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M.Sc. DEGREE EXAMINATION,
DECEMBER 2019.
Second Year
Mathematics

TOPOLOGY AND FUNCTIONAL ANALYSIS

Time : Three hours Maximum : 70 marks

Answer any FIVE of the following questions, selecting at least Two questions from each Section.

All questions carry equal marks.

SECTION – A

1. (a) Let T_1 and T_2 be two topologies on anon-empty set X, and show that $T_1 \cap T_2$ is also a topology on X.

   (b) Show that a subset set of a topological space is dense $\Leftrightarrow$ it intersects every non-empty open set.

2. (a) Prove that a closed subspace of a complete metric space is compact '$\Leftrightarrow$ it is totally bounded.

   (b) Prove that any continuous mapping of a compact metric space into a metric space is uniformly continuous.

3. (a) State and prove Ascolis theorem.

   (b) Prove that a metric space is compact $\Leftrightarrow$ it is totally bounded and complete.

4. (a) Show that any finite T_1–space is discrete.

   (b) Prove that a topological space is a T_1-space $\Leftrightarrow$ each point is a closed set.

5. (a) State and prove a generalization of Tietze's theorem which relates to functions whose values lie in R^n.

   (b) State and prove Urysohn's lemma.

SECTION – B

6. (a) Define Banach Space and give some examples.

   (b) Let N and N' be normed linear space and Ta linear transformation of N into N'. Then show that the following conditions on T are all equivalent to one another :

      (i) T is continuous.

(ii) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$.

(iii) There exist a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\|x\|$ for every $x \in N$.

(iv) If $S = \{x : \|x\| \leq 1\}$ is closed unit space in N, then its image $T(S)$ is bounded set in N'.

7. (a) State and prove open mapping theorem.

(b) State and prove closed graph theorem.

8. (a) Show that the parallelogram law is not true in $l_1^n (n > 1)$.

(b) State and prove the Uniform Boundedness Theorem.

9. (a) Show that $\|TT^*\| = \|T\|^2 9a$. State and prove Bessel's inequality.

(b) State and prove Bessel's inequality.

10. (a) If P and Q are the projections on a closed linear subspaces M and N of H, prove that PQ is a projection if and only if $PQ = QP$. In this case, show that PQ is the projection on $M \cap N$.

(b) If T is an operator on H for which $(T_{x,x}) = 0$ for all x, then prove that $T = 0$.

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Second Year

Mathematics

MEASURE AND INTEGRATION

Time : Three hours Maximum : 70 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) Prove that every subset of a finite set is finite.

   (b) If areal-valued function f is defined and continuous on a closed and bounded set F of real numbers, then show that it is uniformly continuous on F.

2. (a) Prove that the interval $(a, \infty)$is measurable.

   (b) Let $\langle E_i \rangle$ be a sequence of measurable sets, then prove that $m\left(\bigcup E_i\right) \le \Sigma\, mE_i$ and if E_i are pair wise disjoint, then prove that $m\left(\bigcup E_i\right) = \sum mE_i$.

3. (a) State and prove Lusin's theorem.

   (b) Let D and E be measurable sets and f a function with domain $D \cup E$, then show that f is measurable if and only if its restrictions to D and E are measurable.

4. (a) Let f be bounded function defined on $[a,b]$. If f is Riemann integrable on $[a,b]$, then show that it is measurable and

   $R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$.

   (b) State and prove Momotone Convergence theorem.

5. (a) Let f be integrable over E. Then show that, for given $\in > 0$, there is a simple function φ such that $\int_E |f - \varphi\,| < \in$.

   (b) Show that if $\langle f_n \rangle$ is a sequence that converges to f in measure, then each subsequence $\langle f_{nk} \rangle$ converges to f in measure.

6. (a) If f is continuous on $[a,b]$ and one of its derivatives (say D^+) is everywhere nonnegative on (a,b), then show that f is non-decreasing on $[a,b]$; i.e $f(x) \le f(y)$ for $x \le y$.

(b) If f be integrable function on $[a,b]$ and suppose that $F(x) = F(a) + \int_a^x f(t)\,dt$, then prove that $F'(x) = f(x)$ for almost all x in $[a,b]$.

7. (a) State and prove Holder inequality.

(b) Given $f \in L^p, 1 \le p < \infty$ and $\varepsilon > 0$, then prove that there is a step function φ and continuous function ψ such that $\|f - \varphi\|_p < \varepsilon$ and $\|f - \psi\|_p < \varepsilon$.

8. (a) If $E_i \in \mathfrak{B}$, then prove that $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu E_i$.

(b) Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathfrak{B}$ such that $\mu(B_\alpha \sim B_\beta) = 0$ for $\alpha < \beta$. The prove that there is measurable function f on X such that $f \le \alpha$ a.e on B_α and $f \ge \alpha$ a.e. on $X \sim B_\alpha$.

9. State and prove Hahn Decomposition theorem.

10. State and prove Caratheodory theorem.

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M.Sc. DEGREE EXAMINATION, DECEMBER 2019.

Second Year

Mathematics

ANALYTICAL NUMBER THEORY AND GRAPH THEORY

Time : Three hours Maximum : 70 marks

Answer any FIVE of the following questions, selecting at least two questions from each section.

All questions carry equal marks.

SECTION – A

1. (a) For all $x \geq 1$, prove that $\sum_{n \leq x} \sigma_1(n) = \frac{1}{2}\zeta(2)x^2 + O(x \log x)$ Prove that $\sum_{n \leq x} \sigma_\alpha(n) = \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + O(x^\alpha)$, where $\beta = \max\{1, \alpha\}$

   (b) State and prove Euler's summation formula.

2. (a) For all $x > 2$, Prove that $\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + o(x)$ where the sum is extended over all primes $\leq x$.

   (b) State and prove Legendre's identity.

3. (a) State and prove Shapiro's Tauberian theorem.

   (b) For a $x \geq 2$, Prove that $v(= \pi(x)\log x - \int_2^x \frac{\pi(t)}{t} dt$ and $\pi(x) = \frac{v(x)}{\log x} + \int_2^x \frac{v(t)}{t \log^2 t} dt$.

4. (a) Prove that the prime number theorem implies $\lim_{n \to \infty} \frac{M(x)}{x} = 0$

   (b) State and prove Selbergs asymptotic formula.

5. (a) Prove that, In a connected graph G with exactly $2k$ odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

   (b) Prove that, a complete graph with n vertices there are (n – 1)/2 edge-disjoint Hamiltonian circuits ,if n is an odd number $\geq$ 3.

SECTION – B

6. (a) Explain Traveling-Salesman problem.

   (b) Prove that, an Euler graph G is arbitrary traceable from vertex y in G if and only if every circuit in G contains v.

7. (a) Prove that, In any tree(with two or more vertices) , there are at least two pendent vertices.

   (b) Prove that, every connected graph has at least one spanning tree.

8. (a) Prove that, a vertex v in a connected graph G is a cut-vertex if and only if there exist two vertices xx and y in G such that every path between x and y passes through v.

   (b) Prove that, the vertex connectivity of any graph G can never exceeds the edge connectivity of G.

9. (a) Prove that, any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

   (b) Prove that , a connected graph with n vertices and e edges has $e - n + 2$ regions.

10. (a) Prove that ,the set consisting of all the cut-sets and the edge-disjoint union of cut-sets in a graph G is an abelian group under the ring sum operation.

   (b) Explain basis vectors of a graph.

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Mathematics

RINGS AND MODULES

Time : Three hours   Maximum : 70 marks

Answer any FIVE of the following. All questions carry equal marks.

1. (a) Show that Boolean algebra becomes a completed distributive lattice by defining $a \vee b = (a' \wedge b')$, $1 = 0'$.

   (b) Show that in any Boolean ring, we have the identities $a + a = 0, ab = ba$.

2. (a) Prove that the endomorphisms of an abelian group form a ring if addition is defined in a natural way.

   (b) If A, B and C are additive subgroups of R then prove that $(AB)C = A(BC)$. Moreover

   $AB \subset C \Leftrightarrow C : B$

   $\Leftrightarrow B \subset A : C$.

3. (a) Prove that the central idempotents of a ring R form a Boolean algebra $B(R)$.

   (b) If S is a sub-ring of R and K is an ideal, show that $(S+K)/K \cong S/(S \cap K)$.

4. (a) Let R be commutative ring, prove that the following are equivalent

   (i) R has unique prime ideal P.

   (ii) R is local and RadR = radR.

   (iii) non-units are zero-divisors.

   (iv) R is primary and all non-units are zero-divisors.

   (b) Show that the ring of $n \times n$ matrices over a field is a regularring.

5. (a) Prove the following statements concerning the commutative ring R are equivalent.

   (i) Every irreducible fraction has domain R.

(ii) For every f there exist an element $s \in R$ such that $fd = sd$ for all $d \in D$, the domain of f.

(iii) $Q(R) \cong R$ canonically.

(b) Determine all prime and maximal ideals as well as both radicals of $Z(n)$, the ring of integers modulo n.

6. (a) If A_R is an irreducible module, then its ring of endomorphisms $D = Hom_R(A, A)$ is a division ring.

(b) Show that a prime ring with a minimal right ideal is (right) primitive.

7. (a) Prove that, the radical is the largest ideal K such that, for all $r \in K$.

(b) If K and P are ideals such that $K \subset P \subset R$, show that P/K is prime if and only if P is prime.

8. (a) Prove that, a vector space is completely reducible.

(b) If R is right Artinian then, prove that $\text{Rad}\, R = \text{rad}\, R$.

9. (a) Prove that, every R-module is projective if and only if R is completely reducible.

(b) Prove that, M is projective if and only if every ephimorphism $\pi : B \to M$ is direct.

10. (a) Show that every R-module is injective if and only if R is completely reducible.

(b) Prove that, M is injective if and if only M has no proper essential extension.