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M.Sc. (Second) DEGREE EXAMINATION, DEC. – 2016

(Second Year)

MATHEMATICS

Topology and Functional Analysis

Time : 3 Hours **Maximum Marks : 70**

Answer any five questions selecting at least Two from each Section.
All questions carry equal marks.

SECTION - A

Q1) a) State and prove the Lindelof's theorem. Deduce that any open base for X has a countable subclass which is also an open base.

b) Show that subset of a topological space is dense if and only if it intersects every non empty open set.

Q2) a) Prove that the product of any non empty class of compact spaces is compact.

b) Show that any continuous mapping of a compact metric space into a metric space is uniformly continuous.

Q3) a) Prove that a metric space is compact if and only if it is complete and totally bounded.

b) Show that every compact metric space has the Bolzano-Weirstrass property.

Q4) a) Show that every compact Hausdorff space is normal.

b) State and prove the Tietze Extension theorem.

Q5) a) Show that a subspace of the real line R is connected if and only if it is an interval. In particular, prove that R is connected.

b) State and prove the Urysohn's Lemma.

SECTION - B

Q6) Let N and N^1 be normed linear spaces and T is a linear transformation of N into N^1. Then show that the following conditions on T are equivalent.

a) T is continuous

b) $x_n \to 0 \Rightarrow \mathrm{T}(x_n) \to 0$

c) There exists $\mathrm{K} \geq 0, \mathrm{K} \in \mathrm{R}$ such that $\| \mathrm{T}(x) \| \leq \mathrm{K} \| x \|$ for all $x \in \mathrm{N}$.

d) If $\mathrm{S} = \{x \| \| x \| < 1\}$ is the closed unit sphere in N, then its image T(S) is bounded in N^1.

Q7) a) State and prove the uniform boundedness theorem.

b) State and prove the closed graph theorem.

Q8) a) State and prove the Schwartz inequality. Deduce that the inner product in a Hilbert space is jointly continuous.

b) If M is a proper closed linear subspace of a Hilbert space H, then show that there exists a non zero vector Z_0 in H such that $Z_0 \perp \mathrm{M}$.

Q9) a) If T is an operator on H for which $(\mathrm{T}x, x) = 0$ for all x, then show that $\mathrm{T} = 0$. Using this result show that an operator T on H is self adjoint iff $(\mathrm{T}x, x)$ is real $\forall x$.

b) Prove that an operator T on H is unitary if and only if it is an isometric isomorphism of H onto itself.

Q10) a) If P is a projection on H with range M and null space N, then prove that $\mathrm{M} \perp \mathrm{N}$ iff P is self adjoint; and in this case $\mathrm{N} = \mathrm{M}^{\perp}$.

b) If P and Q are the projections on closed linear subspaces M and N of a Hilbert space H, then show that $\mathrm{M} \perp \mathrm{N} \Leftrightarrow \mathrm{PQ} = 0 \Leftrightarrow \mathrm{QP} = 0$.

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M.Sc. (Second) DEGREE EXAMINATION, DEC. – 2016

Second Year

MATHEMATICS

Measure and Integration

Time : 3 Hours **Maximum Marks : 70**

Answer any Five questions.
All questions carry equal marks.

Q1) a) Define a countable set. If A is a countable set then prove that the set of all finite sequences from A is also countable.

b) State and prove the Heine-Borel theorem.

Q2) a) If $m^*E = 0$, show that E is measurable.

b) If E_1 and E_2 are measurable, then show that $E_1 \cup E_2$ is measurable.

Q3) a) Let $<E_n>$ be an infinite decreasing sequence of measurable sets (ie). $E_{n+1} \subset E_n \; \forall n$. If $m\, E_1$ is finite, then prove that $m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$.

b) State and prove the Little Woods third principle.

Q4) a) If f and g are non negative measurable functions then prove that

i) $\int_E cf = c\int_E f$

ii) $\int_E f + g = \int_E f + \int_E g$

iii) If $f \le g$ a.e, then $\int_E f \le \int_E g$.

b) State and prove the Fatou's Lemma. Deduce the monotone convergence theorem.

Q5) a) State and prove the Labesgue convergence theorem.

b) Define convergence in measure. If $f_n \to f$ a.e then show that $f_n \to f$ in measure. Also if $\{f_n\}$ is a sequence that converges to f in measure, then show that $\exists$ a subsequence $\{f_{nk}\}$ that converge to f a.e.

Q6) a) Prove that a function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real valued functions on $[a, b]$.

b) Let f be an integrable function on $[a, b]$ and that $F(x) = F(a) + \int_a^x f(t)dt$. Then prove that $F'(x) = f(x)$ for almost all x in $[a, b]$.

Q7) a) State and prove the Holder's inequality.

b) Show that the L^p spaces are complete.

Q8) a) Let E be a measurable set such that $0 < \nu E < \infty$. Then prove that there is a positive set A contained in E with $\nu A > 0$.

b) State and prove the Lebesgue decomposition theorem.

Q9) a) Suppose that for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathscr{B}$ such that $\mu(B_\alpha \sim B_\beta) = 0$ for $\alpha < \beta$. Then show that there is a measurable function f such that $f \leq \alpha$ a.e on B_α and $f \geq \alpha$ a.e on $X \sim B_\alpha$. If g is any other function with this property then $g = f$ a.e.

b) State and prove the Hahn decomposition theorem.

Q10) a) Prove that the set function μ^* is an outer measure.

b) State and prove the Caratheodory theorem.

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M.Sc. (Second) DEGREE EXAMINATION, DEC. – 2016

Second Year

MATHEMATICS

Analytical Number Theory and Graph Theory

Time : 3 Hours **Maximum Marks : 70**

Answer any Five questions selecting at least 2 from each section

All questions carry equal marks

SECTION - A

Q1) a) For all $x \geq 1$, prove that

$$\sum_{n \leq x} d(n) = x \log x + (2c-1)x + 0\left(\sqrt{x}\right)$$

b) For $x > 1$, prove that

$$\sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + 0(x \log x).$$

Q2) a) For every $x \geq 1$, prove that

$$[x]! = \frac{\pi P^{\alpha(P)}}{P \leq x}$$

Where the product is extended over all primes $\leq x$ and $\alpha(P) = \sum_{m=1}^{\infty} \left[\frac{x}{P^m}\right]$.

b) If $x \geq 2$, prove that

$\log[x]! = x\log x - x + 0(\log x)$

and hence prove that

$$\sum_{n \leq x} \wedge(n) \left[\frac{x}{n}\right] = x \log x - x + 0(\log x)$$

Q3) a) Prove that the following holds

For $x \geq 2$, $\theta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)\, dt}{t}$

b) $\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t} dt$

Q4) a) Let P_n denote n^{th} prime. Then prove that the following asymptotic relations are logically equivalent.

i) $$\lim_{x\to\infty}\frac{\pi(x)\log x}{x}=1$$

ii) $$\lim_{x\to\infty}\frac{\pi(x)\log\pi(x)}{x}=1$$

iii) $$\lim_{n\to\infty}\frac{P_n}{n\log n}=1$$

b) Prove that the relation $M(x)=0(x)$ as $x\to\infty$ implies $\psi(x)\sim x$ as $x\to\infty$.

SECTION - B

Q5) a) Prove that a connected Graph G is Euler Graph if and only if all vertices of G are of even degree.

b) In a connected Graph G with exactly 2K odd vertices, prove that there exist K-edge disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

Q6) a) Prove that a graph is a tree if and only if it is minimally connected.

b) Prove that the number of labeled trees with n-vertices $(n\geq 2)$ is n^{n-2}.

Q7) a) Prove that every circuit has an even number of edges in a common with any cut set.

b) Prove that the ring sum of any two cut sets in a graph is either a third cut set or an edge disjoint union of cut sets.

Q8) a) Prove that the complete graph of five vertices is non planar.

b) Prove that any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

Q9) a) Prove that a connected graph with n-vertices and e-edges has $e-n+2$ regions.

b) Prove that the set consisting of all the circuits and the edge disjoint unions of circuits in a graph G is an abelian group under the ring sum operation $\oplus$.

Q10) a) Prove that the set consisting of all the cut sets and the edge disjoint unions of cut sets in a graph G is an abelian group under the ring sum operation.

b) Prove that in a vector space of a graph the circuit subspace and the cut-set subspace are orthogonal to each other.

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M.Sc. (Second) DEGREE EXAMINATION, DEC. – 2016

Second Year

MATHEMATICS

Rings and Modules

Time : 3 Hours **Maximum Marks : 70**

Answer any five questions

All questions carry equal marks

Q1) a) Define a Boolean algebra

In a Boolean algebra B prove that

i) $(a \wedge b)' = a' \vee b'$

ii) $(a \vee b)' = a' \wedge b'$ for all $a, b \in \mathrm{B}, a'$ stands for complement of a.

b) Let ϕ be a homomorphism of a ring R into another ring. Then Prove that $\mathrm{R}/_{\mathrm{Ker}\phi}$ is isomorphic to $\mathrm{I}m\phi$.

Q2) a) Prove that the set of all subrings of a ring form a complete lattice.

b) Prove that every proper ideal of a ring is contained in a maximal proper ideal.

Q3) a) Let B be a submodule of a A. Then prove that A is Noetherian if and only if B and A/B are Noetherian.

b) Prove that a module has a composition series if and only if it is both artinian and Noetherian.

Q4) a) Prove that an ideal P of a ring R is prime if and only if R/P is an integral domain.

b) Prove that a commutative ring is an integral domain if and only if 0 is a prime ideal of R.

Q5) a) Define radical of a ring R. Prove that radical of a ring R consists of all elements $r \in \mathrm{R}$ such that $1 - rs$ is a unit for all $s \in \mathrm{R}$.

b) Define a semi primitive ring R. Prove that the quotient $R/\text{Rad } R$ is a semi primitive ring where Rad R is the radical of R.

Q6) a) In a commutative ring prove that the following holds.

i) Every non unit is a zero divisor.

ii) Every prime ideal is maximal

iii) Every principal ideal is a direct summand.

b) Prove that every commutative regular ring is semi primitive.

Q7) Prove that the following conditions concerning the module A are equivalent.

a) A is completely reducible

b) A has no proper large submodule

c) L(A) is complemented

Q8) a) In a Noetherian ring prove that every nil radical is nil potent.

b) State and prove Hilbert Basis theorem.

Q9) a) Prove that every free module is projective.

b) If M is the direct sum of a family of modules $\{Mi / i \in I\}$ then M is projective if and only if each Mi is projective.

Q10) a) Define an injective module. Prove that an abelian group is injective module if and only if it is divisible.

b) Show that every R-module is injective if and only if R is completely reducible.

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