(DM 01)

M.Sc. DEGREE EXAMINATION,
DECEMBER 2019.
First Year
Mathematics
ALGEBRA

Time : Three hours　　　　　　Maximum : 70 marks

Answer any FIVE of the following questions.

All questions carry equal marks.

1. (a) If ϕ is a homomorphism of G into $\overline{G}$ with kernel K, then prove that $G/K \approx \overline{G}$.

   (b) State and prove Sylow's theorem for abelian groups.

2. (a) Prove that every permutation is a product of 2–cycle.

   (b) Find the all normal subgroups of S_4.

3. (a) Prove that every finite abelian group is the direct product of cyclic groups.

   (b) If G and G' are isomorphic abelian groups, then prove that for every integer $s, G(s)$ and $G'(s)$ are isomorphic.

4. (a) If U is an ideal of R and $1 \in U$, prove that $U = R$.

   (b) If U is an ideal of ring R, then prove that R/U is a ring and is a homomorphic image of R.

5. (a) If R is a unique factorization domain, then show that $R[x]$ is also unique factorization domain.

   (b) Prove that when F is a field, $F[x_1, x_2]$ is not a principle ideal ring.

6. (a) If a is any real number, Prove that $(a^m / m!) \to 0$ as $m \to 0$.

   (b) If $m > 0$ and n are integers, prove that $e^{\frac{m}{n}}$ is transcendental.

7. (a) Prove that if α, β are constructible, then so are $\alpha \pm \beta \alpha\beta$ and α / β (when $\beta \neq 0$)

   (b) Show that any field of characteristic 0 is perfect.

8. (a) Construct a polynomial of degree 7 with rational coefficients whose Galois group over Q is S_7.

   (b) Show that the polynomial $p(x) = x^5 - 6x + 3$ over Q are irreducible and have exactly two non-real roots.

9. (a) Show that the Lattice of invariant subgroup of any group is modular.

   (b) If a and b are any two elements of a modular lattice, then show that the intervals $I[a \cup b, a]$ and $I[b, a \cap b]$ are isomorphic.

10. (a) Prove that the following two types of abstract systems are equivalent:

(i) Boolean algebra

(ii) Boolean ring with identity

(b) If the elements $a_1, a_2, \ldots a_n$ are independent, then prove that

$$(a_1 \cup a_2 \ldots. \cup a_r \cup a_{r+1}, \ldots \cup a_s) \cap$$
$$(a_1 \cup a_2 \ldots \cup a_r \cup a_{s+1}, \ldots. \cup a_{st}) = a_1 \cup \ldots \cup a_r$$

(DM 02)

M.Sc. DEGREE EXAMINATION,
DECEMBER 2019.

First Year

Mathematics

ANALYSIS

Time : Three hours Maximum : 70 marks

Answer any FIVE of the following questions.

All questions carry equal marks.

1. (a) Let $\{E_n\}$ be a (finite or infinite) collection of sets E_α. Then prove that $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha \left(E_\alpha^c\right)$.

   (b) If X is a metric space and $E \subset X$, then prove that

   (i) E is closed

   (ii) $E = \overline{E}$ if and only if E is closed

   (iii) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

2. (a) Prove that A sub set E of the real line R^1 is connected if and only if it has the following property : If $x \in E,\ y \in E$ and $x < z < y$, then $z \in E$.

   (b) Construct a bounded set of real numbers with exactly three limit points.

3. (a) If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

   (b) Prove that $\frac{a_n}{s_n^2} \le \frac{1}{s_{n-1}} - \frac{1}{s_n}$, hence deduce that $\sum \frac{a_n}{s_n^2}$.

4. (a) If f is continuous mapping of a compact metric space X into Y. And if E is a connected subset of X, then prove that $f(E)$ is connected.

   (b) If f be monotonic on (a, b). Then prove that the set of points of $f(a, b)$ at which f is discontinuous is at most countable.

5. (a) State and prove fundamental theorem of calculus.

   (b) State and prove Integration by parts theorem.

6. (a) If f is continuous on $[a, b]$ then prove that $f \in \mathcal{R}$ on $[a, b]$.

(b) If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then prove that $f \in \mathcal{R}$.

7. (a) Prove that, there exist a real continuous function on the real line which is nowhere differentiable.

(b) Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose $|f_n(x)| \leq M_n$ $(x \in E, n = 1, 2, 3, \ldots)$ then prove that $\sum f_n$ converges uniformly of E if $\sum M_n$ converge .

8. State and prove Stone-Weierstrass theorem.

9. (a) Suppose ϕ is count ably additive on a ring R. Suppose $A_n \in \mathcal{R}$ $(n = 1, 2, 3 \ldots)$, $A_1 \subset A_2 \ldots, A \in \mathcal{R}$ and $A = \bigcup_{n=1}^{\infty} A_n$, then prove that as $n \to \infty$, $\phi(A_n) \to \phi(A)$.

(b) Let f and g are measurable real-valued functions defined on X, let F be real and continuous on R^2, and put $h(x) = F(f(x), g(x))$, $(x \in X)$ then prove that h is measurable.

10. (a) State and prove Lebesgue's dominated theorem.

(b) Suppose that $f = f_1 + f_2$, where $f_i \in \mathcal{L}(\mu)$ on $E(i = 1, 2, 3 \ldots.)$, then prove that $f \in \mathcal{L}(\mu)$ and $\int_E f d\mu = \int_E f_1 d\mu + = \int_E f_2 d\mu$.

(DM 03)

M.Sc. DEGREE EXAMINATION,
DECEMBER 2019.
First Year
Mathematics

COMPLEX ANALYSIS AND SPECIAL FUNCTIONS AND PARTIAL DIF. EQU.

Time : Three hours Maximum : 70 marks

Answer any FIVE of the following questions, selecting at least two questions from each section.

All questions carry equal marks.

SECTION A

1. (a) Prove that

 (i) $c + \int p_n dx = (p_{n+1} - p_{n-1})/(2n+1)$

 (ii) $\int_x^1 P_n dx = (P_{n-1} - P_{n+1})/(2n+1)$

 (b) State and prove orthogonal properties of Legendre's polynomial.

2. (a) Expand x in a series of the form $\sum_{r=1}^{\infty} C_r J_l(\lambda, x)$ valid for the region $0 \le x \le 1$, where λ_r are the roots of the equation $J_1(\lambda) = 0$.

 (b) If λ_i are positive roots of $J_0(\lambda) = 0$, show that $\frac{1-x^2}{8} = \sum_{i=l}^{\infty} \frac{J_0(\lambda, x)}{\lambda_i^3 J_1(\lambda_i)}$, where $-1 < x < 1$.

3. (a) Solve $t(y+z)dx + t(y+z+1)dy + tdz - (y+z)dt = 0$

 (b) Solve $yz^2(x^2 - yz)dx + zx^2(y^2 - zx)dy + xy^2$
 $(z^2 - xy)dz = 0$

4. (a) Solve $\cos(x+p)p + \sin(x+y)q = z$.

 (b) Solve $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)$
 $\sin xy - \cos xy$.

5. (a) Solve $rx^2 - 3sxy + 2ty^2 + px + 2qy = x + 2y$ by Monge's method.

 (b) Solve $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x+y)$.

SECTION B

6. (a) Let $R(z)$ be a rational function of z, show that $\overline{R}(z) = R(\overline{z})$ if all the coefficients in $R(z)$ are real.

(b) Calculate the square roots of $i, \sqrt{3}+3i$ and cube roots of i.

7. (a) If $\sum a_n$ converges absolutely then prove that $\sum a_n$ converges.

(b) If G is open and connected and $f: G \to C$ is differentiable with $f'(z) = 0$ for all z in G then show that f is constant.

8. (a) State and prove Goursat's theorem.

(b) Evaluate $\int_{\gamma} \frac{dz}{z^2 + \pi^2}$ where $\gamma(\theta) = \theta e^{i\theta}$ for $0 \le \theta \le 2\pi$.

9. (a) Show that $\int_0^{\infty} \frac{x^{-c}}{x+1} dx = \frac{\pi}{\sin \pi c}$ if $0 < c < 1$.

(b) State and prove general version of Rouche's theorem for curves other than circle in G.

10. (a) State and prove Maximum Modulus theorem.

(b) Evaluate $\int_0^{\infty} \frac{\cos x - 1}{x^2} dx$.

(DM 04)

M.Sc. DEGREE EXAMINATION,

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Mathematics

THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

Time : Three hours Maximum : 70 marks

Answer any FIVE of the following questions.

All questions carry equal marks.

1. (a) State and prove Uniqueness theorem.

   (b) If $\phi_1, \ldots \phi_n$ be n solutions of $L(y) = 0$ on the interval, then show that they are linearly independent if and only if $W(\phi_1, \ldots \phi_n)(x) \neq 0)$ for all x in I.

2. (a) Verify the functions ϕ_1 satisfies the equation $y'' - 4xy' + (4x^2 - 2\ )y = 0, \phi_1(x) = e^{x^2}$ and find a second independent solution.

   (b) Find two linearly independent power series solutions of $y'' + 3x^2xy' - xy = 0$.

3. (a) Solve $(2ye^{2x} + 2x\cos y)dx + (e^{2x} - x^2 \sin y)dy$.

   (b) Let M, N be two real valued functions which has continuous first partial derivatives on some rectangle $R : |x - x_0| \le a, |y - y_0| \le b$. Then show that $M(x, y) + N(x, y)\, y' = 0$ is exact in R if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

4. (a) Find the first four successive approximations $\phi_0, \phi_1, \phi_2, \phi_3$ for $y' = 1 + xy, y(0) = 1$.

   (b) By computing appropriate Lipschitz constants, show that $f(x, y) = x^2 \cos^2 y + y \sin^2 x$, on S: $|x| \le 1, |y| < \infty$. satisfy Lipschitz conditions on the sets S indicated.

5. (a) Suppose $y = (8 + i, 3i - 2)$; $z = (i, -i, 2)$; $w = (2 + i, 0, 1)$ are vector in C_3, then

      (i) Compute $y + z$

      (ii) Compute $y - z$.

   (b) Let ϕ be the vector-valued function defined for all real x by $\varphi(x) = (x, x^2, ix^4)$, then compute

      (i) $\phi(1)$

(ii) ϕ' and $\phi'(2)$

(iii) $\int_{-1}^{1} \phi(x)\, dx$.

6. (a) State and prove Non-local existence theorem.

   (b) Show that all solutions with values in R_2 of the following system exists for all real

   $x\, y_1' = a(x)\cos y_1 + b(x)\sin y_2,\ y_2' = c(x)\sin y_1 + \ d(x)\cos y_2$ where a, b, c, d are polynomials with real -coefficients.

7. (a) Find the general solution of Riccatis equation $y' = y^2 - 2y + 2$.

   (b) Find the greens function of the boundary value problem $y'' + y = -f(x),\ y(0) = 0,\ y(1) = 0$.

8. (a) Show that if z_1, z_2, z_3 are any four different solutions of the Riccati equation.

   $y' + a(x)\, y + b(x)\, y^2 + c(x) = 0$, then show that $\dfrac{y - y_2}{y - y_1} = \dfrac{y_3 - y_1}{y_3 - y_2}$.

   (b) Find the functions $z(x), k(x)\, m(x)$ such that $z(x)\left[x^2 y'' - 2xy' + 2y\right] = \dfrac{d}{dx}\left(k(x)\, y' + m(x)\, y\right)$ and hence solve $x^2 y'' - 2xy' + 2y = 0,\ x > 0$.

9. (a) State and prove strum seoaration theorem.

   (b) Solve $x^2 y'' - 2xy' + \left(2 + x^2\right) y = 0,\ x > 0$.

10. (a) State and prove Gronwalls inequality.

    (b) Discuss the oscilation of Bessel equation

    $x^3 y'' - xy' + \left(x^2 - n^2\right) y = 0$.

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