Total No. of Questions : 10]

DM21

M.Sc. DEGREE EXAMINATION, JUNE/JULY - 2019

(Second Year)

MATHEMATICS

Topology and Functional Analysis

Time : 3 Hours **Maximum Marks : 70**

Answer any Five of the following questions.
selecting at least two questions from each section.
All questions carry equal marks.

SECTION – A

Q1) a) Show that a subspace of a topological space is itself a topological space.

b) Show that a topological space X is metrizable there exist a homeomorphism of X onto a subspace of some metric space Y.

Q2) a) State and prove Tychonoff's Theorem.

b) Prove that every sequentially compact metric space is totally bounded.

Q3) a) State and Prove Ascoli's theorem.

b) Show that R^{∞} is not locally compact.

Q4) a) Prove that the product of any non-empty class of Hausdorff space is a Hausdorff space.

b) Prove that compact subspace of a Hausdorff space is closed.

Q5) a) State and prove the Urysohn Imbedding theorem.

b) State and prove a generalization of Tietze's theorem which relates to functions whose values lie in R^n

SECTION – B

Q6)

a) Define Banach Space and give some examples.

b) Let N be a non-zero normed linear space, and prove that N is a Banach space $\Leftrightarrow$ $\{x : \|x\| = 1\}$ is complete.

Q7)

a) State and prove the Hahn-Banach Theorem.

b) If N is a normed linear space and x_0 is a non-zero vector in N, then there exist a functional f_0 in N^* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.

Q8) a) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

b) Define Hilbert Space and give some examples.

Q9) a) State and prove Bessel's inequality.

b) Prove that a Hilbert space H is separable if and only if every orthonormal set in H is countable.

Q10) a) If T is an operator on H for which $(T_{x,\, x}) = 0$ for all x, then prove that T = 0.

b) If T is an operator on H, then prove that the following conditions are all equivalent to each other.

i) $TT^* = I$.

ii) $(T_x, T_y) = (x, y)$ for all x and y.

iii) $\|Tx\| = \|x\|$ for all x.

Total No. of Questions : 10]

DM22

M.Sc. DEGREE EXAMINATION, JUNE/JULY - 2019

(Second Year)

MATHEMATICS

Measure and Integration

Time : 3 Hours **Maximum Marks : 70**

Answer any Five of the following questions.
All questions carry equal marks.

Q1) a) The union of a countable collection of countable sets is countable.

b) Show that every bounded infinite sequence has a subsequence that converges to a real number.

Q2)

a) Let $\{A_n\}$ be a countable collection of sets of real numbers, then prove that $m^*(\bigcup A_n) \leq \sum m^* A_n$.

b) If f is measurable function and $f = g$ a.e, then prove that g is measurable.

Q3) a) State and prove Egoroff's theorem.

b) State and prove Lusin's theorem.

Q4)

a) Let φ and ψ be simple functions which vanish outside a set of finite measure, then prove that $\int(a\varphi + b\Psi) = a\int\varphi + b\int\Psi$.

b) State and prove Fatou's Lemma.

Q5)

a) Show that if f is integrable over E, then so is $|f|$ and $\left|\int_E f\right| \leq \int_E |f|$

b) Let $\langle f_n \rangle$ be asequence of measurable functions that converges in measure to f. Then prove that,there is subsequence $\langle f_{n_k} \rangle$ that converges to f almost everywhere.

Q6)

a) Let f be an increasing real-valued function on the interval [a,b]. Then prove that f is differentiable almost everywhere. The derivative f' is measurable, and $\int_a^b f'(x)\,dx \leq f(b) - f(a)$.

b) If f is integrable on $[a, b]$, then show that the function F defined by $F(x) = \int_a^x f(t)dt$ is a continuous function of bounded variation on $[a, b]$.

Q7)

a) Prove the Minkowski Inequality for $1 \leq p < \infty$.

b) Given $f \in L^p$, $1 \leq p < \infty$, and $\varepsilon > 0$, then prove that there is a bounded measurable function f_M with $|f_M| \leq M$ and $\|f - f_M\| < \varepsilon$.

Q8)

a) If $E_i \in \mathfrak{B}$, $\mu E_1 < \infty$ and $E_i \supset E_1$, then prove that

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} \mu E_n.$$

b) Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathfrak{B}$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. The prove that there is unique measurable extended real valued function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \sim B_\alpha$

Q9) State and prove Hahn Decomposition theorem.

Q10)

The class of $\mathfrak{B}$ of μ^* - measurable sets is a σ -algebra. If $\bar{\mu}$ is μ^* restricted to $\mathfrak{B}$, then prove that $\bar{\mu}$ is complete measure on $\mathfrak{B}$.

Total No. of Questions : 10]

DM23

M.Sc. DEGREE EXAMINATION, JUNE/JULY - 2019

(Second Year)

MATHEMATICS

Analytical Number Theory and Graph Theory

Time : 3 Hours **Maximum Marks : 70**

Answer any Five of the following questions,
selecting at least two questions from each section.
All questions carry equal marks.

SECTION – A

Q1)

a) For all $x \geq 1$, Prove that $\sum_{n \leq x} \sigma_1(n) = \frac{1}{2}\zeta(2)x^2 + O(x \log x)$. Prove that $\sum_{n \leq x} \sigma_\alpha(n) = \frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha+1} + O(x^\alpha)$, where $\beta = \max\{1, \alpha\}$

b) State and prove Euler's summation formula.

Q2)

a) For all $x \geq 1$, prove that $\left|\sum_{n \leq x} \frac{\mu(n)}{n}\right| \leq 1$ with equality holding only if $x < 2$.

b) State and prove Legendre's identity.

Q3) a) State and prove Abel's identity.

b) For a $x \geq 2$, prove that $\nu(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\, dt$ and $\pi(x) = \frac{\nu(x)}{\log x} + \int_2^x \frac{\nu(t)}{t \log^2 t}\, dt$.

Q4) a) Prove that the prime number theorem implies $\lim_{n\to\infty}\frac{M(x)}{x}=0$.

b) State and prove Selberg's asymptotic formula.

Q5) a) Prove that, a simple with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

b) Prove that, a connected graph G is an Euler graph if and only if it can be decomposed into circuits.

SECTION – B

Q6) a) Explain Konigsberg Bridge problem.

b) Prove that, an Euler graph G is arbitrary traceable from vertex v in G if and only if every circuit in G contains v.

Q7) a) If in a graph G there is one and only one path between every pair of vertices, then G is a tree.

b) Prove that the distance between vertices of a connected graph is a metric.

Q8) a) Prove that the ring sum of any two cut-sets in a graph is either a third cut-set or an edge-disjoint union of cut-sets.

b) Prove that every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.

Q9) a) Prove that Kuratowski's is second graph is non-planar.

b) Prove that, a connected graph with n vertices and e edges has $e-n+2$ regions.

Q10) a) Prove that, the ring sum of two circuits in a graph G is either a circuit or an edge-disjoint union of circuits.

b) Explain about Modular arithmetic and Galois fields.

Total No. of Questions : 10]

DM24

M.Sc. DEGREE EXAMINATION, JUNE/JULY - 2019

(Second Year)

MATHEMATICS

Rings and Modules

Time : 3 Hours **Maximum Marks : 70**

Answer any Five of the following questions.

All questions carry equal marks.

Q1)

a) The class of semi-lattices can be equationally defined as the class of all semi-groups $(S, \wedge)$, then prove that $(S, \wedge)$ satisfying the commutative and idempotent laws.

b) Prove that a Boolean algebra $(S, 0', .)$ can be turned into a Boolean ring $(S, 0, 1, -, +, .)$ by defining $1 = 0', -a = a, a + b = ab' \vee ba'$ where $a \vee b = (a'b')'$.

Q2)

a) Prove that the sum $\sum_{i \in I} B_i$ of submodules of A_R is direct if and only if, for all $i \in I$, $B_i \cap \sum_{i \neq j} B_j = 0$.

b) Prove that the following statements are equivalent.

(1) R is isomorphic to a finite direct of rings $R_i (i = 1, 2, ..., n)$

(2) There exist central orthogonal idempotents, $e_i \in R$ such that $1 = \sum_{i=1}^{n} e_i$ and $e_i R \cong R_i$.

(3) R is a finite direct sum of ideals $K_i \cong R_i$.

Q3)

a) If B and C are sub-modules of a prove that $(B+C)/B \cong C/(B \cap C)$.

b) Prove that a module is Noetherian if and only if every sub-module is finitely generated.

Q4) a) Prove that radical of a commutative ring R consists of all nilpotent elements of.

b) Prove the following statements concerning the Boolean ideal K are equivalant.

(1) K is maximal

(2) K is prime

(3) For every element s, either $s \in K$ or $s' \in K$ but not both.

Q5) a) Prove that every equivalent class of factions exactly one irreducible fraction, and this extends all fractions in the class.

b) If R is a Boolean ring then prove that Q(R) is a Boolean ring.

Q6) a) Prove that, the ring R is prime if and only if there exist a faithful irreducible module A_R.

b) Show that a dense sub-ring of the ring of linear transformations of a vector space is primitive.

Q7) a) Show that, the prime radical of R is the set of all strongly nilpotent elements.

b) Prove that, the radical of R is the set of all rR such that 1-rs is right invertible for all $s \in R$.

Q8) a) Prove the following conditions concerning the module A are equivalent.

(1) A is completely reducible

(2) A has no proper large sub-module.

(3) L(A) is complemented.

b) If $e^2 = e \in R$ and $f^2 = f \in R$, then $eR \cong fR$ if and only if there exist $u, v \in R$ such that $vu = e$ and $uv = f$.

Q9) a) Prove that every free module is projective.

b) Prove that, M is projective if and only if every ephimorphism $\pi : B \to M$ is direct.

Q10)

a) If M is the direct product of a family of modules $\{M_i / i \in I\}$, then prove that, M is injective if and only if each M_i is injective.

b) Prove that, M is injective if and only if M has no proper essential extension.