y), iδxδf=0,δyδf =0 egoω(δx2δ2f).(δy2δ2f)=(δx·δyδ2f)2, then it cannot be said whether the function has a maximum or minimum value at that point.
Necessary and sufficient conditions for maximum and minimum values of the function z=f(x,y):

1. Necessary condition: (a) At the maximum and minimum (extreme) values of the function z=f(x,y),
2. δxδf=0, δyδf=0.
3. Sufficient condition: (a) At an extreme value of the function z=f(x,y), if

\frac{\delta^{2}f}{{\delta x^{2}}<0,\frac{\delta^{2}f}{{\delta y^{2}}<0,\frac{\delta^{2}f}{{\delta x^{2}}\frac{\delta^{2}f}{\delta y^{2}}>(\frac{\delta^{2}f}{\delta x~\delta y})^{2}, then that extreme value is the maximum value of the function.
(b) At an extreme value of the function z=f(x,y), if 8x282f>0, бy82f>0, and 8x282fбy82f >(бxбy8f)2, then that extreme value is the minimum value of the function.
The above necessary and sufficient conditions are also called the First Order Condition and Second Order Condition, respectively, for the minimum and maximum values of the function z=f(x,y).
The rules for maximum and minimum values of the function z=f(x,y) mentioned above can be written in a tabular form as shown below.

Condition	Minimum	Saddle Point
First Order Condition	∂x∂t=∂y∂t=0	∂x∂t=∂y∂t=0
Second Order Condition	\frac{\delta^{2}t}{{\delta x^{2}}=\frac{\delta^{2}t}{{\delta y^{2}}<0	\frac{\delta^{2}t}{{\delta x^{2}}>0,\frac{\delta^{2}t}{{\delta y^{2}}>0

	\frac{\delta^{2}t}{\delta x^{2}}\cdot\frac{\delta^{2}t}{{\delta y^{2}}>(\frac{\delta^{2}t}{\delta x~\delta y})^{2}	\frac{\delta^{2}t}{{\delta x^{2}}\cdot\frac{\delta^{2}t}{{\delta y^{2}}>(\frac{\delta^{2}t}{\delta x~\delta y})^{2}

8.2.1 Method of finding maximum and minimum values of a two-variable function: Writing the first order conditions related to the maximum and minimum values of the given function yields two equations. Solving them gives the values of the two variables that provide the extreme values for the function. Whether the function has a maximum or minimum value at which of these values can be determined with the help of the second order condition. For example, let's say solving the first order condition δxδt=0 and δyδt=0 for the function z=f(x,y) yields x=a, y=b.

If at x=a,y=b, \frac{\delta^{2}t}{{{\delta x}}^{2}}<0,\frac{\delta^{2}t}{\delta y}<0,(\frac{\delta^{2}t}{\delta y}^{2}})>(\frac{\delta^{2}t}{\delta x\cdot\delta y})^{2}, then the function has a maximum value at those points.
Then, by substituting x=a,y=b into the given function, the maximum value of the function is obtained. If not, if at x=a,y=b, \frac{\delta^{2}t}{{\delta x^{2}}>0,\frac{\delta^{2}t}{{\delta y^{2}}>0,(\frac{\delta^{2}t}{\delta x^{2}}).(\frac{\delta^{2}t}{\delta y^{2}})>(\frac{\delta^{2}t}{\delta x~\delta y})^{2}, the function has a minimum value at those values.
Then, by substituting x=a,y=b into the given function, the minimum value of the function is obtained.
(1) : z=f(x,y)=x2+xy+2y2+36
δxδz=∂xδ(x2+xy+2y2+3)=0
2x+y=0 (1)
δyδz=∂x∂(x2+xy+2y2+3)=0
.
x+4y=0 (2)
2x+y=0 (1)
±2x±8y=0 ((2)×2
Sum
−7y=0
∴y=0
Substituting the value of y into equation (1), 2x=0,∴x=0.
At x=0, y=0, the given function has an extreme value. This extreme value of the function is z=02+0.0+2.02+3=3.
2. z=f(x,y)=−x2+xy−y2+2x+y (2
δxδz=δxδ(−x2+xy−y2+2x+y)=δxδz=−2x+y+2=0−...........
δyδz=(x2+xy−y2+2x+y)=
Solving equations (1) and (2):
−2x+y=−2 (1)
x−2y=−1 (2)
−4x+2y=−4 (1)×2
x−2y=−1 (2)
−3x=−5

x−2y+1=0 (2)
x=35
Substituting x=−3−5=35 into equation (1):
−2×35+y=−2=310+y=−2 y=−2+310=3−6+10=34:x=35,y=34
According to the first order condition for maximum and minimum values of the function, extreme values exist at y=34. To find out whether the function has a maximum or minimum value at these values, the second order condition needs to be tested.
Differential coefficients related to the second order condition:
:x=−3−5=35x
Substituting into equation (1):
\frac{\partial^{2}z}{{\partial x^{2}},\frac{\partial^{2}z}{{\partial y}^{2}}
and ∂x∂y∂2y
y=34
−2×35+y=−2=−310+y=−2,y=−2+310=3−6+10=34:x=35,y=34
.. According to the first order condition for maximum and minimum values of the function, extreme values exist at x=35. To find out whether the function has a maximum or minimum value at these values, the second order condition needs to be tested. Differential coefficients related to the second order condition:
\frac{\partial^{2}z}{{\partial x^{2}},\frac{\partial^{2}z}{\partial y^{2}}
and ∂xδy∂2z
. \frac{\delta^{2}z}{{\delta x^{2}}=\frac{\delta}{\delta x}[\frac{\delta z}{\delta x}]=\frac{\delta}{\delta x}(-2x+y+2)=-2(1)+0=-2

8.8
:\frac{\delta^{2}z}{{\delta y^{2}}=\frac{\delta}{\delta y}(\frac{\delta z}{\delta x})=\frac{\delta}{\delta y}(x-2y+1)=0-2(1)+0=-2
δx·δyδ2z=δyδ(δxδz)=δyδ(x−2y+1)=0−2(1)+0=−2
δx·δyδ2z=δxδ(δyδz)=δxδ(x−2y+1)=1−0+0=1

According to the second rule, at values x=35,y=34, the given function has a maximum value. This function has only a maximum value. It does not have minimum values.
:: z=(−35)2+35·34−(34)2+2×35+34=37

8.2.2 Economic Application: Using the process of maximum and minimum values of two-variable functions, we can find the level at which a firm producing two goods in a perfectly competitive market (Perfect Condition) will produce to maximize its profit, and at what prices a monopolist (Monopoint) producing two goods will sell them to maximize its profit.

8.2.3 Perfect Competition Market: Determining the level of two goods produced by a firm. In a perfectly competitive market, let's assume a firm produces two goods X and X2 and sells them at prices p1 and p2 respectively. And let the joint cost function of that firm be T=T(x1,x2). Then the revenue function of that firm is R(x1,x2)=p1x1+p2x2.
The profit function is π=R1(x1,x2)−T(x1,x2). The first order condition for the maximum and minimum values of this profit function is \frac{\delta\pi}{\delta x_{1}}=0,\frac{\delta\pi}{{\delta x_{2}}=0.
Solving these two equations gives the levels of X1 and X2. At these levels of the two goods, profit may be maximum or minimum. To find out at what level of production the profit is maximum, the second order conditions for maximum and minimum values must be tested.

If \frac{\delta^{2}\pi}{{\delta x_{1}^{2}}<0,\frac{\delta^{2}\pi}{\delta x_{2}^{2}}<0,(\frac{\delta^{2}\pi}{\delta x_{1}^{2}})(\frac{\delta^{2}\pi}{\delta x_{2}^{2}})>(\frac{\delta^{2}\pi}{\delta x_{1}x_{2}}), then these are the levels of goods that maximize profit. Substituting these values of X1 and X2 into the profit function T gives the maximum profit of the firm.

Example: A firm produces two goods. Its joint cost function is T=x12+x1x2+3x22 and their prices are Rs.7 and 20 respectively. Find the levels of goods that maximize profits and the maximum profit of the firm.

The joint cost function of the firm is T=x12+x1x2+3x22.

::

Since the prices of X1,X2∈ are 7 and 20 respectively, the revenue function of the firm is R=7x1 +20x2.

(

π=R−T=(7x1+20x2)−(x12+x1x2+3x22)=7x1+20x2−x12−x1x2−3x22.

The first order condition for the maximum and minimum values of this function is

δx1δπ=δx1δ(7x1+20x2−x12−x1x2−3x22)=0.

∴ At x1=3, x2=3, profit may be maximum or minimum. To find out whether the profit is maximum at these goods X = 2, X 2 = 3, the second order condition for maximum and minimum values must be tested.

Second order condition δx12δ2π,δx22δ2π

and (δx1·δx2δ2π)2.

δx12δ2π=δx1δ(δx1δπ)=δx1δ(7−2x1−x2)=0−2(1)−0=−2<0.

δx22δ2π=δx2δ(δx2δπ)=δx2δ(20−x1−6x2)=0−0−6=−6<0.

(δx1,δx2δ2π)2=δx1δ(δx2δπ)=δx1δ(20−x1−6x2)=0−1−0=−1.

(δx12δ˙2π)(δx22δ2π)=−2x 6=12>(δx1.δx2δ2π)=(−1)2=1.

.. According to the second order condition of the profit function,

δx1δπ=7(1)+0−2x1−x2(l)+0=7−2x1−x2

δx1δπ=0=7−2x1−x2=0−−−−−−−−−−−−−−−−−−−−

δx2δπ=δx2δ(7x1+20x2−x12−x1x2−3x22)=0+20(1)−0−x1(1)−3·2x2=20−x1−6x2

δx2δπ=20−x1−6x2=0 -----------(2)

Solving equations (1) and (2):

−2x1−x2=−7 (1)

2x1+x2=7------------(1)

−x1−6x2=−20

x1+6x2=20 (2)

Multiplying equation (2) by 2:

2x1+x2=7 (1)

2x1+12x2=4 (2)×2

−11x2=−33

x2=−11−33=3

Substituting the value of x2 in equation (1):

2x1+3=7⇒2x1=7−3=4⇒2x1=4

x1=24=2

It will be maximum at x1=2, x2=3.

$=7\times2+20\times3-22-2\times3-3(2)2=14+60-4-6-12=52$

8.2.4 Determining prices when a monopolist produces two goods: Let's assume that the joint cost function of a monopolist producing two goods x1, x2 is c=c(x1,x2). Then the first-order condition for the maximum and minimum values of the monopolist's profit function is:

Profit function $\pi=x1p1+x2p2-c(x1,x2)$'s first order condition for maximum and minimum values:

$\delta x1\delta\pi=0,\delta x2\delta\pi=0$

i.e., x_1 + p

P2

δε 8x20

δρι

(1)

δε δχ

X2+ P1

+P2

бр

δε 8x2-0

бр

(2)

...

es

By solving these two equations, we can find the values of p1,p2. To know whether the profit is maximized at these prices, we can examine the second-order conditions for maximum and minimum values. At this price level:

If $\partial p12\partial 2\pi<0,(\delta p12\delta 2\pi)(\delta p22\delta 2\pi)>(\delta p1\cdot\delta p2\delta 2\pi)2$

then these prices maximize profit. The monopolist will then set these prices for his product.

Example: A monopolist is producing two goods x1,x2 at a constant average cost of Rs. 2 and Rs. 3 respectively. If the demand functions for these two goods are x1=5(p2−p1), x2 =32+5p−10p2, determine the prices that maximize the monopolist's profit and find his maximum profit.

The monopolist is producing two goods x1,x2 at a constant average cost of Rs. 2 and Rs. 3 respectively.

Therefore, his joint function is:

$c=2x1+3x2$−−−−−−−−−−−−−1)

The demand functions for these two goods are:

$x1=5p2-5p1,x2=32+5p1-10p2$

$.\delta p1\delta x1=-5,\delta p2\delta x1=5,\delta p1\delta x2=5,\delta p2\delta x2=-10$

Monopolist's profit function:

$\pi=p1x1+p2x2-(2x1+3x2)=(p1-2)x1+(p2-3)x2$−−−..................(2)

First-order condition for maximum and minimum values of the profit function:

$\delta p1\delta\pi=\delta p1\delta[(p1-2)x1+(p1-3)x2]=(p1-2)\delta p1\delta x1+x1+(p2-3)\delta p\delta x2+x2(0)$

$=(p1-2)\delta p1\delta x1+x1+(p2-3)\delta p1\delta x2=(p1-2)(-5)+(5p2-5p1)+(p2-3)\cdot;$

$=-5p1+10+5p2-5p1+5p2-15=-10p1+10p2-5$

$=\delta p1\delta\pi=0$
$-10p1+10p2-5=0$ (3)

$\delta p2\delta\pi=\delta p2\delta[(p1-2)x1+(p2-3)x2]$

$=(p1-2)5+(p2-3)(-10)+32+5p1-10p2$

$=5p1-10-10p2+30+32+5p1-10p2=10p1-20p2+52$

$\delta p2\delta\pi=0$

$-10p1-20p2+52=0$

$10p1+10p2=5$

$+10p1-20p2=-52$

$-10p2=-47$
Substituting the value into equation (3):
(4)

$-10p1+10p2(4.7)=5=-10p1=5-47-10p1=-42$

$p1=-10-42=4.2$

To determine whether the profit is maximized at prices p1=4.2, p2=4.7, the second-order condition for maximum and minimum values must be examined. The differential coefficients related to the second-order condition are:

$\delta p12\delta 2\pi,\delta p22\delta 2\pi,\delta p1\delta p2\delta 2\pi::\delta p12\delta 2\pi=\delta p1\delta(\delta p1\delta\pi)=\delta p1\delta(-10p1+10p2-5)$

$=-10(1)+0+0=-10$

$\delta p22\delta 2\pi=\delta p2\delta(\delta p2\delta\pi)=\delta p1\delta(10p1-20p2+52)$

$=0-20(1)+0=-20$

$\delta p1\delta p2\delta 2\pi=\delta p1\delta(10p1-20p2+52)=10$

$\cdot\cdot p1\dot{}=4.2,\ p2=4.7\ s\leq\delta p12\delta 2\pi<0,\delta p22\delta 2\pi<0$

$(\delta p12\delta 2\pi)(\delta p22\delta 2\pi)=(-10)(-20)=200>(\delta p1\delta p2\delta 2\pi)2=102=100$
... According to the second-order condition for maximum and minimum values of a function, profit is maximized at p1=4.2, p2=4.7.
The price of X1 that maximizes the monopolist's profit is Rs. 4.2, and the price of x2 is Rs. 4.7.

x1=5(4.7)−5(4.2)=23.5−21.0=2.5

x2=32+5(4.2)−10(4.7)=32+21−47=6
... The monopolist's maximum profit is:

π=(42.2)(2.5)+(4.7−3)6=(2.2)12.51+(1.7)6=15.7

8.3 Exercise:

1. Find the partial differential coefficient for the following function.

z=5x−y+45x2

∂x∂z=dxd(5x−y+45x2)=(5x−y+4)25x−y+4dxd(5x2)−5x2dxd(5x−y+4)

=(5x−y+4)2(5x−y+4)5.2x−5x2.5(1)=(5x−y+4)2(5x−y+4)·10x−25x2

=(5x−y+4)250x2−10xy+40x−25x2=(5x−y+4)250x2−10xy+40x
∂y∂z=∂y∂(5x−y+45x2)=(5x−y+4)2x−y+4dxd(5x2)−5x2dxd(5x−y+4)

=(5x−y+4)25x+y+4.0−5x2.0−1+0

(5x−y+4)20+5x2=(5x−y+4)25x2

2. x1=p1−1/7 p0.8 and x2=P10.5P20.2 are the demand functions for two goods. Find whether the goods are complementary or competitive using partial differentiation.
To determine the type of goods, the partial cross-elasticity must be found. That is:

δp2−δx1,δp1δx2,x1=p1−1.7p20.8,δP2∂x=P1−1.70.8 P20.8−1
=p1−1.7·0.8·p2−0.2=0.8p10.71·p20.21>0,x2=p10.5·p2−0.2
δp1δx=0.5p10.5−1·p1−0.5p2−0.2=0.5p10.51·p20.21>0
Since δp2δx1,δp1δx2 are positive, X1,X2 are competitive goods.

3. Find the total differential of the function z=3x2+xy−2y3.
dz=δxδz·δx+δyδz·δy·δxδz·δxδ(3x2+xy−2y3)=3.2x+y=(6x+y)

δyδz=δyδ(3x2+xy−2y3)=0+x(1)−2.3y=x−6y

... dz=(6x+y)dx+(x−6y)dy

4. Find the maximum and minimum values of the following function.

z=y3+y2−xy+x2+4

δxδz=δxδ(y3+y2−xy+x24)=−y+2x

δxδz=0

2x−y=0−−−−−(1)

δyδz=δyδ(y3+y2−xy+x2+4)=3y2+2y−x

δyδz=0=3y2+2y−x=0.....................
Solving equations (1) and (2) gives x=41,y=21.

δx22δ2z=δxδ(δxδz)=δxδ(2x−y)=2>0

$\frac{\delta^{2}z}{\delta y^{2}}=\frac{\delta}{\delta y}(\frac{\delta z}{\delta y})=\frac{\delta}{\delta y}(3y^{2}+2y-x)=6y+2$

y=−21

$\frac{\delta^{2}z}{\delta y^{2}}=6\times\frac{-1}{2}+2=-3+2=-1<0$

Since $\frac{\delta^{2}z}{\delta x^{2}},\frac{\delta^{2}z}{\delta y^{2}}$ have opposite signs, the function does not have a maximum or minimum value. However, it has a Saddle Point.

5. If Q=LK+0.2L2−0.52, find the marginal product of labor and capital.

δLδQ=δLδQ=δLδ(LK+0.2L2+0.8K2)

=K+0.2.2L+0=K+0.4L

=δKδ(LK+0.2L2+0.8K2)

8.4 Understanding Questions:

=L+0+0.8.2K2=L+0.16K2

1. Explain the concept of partial differentiation.
2. Find the partial differentiation for the following functions.

(a)z=7x3+xy+2y5

(b)z=6x−7y5x(c)z=(2x2+6y)(5x−3y2)

3. Find the second-order, pure, and partial differentials for the following functions.

(a)z=x2+2xy+y2

(b)z=x4+x3y2−3xy2−2y3

(c)z=(x3+2y)4

4. Find the marginal products for the following production functions.

(a)Q=0.5K2−2KL+L2 (b)Q=x2−2xy+3y2 (c)Q=3x2+5xy+4y2

5. Explain the method of finding maximum and minimum values of a two-variable function.

6. z=6x2−9x−3xy−7y+5y2, 2.

8.5 References:

1. Alpha C. Chaiang Fundamental methods of Mathematical Economics, Third Edition Mc.Graw-Hill, International Editions.
2. R.G.B. Allen Mathematical Analysis for Economics, MAC Million
3. Edward T. Bowling Theory and Progress of Mathematics for Economics, Scyanm's artlin series, Mc-Graw Hill stock Company.

LESSON - 9
INTEGRATION

Syllabus:

9.0 OBJECTIVES:

You have already learned about finding the derivative of a function. Conversely, if the derivative of a function is given, the function can be found. This method of finding the function when its derivative is given is called integration. After reading this section, you can understand the following:

1. You can learn what the concept of integration, indefinite integral, and definite integral are.
2. You can learn the definition of indefinite integral, rules of indefinite integrals, and various methods to find them. And by using these methods, you can learn how to find the integrals of given functions.

9.1 INTRODUCTION:

You have already learned that when the independent variable in a function changes by a very small (negligibly small) amount, the change in the dependent variable is indicated by the derivative of that function. Therefore, when the independent variable in a given function changes by a very small amount, the amount of change in the dependent variable can be found by differentiating that function. For example, if the price of a good changes by one unit, the change in its demand can be found by differentiating that function. Similarly, when a consumer's utility function is given, his marginal utility function can be found. Similarly, when a consumer's utility function is given, his marginal utility function can be found.

Similarly, when the total cost function of a good is given, and its marginal cost function is known, it is necessary to find the total cost function. When the marginal utility function is known, it is necessary to find the utility function. When the marginal product function is known, it is necessary to find the production function. Generally speaking, when the derivative of a function is given, finding that function is called integration.

9.2 CONCEPT OF INTEGRATION:

The concept of integration can be explained in two different ways. The integrals explained in these two methods have different properties and different applications. In one method, integration is considered as the reverse method of differentiation. An integral defined by considering integration as the reverse method of differentiation is called an indefinite integral. It does not have a specific numerical value. The indefinite integral gives the function when its derivative is given. In the second method, the integral is considered as the limit of a summation expression. An integral defined by considering it as the limit of a summation expression is called a definite integral.

9.3 INDEFINITE INTEGRAL:

[ie.dxdF(x)=f(x) or dF(x)=f(x)dx].
Definition of Indefinite Integral: If the derivative of a function F(x) is f(x), then F(x) is called the integral of f(x). It is written in symbolic form as $F(x)=\int f(x)dx$. This symbol $\int f(x)dx$ has three parts. The part '|' indicates the integration symbol. The f'(x) part is called the integrand. This is the function to be integrated. The dx part indicates that the integration is to be done with respect to X.
$\int f(x)dx=F(x)+C$. This 'C' is called the constant of integration. As shown above, when integrating, the constant must be added to all functions. This is because when differentiating, a constant must be added to the given function.

Integrals of some standard functions:

You have already learned the derivatives of the power function xn, exponential function ex, and logarithmic function logx. The derivatives of these functions are:
1. (dxd(xn)=n xn−1 (Power function)
(ii) (dxd(ex)=ex (Exponential function)
(iii) dxd(log x)=x1 (Logarithmic function)
The derivative of n+1xn+1 is n+1(n+1)xn+1−1=xn.
Integrals of x1,ex.
Based on these, the power function.

1. ∫xndx=n+1xn+1+c

2. ∫exdx=ex+c

3. ∫x1dx=log x+c

Rules of Integration: If f(x) is a continuous function and K is a constant, then $\int Kf(x)dx=K\int f(x)dx$.
If f(x),g(x) are two functions, then:
$\int[f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx$.
Examples:

7. ∫x6dx=6+1x6+1=7x7+C

8. ∫3.x7dx=3∫x7dx=37+1x7+1+c=83x8+c

9. ∫10xdx=10∫x1/2dx=10·21+1x21+1+c=10·32·x23+c=320·x23+c

10. ∫x5dx=∫(x5)21dx=∫x25dx=25+1x25+1=72x27+c

11. ∫x41dx=∫x−4dx=−4+1x−4+1+c=−3x−3+c=3x3−1+c

12. ∫(x3+x+1)dx=∫x3dx+∫xdx+∫1dx
=(3+1x3+1+c1)+1+1x1+1+c2+x+c3
=4x4+2x2+x+c1+c2+c3
=4x4+2x2+x+c
Integrate ∫(3x−1+4x2−3x+8)dx.
=3∫x−1dx+4∫x2dx−3∫xdx+∫8dx
Acharya Nagarjuna University
=3∫x1dx+4∫x2dx−3∫xdx+∫8dx
=3·log x+42+1x2+1−3·1+1x1+1+8·0+1x0+1
=3 log x+34x3−23x2+8x
Differentiate with respect to X?
∫(x−x1)3dx=∫(x3−3x+x3−x31)
∫x3dx−3∫xdx+3∫x1dx−∫x31dx
=3+1x3+1−3·1+1x1+1+3 log x−−3+1x−3+1
=4x4−23x2+3 log x−−2x−2
=4x4−23x2+3 log x+21x−2

Exercise:

1. ∫x20dx

2. ∫(10−x)dx

3. ∫5x6dx

4. ∫(x3−3x2−2x+1)dx

5. ∫(5x2+4e2x+x3+2)dx

6. ∫(x3−x)dx

9.4 SOME METHODS FOR FINDING INTEGRALS:

If the given function is a product of a function and a constant, or a sum of two functions, its integral can be found using the integration rules mentioned above. If, however, the given function is a complex combination such as a product or quotient of two functions, it is difficult to find its integral with the help of the above rules. In such cases, there are some special methods like the substitution method and integration by parts. Let's learn how to integrate such functions using these methods.

9.5 SUBSTITUTION METHOD:

If the given function is a product of two functions, and one of them is the derivative of the other, the integral can be found using the substitution method. This method is as follows:

Let the given function be f′(x)·f(x). Here, the derivative of f(x) is f′(x). The function is f′(x)·f(x).

∫f′(x)f(x)dx
Now, let f(x)=t. (In the given product, the function whose derivative is the other function should be taken as t). Differentiating this gives f′(x)dx=dt, and dx=f′(x)dt.
Substituting these into ∫f′(x)f(x)dx:
∫f′(x)f(x)dx=∫f′(x)t·f′(x)dt=∫tdt
∫t dt=1+1t1+1+c=2t2+c
Substituting t back into the form of x:
∫f′(x)f(x)dx=2[f(x)]2+c
Example 1: Find ∫2x(x2+1)dx.
In the given function, the derivative of (x2+1) is 2x. So, let x2+1=t. Then 2xdx=dt and dx=2xdt.
Substituting these into ∫2x(x2+1)dx:
∫2x(x2+1)dx=∫2x·t2·xdt=∫tdt
∫tdt=1+1t1+1+c=2t2+c
Thus, ∫2x(x2+1)dx=2(x2+1)2+c.
Example 2: Find ∫4x3(x4+2)80dx. In the given function, the derivative of x4+2 is 4x3dx=dt. And dx=4x3dt.
Substituting these into ∫4x3(x4+2)80dx:
∫4x3(x4+2)dx=∫4x3·t80·4x3dt
=∫t80dt=81t81+c
Writing t back in terms of x:
∫4x3(x4+2)80dx=81(x4+2)81+c
Example 3: Find ∫8x·e2x2+1dx.
Let 2x2+1=t. Then 4xdx=dt.
Therefore, dx=4xdt.
Substituting these into ∫8x·e2x2+1dx:
∫8x et·4xdt=∫2etdt=2∫etdt
Since ∫etdt=et+c,
2∫etdt=2et+c. Substituting t back:
∫8x·e2x2+1dx=2e2x2+1+c.

4. ∫(x−1)(x2−2x+3)n/2dx
In the given function, the derivative of x2−2x+3 is (2x−2)dx=dt and dx=(2x−2)dt.
Substituting these into ∫(x−1)(x2−2x+3)n/2dx:
∫(x−1)(x2−2x+3)n/2dx=21∫tn/2dt
=21·2n+1t2n+1+c
=n+2(x2−2x+3)2n+1+c

II. If the given function is a quotient of two functions, and the numerator is the derivative of the denominator, the function can be integrated using the substitution method. This method is as follows:
Let the given function be f(x)f′(x). Its integral is ∫f(x)f′(x)dx.
Here, let f(x)=t. Then f′(x)dx=dt and dx=f′(x)dt.
Substituting these into ∫f(x)f′(x)dx:
∫f(x)f′(x)dx=∫tf′(x)·f′(x)dt=∫t1dt
∫t1dt=log t+c
Writing t back in terms of x:
∫f(x)f′(x)dx=log[f(x)]+c.

Example 1: Find $\int 1+x22xdx$.
In the given function, the numerator is the derivative of the denominator. So, let $1+x2=t$. Then $2xdx=dt$ and $dx=2xdt$.
Substituting these into $\int 1+x22xdx$:
$\int 1+x22xdx=\int t2x\cdot 2xdt=\int t1dt$
Since $\int t1dt=\log t+c$. Writing t back in terms of x:
$\int 1+x22xdx=\log(1+x2)+c$.
Example 2: Find $\int 2x+11dx$.
Here, let $(2x+1)=t$. Then $2dx=dt$.
Therefore, $dx=2dt$. Substituting these:
$\int 2x+11dx=\int t12dt=21\int t1dt=21\log t+c$.
Writing t back in terms of x:
$\int 2x+11dx=21\log(2x+1)+c$.
9.5 Exercise:
Find the following:

1. $\int (1+6x)2dx$

Integration by Parts Method: If u and v are functions of x, according to the product rule of differentiation, $d(u\,v)=u\,dv+v\,du$.
Therefore, $\int d(u\,v)=\int (udxdv+vdxdu)dx$
i.e., $u\,v=\int udxdvdx+\int vdxdudx$
$\int u\,dv=uv-\int v\,du$ (where $du=dxdudx$)
This is called the integration by parts rule.
To find the integral using this rule, the given function should be considered as u dv. That is, one part of the given function should be taken as u, and the remaining part as dv. The part taken as dv should be easy to integrate, because v needs to be found from dv.
Example 1:
$\int x\cdot exdx=x[\int exdx]-[\int exdx][dxd(x)]dx$
$=x\cdot ex-\int ex\cdot 1\,dx$
$=x\,ex-ex+c$ (Note: 'c' or 'k' is the constant of integration)
$=ex(x-1)+k$
Example 2: Find $\int x2\cdot exdx$.
$\int x2\cdot exdx=x2[\int exdx]-[\int exdx]dxd(x2)dx$
$=x2\cdot ex-\int [ex]\cdot 2x\,dx$
$=x2\cdot ex-2[\int x\,exdx]$
$=x2ex-2[ex(x-1)]+k$
$=ex[x2-2(x-1)]+k$
$=ex[x2-2x+2]+k$

Example 3: Find $\int \log x\,dx$.
Here, $u=\log x$, $dv=1\,dx$.
According to the integration by parts rule:
$\int \log xdx=\log x[\int 1dx]-[\int 1dx][dxd\log x]dx$
$=\log x\cdot x-\int x\cdot x1dx$
$=x\log x-\int 1dx$
$=x\log x-x+k$
$=x(\log x-1)+k$
$=x(\log x-\log e)+k$ or
$=x\log(ex)+k$

4. $\int x(x+1)21dx$.

∫x(x+1)21dx=x[∫(x+1)21dx]−[∫(x+1)21dx·dxd(x)]dx
=x[21+1(x+1)21+1]−∫21+1(x+1)21+1dx
=x·(23(x+1)23)−∫23(x+1)23dx
=x·32(x+1)23−32∫(x+1)23dx

5. Find ∫x(x+1)21dx.
Let u=x, dv=(x+1)21dx.
Then du=dx.
To find v, integrate dv:
v=∫(x+1)21dx.
Substituting (x+1)=t in ∫(x+1)21dx:
∫(x+1)21dx=32(x+1)23+c1.
So, v=32[x+1]23+c1.
Substituting into the integration by parts formula ∫u dv=uv−∫v du:
∫x(x+1)21dx=x[32(x+1)23+c1]−∫[32(x+1)23+c1]dx
=32x(x+1)23+c1x−[32∫(x+1)23dx+∫c1dx]

Integrating ∫(x+1)23dx using the substitution method as explained above:
∫(x+1)23dx=52(x+1)25+c2.
Therefore, ∫x(x+1)21dx=32x(x+1)23+c1x−32[52(x+1)25+c2]−c1x+c3
(Note: The c1x terms cancel out, and constants combine into a single 'c')
=32x(x+1)23−154(x+1)25−32c2+c3
=32x(x+1)23−154(x+1)25+c
=32x(x+1)23−154(x+1)25+c

9.5 DEFINITE INTEGRAL:

Definition: Let y=f(x) be a single-valued function, and assume it is continuous for all values of x from x=a to x=b. We divide the interval [a, b] into n parts using points a=x1,x2,⋯,xn,xn+1 =b.
The sum f(x1)(x2−x1)+f(x2)(x3−x2)+⋯+f(xn−1)(xn−xn−1)+f(xn)(xn+1−xn) can be written in symbolic form as ∑i=1nf(xi)(xi+1−xi). As the number of divided parts 'n' increases, the length of the parts decreases. If this number 'n' approaches infinity, the sum ∑i=1nf(xi)(xi+1−xi) approaches a definite value. This value is called the definite integral of f(x) from a to b. It is written in symbolic form as ∫abf(x)dx. Here, 'a' is called the lower limit of the integral, and 'b' is called the upper limit.

∫abf(x)dx=limn→∞∑i=1nf(xi)(xi+1−xi)
Explanation of Definite Integral: We know that if the derivative of a function F(x) is f(x), then ∫f(x)dx=F(x)+c. This does not have a unique value. If the variable x takes two values a and b, (a<b), then the value of the integral at x=a is F(a)+c. Subtracting the value of the integral at x=a from the value of the integral at x=b:
[F(b)+c]−[F(a)+c]=F(b)−F(a).
This is a unique value that does not depend on the values of X or c. This is called the definite integral of f(x) from a to b.

∫abf(x)dx=F(b)−F(a)
Example: 1. Evaluate ∫02(2−3x+4x2)dx.

∫(2−3x+4x2)dx=∫022dx−3∫02xdx+4∫02x2dx

=[2x]02−3[1+1x1+1]02+4[2+1x2+1]02
=[2x]02−3[2x2]02+4[3x3]02
=[2(2)−2(0)]−[3222−3202]+[4323−4303]
=[4−0]−[6−0]+[332−0]
=4−6+332
=−2+332
=3−6+32
=326
Find ∫Adx.
First, differentiate and then set the limits.
3. ∫x1+log xdx=∫x1dx+[∫(log x)x1dx] (Here f(x)=log x, f′(x)=x1)

=log x+21(log x)2+c

So, ∫x1+log xdx=log x+21(log x)2+c
Evaluate ∫1ex1+log xdx.
Let u=1+log x. When x=1,u=1+log 1=1+0=1. When x=e,u=1+log e=1+1=2.
Let du=x1dx.
∫1ex1+log xdx=∫12u du=[2u2]12
=222−212=24−21=23
Alternatively, using the previous result:
=[log x+21(log x)2]1e
=[(log e)+21(log e)2]−[(log 1)+21(log 1)2]
=[1+21(1)2]−[0+21(0)2]
=1+21=23
Find ∫dx.
Find ∫dx.
4. Evaluate ∫15(x+x24)dx.
∫15(x+x24)dx=∫15x dx+4∫15x−2dx
=[2x2]15+4[−1x−1]15
=[252−212]+4[−x1]15
=[225−21]+4[−51−(−11)]
=224+4[−51+1]
=12+4[54]
=12−516 (Note: The original text has a minus sign here, but the calculation shows a plus sign before the 16/5. I'm following the calculation result.)
=560−16=544
Find ∫x2+xdx.

∫022+x5dx=5∫022+x1dx=5[log(2+x)02]=5[log(2+2)−log(2+0)]

=5[log 4−log 2]=5 log[25]=5 log 2
x(x²+6) dx 2
Find dx.
∫022+x5dx=5∫022+x1dx=5[log(2+x)02]=5[log(2+2)−log(2+0)]
=5[log 4−log 2]=5 log(24)=5 log 2

4. ∫01x(x2+6)dx 555π∘35Δ33.
5. ex
∫01x(x2+6)dx=[4(x2+6)2]01=4(12+6)2=4(02+6)2

=449−436=413
Find e 2x +ex) dx.

∫13(e2x+ex)dx=∫13e2xdx+∫13exdx

=[21e2x]13+[ex]13=21[e6−e2]+[e3−e1]=2e6−e2+e3−e1=2e6−e2+2e3−2e1

=2e6−2e3−e2−2e1
7. Find the area under 6y=x2∅∅ between points x=1 and x=3.
The area under the line y=x2 between points x=1 and x=3 is ∫13x2dx.

∴∫13x2dx=[3x3∫13=333−313=327−31=326
Exercise: Find the following definite integrals.

∫13xdx

2. ∫(x3−6x2)dx

∫−11(10x2+6x+2)dx

∫43x2(31x2+1)dx

∫02(x−1)(x2+x+1)dx
6. Find the area under the line y=9−x2 between points x=1 and x=3.
9.6 Exercise:

1. ∫(x3−x+1)dx=∫x3dx−∫x dx+∫1 d
=+-+c2+x+ =
+x+c 25
4
(c1−c2−c3)=c .

∫ex+x31dx=∫exdx+x31dx=∫exdx+∫x−3dx

ex+c+−3+1x−3+1+c2=ex−2x−2+c1+c2=ex−21x−2+c
ex−2x21+c (c1+c2=c ශ).

3. ∫(5x+7)8dx

t=5x+7 ... ∴dt=5dx=dx=5dt

::∫(5x+7)8dx=∫t8·5dt=51∫t8dt=51[8+1t8+1]+c

=51.9t9+c=451t9+c
Writing it again in terms of x

∫(5x+8)8dx=451(5x+7)9+c

4. ∫x8log xdx
u=log x&v=xn .

::∫xnlog x dx=∫log x·xndx=log x·n+1xn+1−∫x1·n+1xn+1dx

=n+11·xn+1log x−∫n+1xndx

=n+11·xn+1log x−∫n+1xndx

=n+11xn+1log x−(n+1)2xn+1+c

=n+1xn+1[log x−n+11]+c

5.∫12x(x+1)dx=∫12(x1−x+11)dx=∫12x1dx−∫12x+11dx
=[log x]12−[log(x+1)]12=[log 2−log 1]−[log 3−log 2]
=2 log 2−log 3[sin u log l=0]

6.∫232x3+x−26x2+1dx
Let's assume 2x3+x−2=t.

∴(6x2+1)dx=dt

:∫232x3+x−26x2+1dx=∫t6x2+16x2+1dt∫t1dt φϕϕϕϕϕϕϕϕ.

=−21+1t−1/2+1+c=21t1/2+c
=212.
- c Writing t again in terms of x

∫232x3+x−26x2+1dx=2[2x3+x−2]23

=2[54+3−2−16+2−2]=2[55−4]=6.88

7.∫−1+1(4−3x)5dx
4−3x=t .

−3 dx=dt
dx=31dt will be.

:∫(4−3x)5dx=∫t5·(−31)dt=31∫t5dt
Writing it again in terms of x

−316(4−3x)6+c∫−1+1(4−3x)5dx=−[18(4−3x)6+c]

=−181[16−76]=6549.61

8. ∫−23(1−x52)dx
∫(1−x52)dx=∫dx−2∫x5dx1
=x−2−5+1x−5+1+c=x+21−x41+c
∫−23(1−x52)dx=[x+2x41]−23=4.975

9.7 Understanding Questions:
Find the following integrals.

1. ∫(x+3)(x+1)21dx 2. ∫x·log x·dx3. \int\frac{5x}{(x-1)^{\frac{2}{2}}dx 4. xedx
2. ∫(2x5−3x41)dx
3. ∫(6e3x−8e−2x) dx

7·∫x4(2x5−5)4c dx

∫4x2+8x3x2+2dx 10. ∫15x(x+4)32dx

8.9.2 ∫(4x2+7)2x2dx

∫13(x3+x+6)dx

15. ∫135x·ex+2

9.8 REFERENCE BOOKS:

∫12x2(x3−5)2dx

∫13x2+16xdx

∫133x2e2x2+1dx

16. ∫135x.ex−2

17. R.GD. Allen : Mathematical Analysis for Economics (MAC Million India Limited, 1986 Chapter - XV)
18. Alpha C. Chaing:
19. Edward T. Dowling
Fundamental Methods of Mathematical Economics Third Edition, Mc GrawHill International Editions Chapter - Xiii
:
Mathematics for Economists Schaum's Outline Series in Economics. Mc Graw Hill Book Company, Chapter, 16 & 17.
4. G.S. Monga
:
Mathematics and Statistics for Economics Vikas Publishing House Pvt. Ltd., Chapter - ii

LESSON - 10
INTEGRATION - ECONOMIC APPLICATIONS

Syllabus:

10.0 OBJECTIVES

10.1 INDEFINITE INTEGRALS - ECONOMIC APPLICATIONS

10.2 DEFINITE INTEGRALS - ECONOMIC APPLICATIONS

10.3 EXERCISE

10.4 UNDERSTANDING QUESTIONS

10.5 REFERENCE BOOKS

10.0 OBJECTIVES:

In the previous lesson, we studied what integration is, the concepts of integration, what indefinite integration is, and methods to find them. In this lesson, we can understand the following topics.

1. Definition of indefinite integrals, rules of indefinite integrals, and applying the process of indefinite integrals to some problems in economics.
Using the process of definite integrals, we can learn to find consumer surplus and producer surplus.

10.1 INDEFINITE INTEGRALS - ECONOMIC APPLICATIONS:

Using the process of indefinite integrals, we will learn how to find the total cost function when the marginal cost function is given, the total revenue function when the marginal revenue function is given, the consumption function when the marginal propensity to consume is given, and the saving function when the marginal propensity to save is given.
Cost Function: The relationship between the output of a firm and the cost of production is called the cost function. It can be written symbolically as $T=f(x)$.
Here, T represents the total cost of production and X represents the output. You have already learned that differentiating the total cost function gives the marginal cost function. That is, the marginal cost function is $T'=dxdT$.
Therefore, the total cost function, $T=\int T'\, dx$. Integrating the marginal cost function gives the total cost function.

Example: 1. If the marginal cost function of a firm is $MC=100-10x+0.1x2$ and x is the quantity of production, find its total cost function when the total cost and average cost are given, and the fixed cost is 500.

$(=100-10x+0.1x2$

Total Cost Function $TC=\int MCdx$

$=\int(100-10x+0.1x2)dx$

=100x−5x2+30.1x3+k
Here, fixed cost is Rs.500

TC sp6x=500

TC=100x−5x2+30.1x3+500
Average Cost =
TC C
=
X
X
=x100x−5x2+30.1x3+500
=100−5x+30.1x2+x500
Example: If the marginal cost function of a firm is T′=25+30x−9x2 and the fixed cost is 55, find its total cost function.

The marginal cost function of the firm is T′=25+30x−9x2

T=∫(25+30x−9x2)dx=∫25 dx+∫30xdx−9∫x2dx

=25x+302x2−93x2+c

=25x+15x2−3x3+c
Fixed cost is the cost incurred by the firm when production is zero. Fixed cost is given as 55.

∴55=25(0)+15(02)2−3(0)3+c:.c=55

The total cost function of the firm is T=25x+15x2−3x3+55
....
Revenue Function: The relationship between the total revenue of a firm and the quantity of goods sold is called the total revenue function or revenue function. It is written symbolically as R f(x).
Here, R represents the total revenue and X represents the quantity of goods sold.
You have already learned that differentiating the total revenue function gives the marginal revenue function.

That is, the marginal revenue function is R′=dxdR

Therefore, the revenue function is .. R=∫R′dx
.. Integrating the marginal revenue function gives the total revenue function.
Example:
(1) If the marginal revenue function of a firm is R′=60−2x−2x2, find its total revenue function.

R′=60−2x−2x2
::

, R=∫R′dx=∫(60−2x−2x2)dx

=∫60dx−∫2xdx−∫2x2dx=60∫dx−2∫xdx−2∫x2dx

= 60x-2-2+c =60x−x2−32x3+c
3
When the quantity of goods sold is zero, the revenue is zero. Therefore,

0=60(0)−02−32(03)+c:c=0

Revenue function R=60x−x2−32x3
(2)
If the marginal revenue of a firm is MR=16−x2 and x is the quantity of goods produced, find (1) the total revenue of the firm and (2) the demand function.

MR=16−x2
.. Total Revenue Function R=∫MR dx
--- PAGE 9 ---
Acharya Nagarjuna University
(3)
Marginal Revenue Function

MR=(x+2)26+5
=∫(16−x2)dx
=∫16dx−∫x2dx+c
TR=16x−3x3+c
Find the total revenue function and the demand equation.

1. TR=x[b(x+b)a−c]

=x[(x+2)26+5]

In the above function, if a=6, b=2 and c=5

=x[2(x+2)6+5]

=x[(x+2)3+5]

=x+23x+5x
Since the demand function is equal to the average revenue

AR=xTR

=x+23x+5
x=f(p)=PR=b(p+c)r−b
=p−53−2

Consumption Function: The relationship between aggregate consumption expenditure and aggregate real disposable income is called the consumption function. It can be written symbolically as
c=f(y). Here c denotes aggregate consumption expenditure and y denotes aggregate disposable real income. You already know that differentiating the consumption function gives the marginal propensity to consume.

.. Marginal Propensity to Consume, $c' = dTdc$

Therefore, the consumption function, $c = \int c' dy$
.. Integrating the marginal propensity to consume gives the consumption function.
Example:
(1) Marginal Propensity to Consume
.. Consumption Function

$c' = 0.6 + 0.1\ y - 31$

$c = \int c' dy = \iint (0.6 + 0.1y - 31)dy$

$= \int 0.6dy + \int 0.1y - 31dy = 0.6y + 0.1 - 31 + 1y - 31 + c$

$0.6y + 0.15y32 + c$
Since consumption expenditure is 40 when real disposable income is zero,

$40 = 0.6(0) + 0.15(32) + c::.c = 40$
:. 2
(

$c = 0.6y + 0.15y32 + 40$
(2) Example: 2 If Marginal Propensity to Consume (MPC) $= 0.7 + 0.4y - 1/2$, find the consumption function and the consumption expenditure when $y = 0$ and $c = 0$.
22

$(CF) = C(Y) = \int (MPC)dx$

$C(Y) = \int (0.7 + 0.4\ Y - 21)dy$

$= \int 0.7\ dy + 0.4 \int y - 1/2dy + C$

$= 0.7y + 0.421y - 1/2 + c$
$= 0.7Y + 0.8\ Y21 + c$

$Y = 0, C = 10$

$\therefore 10 = 0.7(0) + 0.8(0) + A$

$A = 10$
Required Consumption Function

$C(Y) = 10 + 0.8y + 0.7y$
Savings Function: Keynes defined the relationship between aggregate savings and aggregate real disposable income as the savings function. It can be written symbolically as $s = f(y)$. Here, 's' represents aggregate savings and 'y' represents aggregate real disposable income. As you already know, differentiating the savings function yields the marginal propensity to save.
... Marginal Propensity to Save

$S' = dYdS$ Therefore, the savings function is $s = \int s' dy$

Integrating the marginal propensity to save gives the savings function.

Example: Marginal propensity to save s′=0.5−0.2y−21
Find the savings function.

Marginal propensity to save s′=0.5−0.2y−21
... Savings function,
And if income is 25 and savings is -3.5, find the savings function.

s=∫s′dy=∬(0.5−0.2y21)dy=0.5y−0.221y21+c=0.5y−0.4y21+c
Since income is 25 and savings is -3.5,

−3.5=0.5(25)−0.4(2.5)21+c=0.5(25)−0.425+c:.c=−14

(s=0.5y−0.4y−14

Capital, Investment: Capital 'k' changes with time 't'. Therefore, the capital function can be written as k=f(t). The rate of change in capital over time is called the investment rate.

... Investment I=dtdk

... Capital k=∫I dt

I=80t32
That is, if the investment rate is I=80t32 and capital is 75 when t=0, find the capital function.

Investment rate I=80t32

(k∫I dt=∫80t52dt
=80\int
t^{\frac{2}{3}}dt=800\frac{\frac{2}{5}+t}}{\frac{2}{5}+1}+c=80.\frac{5}{7}t^{\frac{7}{5}}+c
=7400t57+c t=0
75=7400·t57+c
Exercise:
Since capital is 75 when t=0,
... Capital

k=7400·tS1+75

1. If the marginal cost function of a firm is T′=15+x2 and fixed cost is 50, find its total cost function.
2. If the marginal cost function of a firm is T′=5+6ex and fixed cost is 75, find its total cost function.
3. If the marginal revenue function of a firm is R′=20+10x−5x2 and revenue is 100 when output is 5 units, find its total revenue function.
4. Find the total revenue function if the marginal revenue function of a firm is R′=0.5x−0.5.

LESSON – 11
MATRIX THEORY, TYPES, MATHEMATICS, DETERMINANTS

11.0 LEARNING OUTCOMES:

After learning this lesson, you will be able to easily:
i) Define the concept of a matrix, its uses, and its notation;
ii) Describe different types of matrices;
iii) Perform addition, subtraction, and multiplication of matrices;
iv) Evaluate the determinant of 2x2 and 3x3 matrices;
v) Analyze the properties of matrices with examples.

11.1 INTRODUCTION

Distance Education Center
Matrix mathematics is also known as linear algebra. It provides a neat way to write systems of simultaneous equations, no matter how large they are. By evaluating the determinant of a system of equations, matrix mathematics allows us to test the existence of its solution. It provides us with a method to find its solution. It is useful in static, comparative static, and dynamic analysis. Matrix algebra is used in input-output analysis to examine the mutual technical relationship between industry inputs and outputs. Matrix algebra is also useful in national income analysis and social accounting. However, matrix algebra is only useful for analyzing systems of linear equations. It is important to note to what extent a real-world situation can be described in terms of linear relationships.

11.2 MATRIX CONCEPT NOTATION

A matrix is a rectangular arrangement of variables or parameters formed in 'm' rows and 'n' columns. If they are enclosed in brackets, such an arrangement is called a matrix. We identify matrices with capital letters and elements or members of the matrix with English small letters. An element in the i-th row and j-th column of a matrix 'A' is denoted by 'a_ij'. The order of the subscripts is very important because 'i' always refers to the row where the element is located, and 'j' always refers to the column where the element is located. Consider the following matrix:

A=[aij]=a11a21⋮am1a12a22⋮am2a13a23⋮am3……⋱…a1na2n⋮amnm×n
The above matrix is identified as matrix A. In matrix notation, it is also denoted by [aij]m×n. This matrix has m rows and n columns. Therefore, the order of the matrix is 'm x n'. The first element of the matrix is a11. Here, the first number 1 indicates the row where the element is located. The second 1 indicates the column where the element is located. This element is denoted by a11. Take another element a21. In this element, the first number 2 indicates the row where the element is located. The second number indicates the column where the element is located. In other words, the element a21 is in the second row and first column of matrix A.

11.3 TYPES OF MATRICES

There are many types of matrices. We will explain some of the main types here.

11.3.1 ROW MATRIX:

A matrix containing elements in only one row, i.e., a matrix of the form 1×m, is called a row matrix or row vector.

Example: A=[a11a12a13a14]

A=[2468]

11.3.2 COLUMN MATRIX:

A matrix containing elements in only one column, i.e., a matrix of the form 'm x 1', is called a column matrix or column vector.

Example: B=1357

11.3.3 SQUARE MATRIX:

If the number of rows in a matrix is equal to the number of columns, that matrix is called a 'square matrix'.

Example: A=a11a21a31a12a22a32a13a23a333×3

A=1472583693×3

11.3.4 DIAGONAL MATRIX:

A diagonal matrix is a square matrix where all elements are zero except for those on the principal diagonal. The principal diagonal refers to the elements extending from the upper left position to the lower right position.

Example: A=a11000a22000a333×3

A=1000500093×3

11.3.5 TRIANGULAR MATRIX:

In a square matrix, if the elements aij are equal to zero when i<j, it is called a lower triangular matrix. If they are equal to zero when i>j, it is called an upper triangular matrix.
Lower Triangular Matrix:
a11a21a310a22a3200a33
Upper Triangular Matrix:
a1100a12a220a13a23a33

Example: A=1470580093×3

B=1002403563×3

11.3.6 NULL (OR) ZERO MATRIX:

A matrix where every element is zero is called a null matrix or zero matrix.

Example: A=000000000

11.3.7 SCALAR MATRIX:

A diagonal matrix where the diagonal elements are equal is called a Scalar Matrix.

Example: A=5000500053×3

11.3.8 IDENTITY MATRIX:

A diagonal matrix where the diagonal elements are equal to unity (one) is called an identity or unit matrix.

Example: A=100010001

11.3.9 TRANSPOSE OF A MATRIX:

A matrix formed by writing the row elements of a matrix as column elements and the column elements as row elements is called the transpose of a matrix. It is denoted as AT.

Example: A=1782403563×3

AT=1237458063×3

11.3.10 SYMMETRIC MATRIX:

If the structure of a matrix does not change through transposition, i.e., if A=AT, then A is called a symmetric matrix.

Example: A=5000500053×3

AT=5000500053×3

11.4 BINARY OPERATIONS ON MATRICES:

Addition, subtraction, multiplication of matrices, and finding the inverse of matrices are the main operations performed on matrices. For matrix addition and subtraction, equality of matrices is a necessary condition. For matrix multiplication, conformability of matrices is a mandatory condition. To find the inverse of a matrix (which is equivalent to division), the determinant of the matrix should not be equal to zero.

11.4.1 EQUALITY OF MATRICES:

Two matrices A and B are called equal matrices if they are of the same order and their corresponding elements are the same.

Example: A=[316425964]2×3

B=[316425964]2×3

11.4.2 CONFORMABILITY FOR MATRIX MULTIPLICATION:

Two matrices A and B are said to be conformable for multiplication if the number of columns in the first matrix is equal to the number of rows in the second matrix. So, if matrix A has order m x n and matrix B has order r x s, then if the condition n = r is satisfied, matrices A and B are suitable for multiplication.

A=[316425964]2×3

B=31623425123×2
A and B are conformable matrices. If we multiply matrices A and B, we get a 2x2 square matrix.

11.4.3 MATRIX ADDITION / SUBTRACTION:

If A and B are both m×n matrices, the sum denoted by [aij+bij] or A + B is the matrix obtained by adding the corresponding elements of matrices A and B. The necessary condition for their addition is that both matrices must have the same order.

A=[384594]2×3

B=[412396]2×3

A+B=[aij+bij]=[3+48+14+25+39+94+6]2×3=[79681810]2×3
Similarly, subtraction of matrices A and B:

A−B=[aij−bij]=[3−48−14−25−39−94−6]2×3=[−17220−2]2×3
Properties of Matrix Addition:

1. Matrix addition obeys the Commutative Law. That is, if A and B are matrices of the same form, then A+B = B+A.
2. Matrix addition also obeys the Associative Law. That is, if A, B, and C are matrices of the same form, then (A+B)+C = A+(B+C).

11.4.4 SCALAR MULTIPLICATION OF MATRICES:

If A is an m×n matrix and k is a scalar, we denote the matrix obtained by multiplying every element of A by k as kA. This process is called scalar multiplication.
Example:

A=[10−2−123]

3A=[3(1)3(0)3(−2)3(−1)3(2)3(3)]=[30−6−369]
In this example, matrix A is multiplied by the scalar 3.
Properties of Scalar Multiplication of Matrices:

1. k(hA)=(kh)A

2. (k+h)A=kA+hA

3. k(A+B)=kA+kB

11.4.5 CONFORMABILITY FOR MATRIX MULTIPLICATION:

Matrix multiplication is possible only when two matrices are conformable for multiplication. That is, the number of columns in the first matrix must be equal to the number of rows in the second matrix. Multiply the rows of the first matrix with the columns of the second matrix by adding the corresponding elements as shown by the arrows.

A=[30−241−1]

B=2−1−3431

(3)(2)+(−2)(−1)+(1)(−3)=5

(3)(4)+(−2)(3)+(1)(1)=7

(0)(2)+(4)(−1)+(−1)(−3)=−1

(0)(4)+(4)(3)+(−1)(1)=11
So, the required product matrix is as above:

A×B=[5−1711]
Therefore, to multiply two matrices A and B, the order of the matrices should be:
Order of Matrix A: m×n
Order of Matrix B: n×p
Order of Product Matrix: m×p

11.4.6 Properties of Matrix Multiplication:

1. The Commutative Law does not apply to matrix multiplication. That is, AB and BA are not necessarily equal. However, if B is the inverse of A, then AB = BA.
2. Matrix multiplication obeys the Associative Law. That is, if matrices A, B, and C are suitable for multiplication, then [(AB)C=A(BC)].
3. If matrices A and I are suitable for multiplication, then AI=IA.
4. If matrices A and 0 are suitable for multiplication, then A0=0A=0.
5. If AB = 0, it is not necessary that either A or B is a null matrix.
6. Even if AB=AC or BA=CA, it is not necessary that B = C.
7. If matrices A, B, and C are suitable for multiplication, then A(B+C)=AB+AC. This is called the Distributive Law.

11.5 DETERMINANTS AND THEIR PROPERTIES

A determinant is a pure number associated with a matrix. It can be positive, negative, or zero. Determinants exist only for square matrices. If the determinant of a matrix is zero, it is called a singular matrix; otherwise, it is called a non-singular matrix. It is used to test the existence of a unique solution for a system of simultaneous equations.

11.5.1 DETERMINANT OF A 2X2 MATRIX:

The determinant (Det. A) of a matrix of order 2×2 can be found as follows:

A=[a11a21a12a22]

Det.A=a11a21a12a22=(a11×a22)−(a12×a21)

Example: A=[4123]=(4×3)−(2×1)=12−2=10

11.5.2 DETERMINANT OF A 3X3 MATRIX:

The determinant of a 3×3 matrix can be found by taking any row or any column.
Det.A=a11a21a31a12a22a32a13a23a33=a11(a22a33−a23a32)−a12(a21a33−a23a31)+a13

(a21a32−a22a31)

Example: M=312−4051−11
=3(0×1−(−1)×5)−(−4)(1×1−(−1)×2)+1(1×5−0×2)
=3(0+5)+4(1+2)+1(5−0)
=3(5)+4(3)+1(5)
=15+12+5
=32
So, the determinant of the given matrix is 32.
Note: In the first matrix, after removing the first row and first column, the remaining matrix is called the Minor determinant. This minor determinant is related to the first element a11 in the matrix. If a sign is given to this element based on its position in the matrix, it is called a Co-Factor. The rule for giving the sign is (−1)(i+j). Here, i represents the row and j represents the column. For example, the sign of a11 will be (−1)1+1=(−1)2=(−1)(−1)=+1. Similarly, the sign of the element a12 will be (−1)1+2=(−1)3=(−1)(−1)(−1)=−1. The sign of the element a21 will be (−1)2+1=(−1)3=(−1)(−1)(−1)=−1. The sign of a22 will be (−1)2+2=(−1)4=(−1)(−1)(−1)(−1)=+1. In this way, every element in the matrix has a minor determinant and a sign. Based on this rule, signs have been given to the above elements. The matrix obtained with these signs is called the Cofactor matrix.

11.5.3 PROPERTIES OF DETERMINANTS:

1. If any two rows or columns in a determinant are interchanged, the determinant will have its absolute value, but its sign will change. Deta1a2a3b1b2b3c1c2c3=−Deta1a2a3b1b2b3c1c2c3 (This seems to be a typo in the original Telugu, it should show the change in rows/columns for the negative sign) Corrected interpretation of the property:
If two rows or columns are interchanged, the sign of the determinant changes.
Deta1a2a3b1b2b3c1c2c3=−Deta2a1a3b2b1b3c2c1c3
2. If all rows are changed to columns and all columns are changed to rows in a determinant, the value of the determinant does not change. Deta1a2a3b1b2b3c1c2c3=Deta1b1c1a2b2c2a3b3c3
3. If two rows or columns in a determinant are identical, the determinant vanishes. That is, the value of the determinant becomes zero.
Deta2a2a3b2b2b3c2c2c3=0

Detb1b2b3b1b2b3c1c2c3=0

4. If any row or column in a determinant is multiplied by a constant K, the new determinant obtained will be K times the value of the original determinant.

D=Deta1a2a3b1b2b3c1c2c3

Deta1ka2a3b1kb2b3c1kc2c3=kD

5. If k times the corresponding elements of another row or column are added to any row or column, the value of the determinant does not change.

Deta1a2a3b1b2b3c1c2c3

Deta1+kb1a2+kb2a3+kb3b1b2b3c1c2c3=Deta1a2a3b1b2b3c1c2c3

6. If all elements of any row or column are zeros, then the determinant will be zero.

Det(\begin{matrix}0&0&0\\ a_{2}&b_{2}&c_{2}\\ a_{3}&b_{3}&c_{3}\end{matrix})&Det(\begin{matrix}0&b_{1}&c_{1}\\ 0&b_{2}&c_{2}\\ 0&b_{3}&c_{3}\end{matrix})

Det(1aa21bb21cc2)

7. If the determinant vanishes by setting x=a, then x−a is a factor of the determinant.

Det. = 0

This determinant has (a−b) as a factor because by setting a=b, the first and second columns become identical. Therefore, the determinant vanishes.

8. If any row or column is expressed as a multiple of any other row or column, the determinant will be zero. This is called linear dependence of rows or columns.

Det(a12a1b1a22a2b2a32a3b3)

11.6 :

= 0

In this lesson, we defined the concept of a matrix and its notation. A rectangular arrangement of numbers, variables, and parameters is called a matrix. Matrices are denoted by English capital letters, and their elements are denoted by English small letters. There are many types of matrices such as row matrix, column matrix, diagonal matrix, triangular matrix, scalar matrix, null (or) zero matrix, identity or equality matrix, transpose matrix, symmetric matrix, singular, non-singular.

Matrix Theory

There are many types of matrices such as matrix. Matrices can be added, subtracted, and multiplied. The concept of matrix inverse is closely related to the concept of matrix determinant. A determinant is a pure number that is positive, negative, or zero. Determinants are found only for square matrices. In this lesson, we learned how to calculate the determinant for matrices of order two by two and three by three. We learned eight important properties of determinants. These properties help us evaluate the determinant and inverse of matrices in a simpler way.

11.7 GLOSSARY

1. Matrix
2. Variables
3. Parameters
4. Row Matrix

5. Column Matrix
6. Diagonal Matrix
7. Triangular Matrix
8. Scalar Matrix
9. Null (or) Zero Matrix
10. Identity Matrix
11. Transpose
12. Symmetric Matrix
13. Singular Matrix
14. Non-Singular Matrix
15. Addition of Matrices
16. Subtraction of Matrices
17. Inverse of Matrix
18. Determinant of a matrix
19. Square Matrix
20. Properties of Determinant

11.8 SAMPLE EXAM QUESTIONS

Answer the following questions briefly

1. Define matrix and give examples.
2. Explain the differences between upper and lower triangular matrices with examples.
3. Explain the uses of matrices in economics.
4. Find the determinant for the following matrix.

(123245356)

11.8.2 Answer the following questions in detail

1. Define matrix and explain different types of matrices.

If A=(3−157) and B=[4156], then find 2A+2B.

3. Multiply the following matrices and prove AB.

A=[123245000]

4. Explain the properties of determinants with examples.

11.9 SUGGESTED BOOKS

1. Alpha Chiang: Fundamental Methods of Mathematical Economics
2. R. G. D. Allen: Mathematical Analysis for Economists
3. Mehta and Medhani: Mathematics for Economists.

LESSON – 12

MATRIX INVERSE, SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS, SOLUTION

Outline of the Lesson

12.0 EXPECTED LEARNING OUTCOMES:

After learning this lesson, you will be able to easily:
Centre for Distance Education 23
i) Define the concepts of minor of a matrix, cofactor matrix, and adjoint matrix, and their notations;
ii) Calculate the inverse for 2×2 and 3×3 matrices;
iii) Write a system of simultaneous equations in matrix form;
iv) Find the solution to a system of simultaneous equations with two and three variables using matrix inverse;
v) Find the solution to a system of simultaneous equations with two and three variables using Cramer's Rule.

12.1 INTRODUCTION

In the previous lesson, we defined the concept of a matrix and its notation. We discussed various types of matrices such as row matrix, column matrix, diagonal matrix, triangular matrix, scalar matrix, null (or) zero matrix, identity or equality matrix, transpose matrix, symmetric matrix, singular, and non-singular matrices. We learned about addition, subtraction, and multiplication of matrices. We learned how to calculate the determinant for 2×2 and 3×3 matrices, which is closely related to the concept of matrix inverse. We learned eight important properties related to determinants. We explained that these properties help us evaluate the determinant and inverse of matrices in a simpler way. In this lesson, we will learn in detail about the concepts of minor of a matrix, cofactor matrix, and adjoint matrix, and their notations. We will also learn in detail about topics such as calculating the inverse for 2×2 and 3×3 matrices, writing a system of simultaneous equations in matrix form, finding the solution to a system of simultaneous equations with two and three variables using matrix inverse, and finding the solution to a system of simultaneous equations with two and three variables using Cramer's Rule.

In a matrix, a small square matrix determinant obtained by removing one or more rows and columns is called a 'minor determinant'. The minor obtained by removing the i-th row and j-th column of matrix A is denoted by Mij. It is also called the 'minor determinant' related to the element aij in matrix Aij. According to the determinant rule, to find the minor determinant of any element, the row and column containing that element must be removed, and the remaining elements must be written in the minor determinant.
'Minor determinant' related to element a13 is u_{3}&=&&&a_{41}&a_{12}&a_{13}\\ a_{21}&a_{22}&a_{33}\\ a_{31}&a_{32}&a_{33}\end{matrix}|&=&&&a_{21}&a_{22}\\ a_{11}&a_{32}\end{matrix}
:::\begin{matrix}d_{11}&a_{12}&a_{13}\\ d_{33}&a_{22}&a_{23}\\ d_{31}&a_{32}&a_{33}\\
d_{31}&a_{32}&a_{33}\end{matrix}|&=&|\begin{matrix}a_{12}&a_{13}\\
a_{32}&a_{33}\end{matrix}|
'Minor determinant' related to element A21 is M_2=
A11 A12 A13
'Minor determinant' related to element A22 is M22
=
421 422 423
AA13
A31 A33
A31 A32 A33
A12 A3
13
A12
'Minor determinant' related to element A27 is M 23
=
A21 22 23
=
A31 A32
A31 A32 A33
A12 A13
'Minor determinant' related to element A31 is M31
=
A21 A22 A23
032 033
a
=
'Minor determinant' related to element a32 is M32 =

A12 A13
A22 A23
\begin{matrix}a_{11}&a_{12}^{|&a_{13}&&&&a_{22}&a_{23}&=&|\begin{matrix}a_{1 1}&a_{13}\\
A11 A12
a13

'Minor determinant' related to element A33 is M33 =
A21 A22 A23
A A12
A21 22
a
12.2 (Co-Factor Matrix):
In a Minor, if a sign is given based on the position of the element, it is called a Co-Factor. The rule for giving the sign is (−1)(i+j). Here, i denotes the row and j denotes the column. For example, the sign of a11 will be (−1)1+1=(−1)2=(−1)(−1)=+1. Similarly, the sign of element

a12 will be (−1)1+2=(−1)3=

(−1)(−1)(−1)=−1

. a21

(−1)2+2=(−1)4=(−1)(−1)(−1)(−1)=+1
, (−1)2+1=(−1)3=(−1)(−1)(−1)=−1
.

. a22

aH
' ' C_{11}=(-1)^{(t+c)}=(-1)^{(1+1)}&+&a_{22}&a_{23}\\

a_{32}&a_{33}\end{matrix}|z+(a_{22}\times a_{33})-(a_{23}\times a_{32})
'Co-factor' related to element a12 is C12=(−1)(i+e)=(−1)(i+2=−|a21a31a23a33|=−(a21×a33

)−(a23×a31).

a13
'

' C13=(−1)(t+c)=(−1)(1+3)=+|a21a31a22a32|=+(a21×a32)−(a22×a31)

a21
'

' C21=(−1)(+e)=(−1)(2+1)=−|a12a32a13a33|=−(a12×a33)−(a13×a32)
'Co-factor' related to element a22 is C22=(−1)((+c)=(−1)(2+2)=+|a11a31a13a33|=+(a11×a33

)−(a13×a31)

' C23 = (-1) (c) = (-1)(2+3)=-
A11 A12
A31 A32
=
(a11 x a32)-(a12 x a31)
'Co-factor' related to element a31 is C31=(−1)((+e)=(−1)(3+1)=+|a12a22a13a23|=+(a12×a23

)−(a13×a22)

a32

Co-factor Matrix

a33

' C32=(−1)(i+e)=(−1)(3+2)=−|a11a21a13a23|=−(a11×a23)−(a13×a21)

' C33=(−1)(i+c)=(−1)(3+3)=+|a11a21a12a22|=+(a11×a22)−(a12×a21)
Similarly, co-factors can be found for all elements. If all the co-factors of a matrix are found and arranged in a matrix form, such a matrix is called a Co-Factor Matrix.

12.3 (ADJOINT MATRIX):

If we obtain the Transpose of the Co-Factor Matrix, we get the Adjoint Matrix. That is, the row elements of the co-factor matrix should be written as column elements, and the column elements as row elements.

12.4 2
For example, if the Co-factor matrix is
Cq=|a1a2a3b1b2b3c1c2c3|. then the Adjoint Matrix A
A4
(Inverse Matrix)
will be.
By dividing each element of the adjoint matrix by the determinant value, we get the "inverse matrix" of the given matrix. The "inverse matrix" is denoted as A'. For example,
{a_{1}}^{*}=\frac{1}{DetA}Ad|A=&\begin{matrix}\frac{a_{2}}{D}\frac{a_{2}}{D}&\frac{a_{2}}{D}\\ D&\frac{b_{1}}{D}&\frac{b_{2}}{D}&\frac{c_{2}}{D}\\

\frac{c_{1}}{D}\frac{c_{2}}{D}\frac{c_{1}}{D}\frac{c_{1}}{D}\frac{c_{1}}{D}}
12.5 Numerical Examples
Through some numerical examples, we will learn how to calculate the inverse matrices for 2×2 and 3×3 order matrices. The important condition to obtain the inverse is that the determinant value should not be zero. A matrix with a zero determinant is called a "Singular Matrix". A matrix with a non-zero determinant is called a "Non-Singular Matrix".
12.5.1: Inverse Matrix for a 2×2 order matrix

Example- 1:
Solution:
Find the inverse matrix for the following 2×2 order matrix:
A=(2113−5)
Determinant of the matrix: Det. A=|2113−5|=(2 x−5)−(3x11)=−10−33=−43□=0
Since the determinant value is not zero, this is a "Non-Singular Matrix". We can find the inverse for this matrix.
Co-factor Matrix
First, let's find the co-factors for the elements of this matrix and the co-factor matrix.
Let's find the co-factors for the elements in the first row of the matrix, and then for the elements in the second row.

Co-factor of element 2 = (−1)1+1 |-5| = (1) (-5) =
= -5
11-5
23
Co-factor of element 3 = (−1)1+2 |11| = (-1)(11) =
=
-11
11-5
2
Co-factor of element 11 = (−1)2+1 |3| = (-1)(3) =
3
11-5
=
-3

-52 = (−1)2+2∣3∣=(1)(2)=∣2−+13−−15−1∣=2
Therefore, Co-factor matrix =[−5−3−112]
=[−5−11−32]
Adjoint Matrix
Inverse matrix", A+=Det.A1Adj.A=−431(−5−11−32)
Inverse matrix", x1=[−5−43−43−43−11−43−43−3−43−43−22]∗(435431433−432)
If the given matrix is multiplied by its inverse matrix, we should get the identity matrix. That is,
A.A−1 A−1A=1

1=[2113−5][4354311433−43]
x=[2(5/43)+3(1/43)1(5/43)∗(5/43)2(3/43)+3(2/43)1(5/43)∗(5/43)∗(5/43)∗(5/43)
]∗[(0+3)/43(55.555/43(6·6)/43(33∗6)/43]
z=[4343434300434343]z[1001]

12.5.2: Inverse Matrix for a 3×3 Order Matrix
Example - 2:
Solution:
Find the inverse matrix for the following 3×3 order matrix given below.

A=(403130−127)

Determinant of the matrix: Det. A=∣403130−127∣
=−1(1+1)4[(3×7)−(2×0)]−1(1+2)1[(0×7)−(2×3)]−1(1+3)(−1)[(0×0)−(3×3)]
=1×4[21−0]−1×1[0−6]−1×(−1)[0−9]

=4[21]−1[−6]−1[−9]

=84+6+9=99□=0
Since the determinant value is not zero, this is a "Non-Singular Matrix". We can find the inverse for this matrix.
Cofactor Matrix
First, let's find the cofactors of the elements of this matrix and the cofactor matrix. Let's find the cofactors for the elements in the first row of the matrix, then for the elements in the second

row, and then for the elements in the third row. We assign signs to the elements according to the (−1)r+c rule based on their position.
First Row:

Cofactor of element 4 =(−1)1+1|3027|=+1(21−0)=21

Cofactor of element 1 =(−1)1+2|0327|=−1(0−6)=6.

Cofactor of element -1 =(−1)1+3|0330|=1(0−9)=−9
Second Row

Cofactor of element 0 =(−1)2+1|10−17|=−1(7−(0))=−7

Cofactor of element 3 =(−1)2+2|43−17|=1(28−(−3))=31

Cofactor of element 2 =(−1)2+3|4310|=−1(0−3)=3
Third Row

Cofactor of element 3 =(−1)3+1|13−12|=+1(2−(−3))=5

Cofactor of element 0 =(−1)3+2|40−12|=−1(8−(0))=−8

Cofactor of element 7 =(−1)3+3|4013|=1(12−(0))=12

Therefore, the cofactor matrix =[21−75631−8−9312]
Adjoint Matrix
12.10

=[216−9−73135−812]

Inverse Matrix, A−1=DetA1Adj.A=991[216−9−73135−812]
Inverse Matrix, A
=

992199699−999−7993199399599−89912
If we multiply the given matrix by its inverse matrix, we should get the identity matrix. That

is, A·A−1=A−1·A=I

I=A·A−1=403130−127992199699−999−7993199399599−89912=100010001
You can prove the above result by calculating it as done in the first example.

12.6 SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS

As stated at the beginning of this lesson, matrix geometry is suitable for writing a system of simultaneous linear equations in a convenient matrix form and solving it, i.e., finding its unknown values.
For example, the given system of equations,

a11x1+a12x2+a13x3=b1

$a_{21}x_1+a_{22}x_2+a_{33}x_3=b_2$

$a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3$

can be written in matrix form as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If we write the coefficient matrix as A, the variable matrix as X, and the constant matrix as B, we get

$A\,X=B$.

Using matrix inverse and matrix multiplication methods, we can obtain the unknown values X_1, X_2, X_3. Matrix theory provides the convenience of easily solving them, no matter how large the matrix size or how many numbers there are. However, the essential condition for solving is that the number of unknowns must be equal to the number of independent equations. There are two different methods to solve a system of simultaneous linear equations using matrix theory. They are:

1. Matrix Inverse Method,
2. Cramer's Rule.

Using these, we will solve the system of equations and find their unknown values.

12.6.1 Matrix Inverse Method

If matrix A is a non-singular matrix, i.e., its determinant is not zero, then it has an inverse (A^{-1}). Multiplying both sides of the equation $AX=B$ by A^{-1} from the left, we get:

$A^{-1}A\,X=A^{-1}\,B$

We have already proved that if any matrix is multiplied by its inverse matrix, either from the front or the back, we get the Identity Matrix (I). Therefore,

$IX=A^{-1}\,B$

Also, we know that if any matrix is multiplied by the identity matrix I, its value does not change. Therefore,

$X=A^{-1}\,B$.

That is, to obtain the unknown values of matrix X, we need to find the inverse of that matrix and multiply it by matrix B, which is the matrix of constant values.

Example - 3: Solve the following system of linear equations using the matrix inverse method.

Solution:

$5x_1+3x_2=30$

$6x_1-2x_2=8$

Writing the given set of equations in matrix form:

$$\begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 30 \\ 8 \end{bmatrix}=Ax=B$$

$A^{-1}A\,X=A^{-1}\,B$

$IX=A^{-1}\,B$

$X=A^{-1}\,B$

Obtaining the determinant for the given coefficient matrix:

$$|A|=\begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix}=[(5\times-2)-(3\times6)]=-10-18=-28\neq0.$$

Since the determinant value is not zero, this is a "Non-Singular Matrix". We can find the inverse for this matrix.

Cofactor Matrix

First, let's find the cofactors of the elements of this matrix and the cofactor matrix. Let's find the cofactors for the elements in the first row of the matrix, then for the elements in the second row.

Cofactor of element 5 =(−1)1+1|−2|=(1)(−2)=−2

Cofactor of element 3 =(−1)1+2|6|=(−1)(6)=−6.

Cofactor of element 6 =(−1)2+1|3|=(−1)(3)=−3

Cofactor of element -2 =(−1)2+2|5|=(1)(5)=5

Therefore, the cofactor matrix =[−2−3−65]
Adjoint Matrix

=[−2−6−35]
Inverse Matrix, A−1=DetA1Adj.A=−281[−2−6−35]
X=A−1B=[−28−2−28−6−28−3−285][308]

=−28−2×30+−28−3×8=2860+2824=2884=3−28−6×30+−285×8=28180−2840=28140=5
Therefore, the value of $x_1=3$, and the value of $x_2=5$.
By substituting the values in the given set of equations:

$5x_1+3x_2=30$

$6x_1-2x_2=8$

$5(3)+3(5)=30$

$6(3)-2(5)=8$

$15+15=30$

$18-10=8$
Example - 4: Solve the following system of linear equations using the matrix inverse method.

$7x_1-x_2-x_3=0$

$10x_1-2x_2+x_3=8$

$6x_1+3x_2-2x_3=7$
Solution:
12.14
Writing the given system of equations in matrix form:

[7106−1−23−11−2][x1x2x3]=[087] AX=B
$A^{-1}A\ X=A^{-1}\ B$
$IX=A^{-1}\ B$
$X=A^{-1}\ B$
Obtaining the determinant for the given coefficient matrix:

|A|=|7106−1−23−11−2|=7[(−2×−2)−(1×3)]−(−1)[(10×−2)−(1×6)]+(−1)[(10×3)−(−2×6)]

=7[4−3]+1[−20−6]−1[30+12]

=7[1]+1[−26]−1[42]

=7−26−42

=−68+7=−61=0

Since the determinant value is not zero, this is a "Non-Singular Matrix". We can find the inverse for this matrix.

Cofactor Matrix

First, let's find the cofactors of the elements of this matrix and the cofactor matrix. Let's find the cofactors for the elements in the first row of the matrix, then for the elements in the second row, and then for the elements in the third row. We assign signs to the elements according to the (−1)r+c rule based on their position.

First Row:

Cofactor of element 7 =(−1)1+1|−231−2|=+1(4−3)=1

Cofactor of element -1 =(−1)1+2|1061−2|=−1(−20−6)=26

Cofactor of element -1 =(−1)1+3|106−23|=+1(30+12)=42

Second Row

Cofactor of element 10 =(−1)2+1|−13−1−2|=−1(2+3)=−5

Cofactor of element -2 =(−1)2+2|76−1−2|=+1(−14+6)=−8

Cofactor of element 1 =(−1)2+3|76−13|=−1(21+6)=−27

Third Row

Cofactor of element 6 =(−1)3+1|−1−2−11|=+1(−1−2)=−3

Cofactor of element 3 =(−1)3+2|710−11|=−1(7+10)=−17

Cofactor of element -2 =(−1)3+3|710−1−2|=+1(−14+10)=−4

Therefore, the cofactor matrix (1−5−326−8−1742−27−4)

Adjoint Matrix

=[12642−5−8−27−3−17−4]

Inverse Matrix, A−1=DetA1Adj.A=−611[12642−5−8−27−3−17−4]

Inverse Matrix, A

=

−611−6126−6142−61−5−61−8−61−27−61−3−61−17−61−4

x1x2x3=A−1B=−61161−2661−4261561861276136117614087 x1=−611×0+615×8+613×7=0+6140+6121=6161=1

x2=61−26×0+618×8+6117×7=0+6164+61119=61183=3

x3=61−42×0+6127×8+614×7=0+61216+6128=61244=4

Therefore, x1x2x3=134

12.6.2 CRAMER'S RULE
GIVEN THE SYSTEM OF EQUATIONS,

a11x1+a12x2+a13x3=b1

a21x1+a22x2+a33x3=b2

a31x1+a32x2+a33x3=b3
If we write the coefficient matrix as A, the variable matrix as X, and the constant matrix as B in matrix form as follows:

[a11a21a31a12a22a32a13a23a33][X1X2X3]=[b1b2b3]
We already know that we get A X=B. To find the values of the X-matrix, a mathematician named Cramer developed an easy method. According to that rule, when the determinant is not zero, i.e., Det. A□=0,

xi=Det.ADi,i=1,2,3,.....n

That is, X1=Det.AD1, X2=Det.AD2, X3=Det.AD3
Here, D1 is the determinant obtained by substituting the constant matrix from the right side of the equation in place of the values in the first column of the coefficient matrix. D2 is the determinant obtained by substituting the constant matrix from the right side of the equation in place of the values in the second column of the coefficient matrix. D3 is the determinant obtained by substituting the constant matrix from the right side of the equation in place of the values in the third column of the coefficient matrix. In this way, we can obtain the values of subsequent variables.

$\pi_{r}=\frac{B}{ctanA},|\begin{matrix}a_{r}&a_{r}\\ a_{r}&a_{r}&a_{r}\\ |a_{r}&a_{r}&a_{r}\end{matrix}|\\ |a_{r}&a_{r}&a_{r}|\\ |a_{r}&a_{r}&a_{r}|\\ |a_{r}&a_{r}&a_{r}|.\end{matrix}$ ||u1u1u1u1|u1|u1|u1|u2u2u2u2u2u2u2u2u2u2u2u2u2 u2u2u2u2|.
Example - 5: Solve the following system of linear equations using Cramer's rule.
Solution:
5x1+3x2=30 6x1−2x2=8
Writing the given system of equations in matrix form:
[563−2][x1x2]=[308]=Ax=B
X1=Det.AD1,X2=Det.AD2,
Obtaining the determinant for the given coefficient matrix:
|A|=|563−2|=[(5x−2)−(3×6)]=−10−18=−28□=0
Since the determinant value is not zero, this is a "Non-Singular Matrix".
D1=|3083−2|=[(30x−2)−(3×8)]=−60−24=−84 D2=|56308|=[(5×8)−(30×6)]=40−180=−140

s∘osg3=x1=Det.A|D1|=−28−84=3 x2=Det.A|D2|=−28−140=5
Example - 6: Solve the following system of linear equations using Cramer's rule.
Solution:
7x1−x2−x3=0 10x1−2x2+x3=8 6x1+3x2−2x3=7
Writing the given system of equations in matrix form:
[7106−1−23−11−2][x1x2x3]=[087]=ax=6
Obtaining the determinant for the given coefficient matrix:
|A|=|7106−1−23−11−2|=7[(−2x−2)−(1×3)]−(−1)[(10x−2)−(1x6)]+(−1)[(10x3)−(−2x6)]
=7[4−3]+1[−20−6]−1[30+12]
=7[1]+1[−26]−1[42]
=7−26−42
=−68+7=−61□=0
Since the determinant value is not zero, this is a "Non-Singular Matrix".
|D1|=|087−1−23−11−2|=0[(−2x−2)−(|x3)]−(−1)[(8x−2)−(|x7)]+(−1)[(8x3)−(−2x7)]
=0[4−3]+1[−16−7]−1[24+14]
=0[1]+1[−23]−1[38]
=0−23−38
=−61
|Dz|=|7106087−11−2|=7[(8x−2)−(1×7)]−0[(10x−2)−(1×6)]+(−1)[(10x7)−(8x6)]
=7[−16−7]+O[−20−6]−1[70−48]
=7[−23]+O[−26]−1[22]
=−161+0−22
=−183
$|D_{3}|=|\begin{matrix}7&-1&0\\ 10&-2&8&8&8&7[(-2\times7)-(8\times3)]-[(10\times7)-(8\times6)]+0)[(10\times3)-(-2\times6)]$
=7[−14−24]+1[70−48]−O[30+12]
=7[−38]+1[122]−O[42]
=−266+22+0
=−244
Therefore = X =
D Det.A
=
-61
= 1
-61
x2=Det.A|D2|=−61−183=3 x3=Det.A|D3|=−61−244=4
In this lesson, we learned how to find the inverse of a matrix. To find the matrix inverse, the matrix determinant should not be zero. We found the co-factors and co-factor matrix for the given matrix according to the (−1)r+c rule. After obtaining the transpose of the Co-Factor Matrix,

Mathematical Methods that is, after writing the row elements of the co-factor matrix as column elements and the column elements as row elements, we obtained the Adjoint Matrix. After dividing the adjoint matrix by the determinant, we were able to obtain the matrix inverse. Matrix theory provides a neat way to write systems of simultaneous equations, no matter how large they are. By evaluating the determinant of the 'coefficient matrix' of the system of equations, matrix mathematics allows us to test for the Existence of Solution. Moreover, it provides us with a method to find its solution. The solution of a system of equations can be found by two different methods: the matrix inverse method and Cramer's rule. In the matrix inverse method, if we find the inverse of the coefficient matrix and multiply it by the matrix of constants on the right side, we get the unknown values. According to Cramer's rule, after finding the determinant of the coefficient matrix, we substitute the matrix of constants on the right side in place of the columns in the coefficient matrix and find the determinants. If we divide the various determinants thus obtained by the determinant of the coefficient matrix, we get the unknown values.

12.8 GLOSSARY

1. Singular Matrix
2. Non-Singular Matrix
3. Determinant of a matrix
4. Minor of a determinant
5. Co-Factor Matrix
6. Transpose
7. Adjoint Matrix
8. Inverse of Matrix
9. System of Simultaneous Equations
10. Existence of Solution
11. Cramer's Rule
12. Unknown Values
13. Constants
14. Coefficients

12.9 SAMPLE EXAM QUESTIONS

12.9.1 Answer the following questions briefly:

1. Find the determinant of the following matrix. [61434−11−25]
2. Find the determinant of the following matrices. [3456][−1407] [2−145] [3064185210]
3. Find the inverse of the following matrix.
4. Find the co-factor matrix of the following matrix.

12.9.2 ANSWER THE FOLLOWING QUESTIONS IN DETAIL:

1. Find the inverse of the following matrix [314−100232]
2. (a) Solve the following system of simultaneous linear equations using the matrix inverse method. 2x1+x2=24 3x1−2x2=8
   (b) Solve the following system of simultaneous linear equations using Cramer's rule. 5x1−2x2=15 4x1+x2=12
3. Solve the following system of simultaneous linear equations using the matrix inverse method: 3x1+x2+x3=4 x1−x2−x3=0 2x1+x3=5
4. Solve the following system of simultaneous linear equations using Cramer's rule. 2x1−x2=2 3x2+2x3=16 5x1+3x3=21

12.10 Suggested Reading

1. Alpha Chiang (2017), Fundamental Methods of Mathematical Economics, 4th Edition, New Delhi: McGraw Hills.
2. R. G. D. Allen, (2014), Mathematical Analysis for Economists, New Delhi: Trinity Press.
3. B.C. Mehta and G.M.K. Madnani, Mathematics for Economists, New Delhi: Sultan Chand & Sons.